Geometric bounds in spherically symmetric general relativity

Jemal Guven*
Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Apdo. Postal 70-543, 04510 México, D.F., Mexico

Niall Ó Murchadha†
Physics Department, University College Cork, Cork, Ireland
(Received 24 July 1997)

We exploit an arbitrary extrinsic time foliation of spacetime to solve the constraints in spherically symmetric general relativity. Among such foliations there is a one parameter family, linear and homogeneous in the extrinsic curvature, which permit the momentum constraint to be solved exactly. This family includes, as special cases, the extrinsic time gauges that have been exploited in the past. These foliations have the property that the extrinsic curvature is spacelike with respect to the the spherically symmetric superspace metric. What is remarkable is that the linearity can be relaxed at no essential extra cost which permits us to isolate a large nonpathological dense subset of all extrinsic time foliations. We now identify properties of solutions which are independent of the particular foliation within this subset. When the geometry is regular, we can place spatially invariant numerical bounds on the values of both the spatial and the temporal gradients of the scalar areal radius R. These bounds are entirely independent of the particular gauge and of the magnitude of the sources. When singularities occur, we demonstrate that the geometry behaves in a universal way in the neighborhood of the singularity. These results can be exploited to develop necessary and sufficient conditions for the existence of both apparent horizons and singularities in the initial data which do not depend sensitively on the foliation. [S0556-2821(97)05224-7]

PACS number(s): 04.20.Cv, 04.60.Kz

I. INTRODUCTION

In this paper we examine the constraints in general relativity when the spatial geometry is spherically symmetric and possesses just one asymptotically flat region [1,2]. This is the simplest gravitational scenario which exhibits local degrees of freedom.

In [2] we focused on the solution of the constraints when the extrinsic curvature K_{ab} vanishes. Though they are simple, they nonetheless display some of the features of the general problem. Such solutions are, however, very special. For if the extrinsic curvature vanishes, the momentum constraint requires that the current density of the matter fields J must also vanish. The solutions of the Hamiltonian constraint which result therefore correspond to ''momentarily static'' spatial geometries which do not generally occur in a given spacetime [3]. Since, if they occur at all, they occur as isolated objects, we did not need to fix the foliation. Here, we extend our work to cover the general situation where matter flows and, as a result, the extrinsic curvature is nonvanishing. The advantage of having dealt separately with the momentarily static solutions is that we can focus here on the physical feedback on the spatial geometry introduced by extrinsic curvature.

The introduction of extrinsic curvature complicates the analysis substantially. This occurs on two levels. The first is purely technical: the Hamiltonian constraint gets coupled to the momentum constraint—we have to solve a coupled system of equations. The second is conceptual: the constraints do not single out a unique slice through the spacetime—we need to specify some foliation.

In general, the initial data are given by specifying the intrinsic and the extrinsic geometry on some spacelike hypersurface which satisfy the constraints. A spherically symmetric geometry is completely characterized by specifying the areal radius R as a function of the proper radius ℓ. The extrinsic curvature can be expressed in a form consistent with spherical symmetry

$$K_{ab}=n_a n_b K_{\mathcal{L}}+(g_{ab}-n_a n_b)K_R, \tag{1}$$

where $K_{\mathcal{L}}$ and K_R are two spatial scalars and n^a is the outward pointing unit normal to the two-sphere of fixed radius in the slice.

How does one go about fixing the foliation? In principle, any foliation admitting globally regular solutions of the constraints is as good as any other. Ideally, therefore, we would like to consider a completely general slice through the spacetime; realistically, however, if the slicing is too general, it becomes very difficult to prove anything. At very least, the feedback on the spatial geometry introduced by extrinsic curvature should reflect the strength of the material currents flowing—the gauge certainly should not overshadow completely the underlying physics.

We will focus in this paper on an ''extrinsic time'' foliation of spacetime. This involves fixing some spatial scalar function of the extrinsic curvature. For a spherically symmetric geometry, we can cast this relationship in terms of two scalars appearing in Eq. (1) as follows:

$$\mathcal{F}[K_R,K_{\mathcal{L}}]=0, \tag{2}$$

*Email address: jemal@nuclecu.unam.mx
†Email address: niall@ucc.ie

with a possible dependence on R and ℓ which we have not indicated explicitly. Any gauge of this form should (at least implicitly) be solved to fix one of the scalars ($K_{\mathcal{L}}$ say) appearing in Eq. (1), in terms of the other.

All previous work on the constraints in spherically symmetric relativity has focused exclusively on some given foliation of this type. These have been maximal slicing or the so-called polar slicing [4]. The latter slicing mimics the $K_{ab}=0$ form of the Hamiltonian constraint and is the foliation which provides the standard presentation of the Schwarzschild geometry. Unfortunately, in either case, one is at a loss to know just how sensitively the solution depends on the choice of gauge. How will our notions of the size and the energy content change in another foliation? If they change in a way we cannot quantify they are almost useless. To address this kind of question it is desirable to work with as large a class of foliations as possible.

In [1], we introduced a function α, the ratio of the two scalars defining the extrinsic curvature

$$K_{\mathcal{L}}+\alpha K_R=0. \tag{3}$$

By setting α equal to some specified function, $\alpha=\alpha[K_R,R,\ell]$ say, Eq. (3) defines an extrinsic time foliation. If α is a function only of R and ℓ (in particular, if it is constant), the momentum constraint is exactly solvable for K_R.

It was shown in [1] that each constant value of α which is greater than 0.5 provides a globally regular slice for appropriate sources. These values of α correspond to a spacelike extrinsic curvature ''vector'' with respect to the superspace metric. As special cases we recover both the maximal slicing, with $\alpha=2$, and the polar gauge when $\alpha\to\infty$.

Remarkably, one can show that even when α is not a constant the gauge continues to provide regular slices of spacetime so long as $\alpha\leqslant 0.5$ asymptotically. All spacelike extrinsic curvatures in superspace provide regular foliations. The identification of potentially singular geometries will, however, require that the gradients of α be appropriately bounded.

The gauges we consider, in fact, represent a very large class of extrinsic time foliations. Recasting Eq. (2) in the homogeneous form, $K_{\mathcal{L}}=-\alpha[K_R,R,\ell]K_R$, ensures that when the material current J vanishes, $K_{ab}=0$. In particular, flat spacetime will be foliated by flat spatial hypersurfaces. Indeed, when the momentum constraint is satisfied, the extrinsic curvature is (quasi)linear in J, albeit in a nonlocal way. In this way the extrinsic curvature of the hypersurface responds directly to the movement of matter on it—a physically reasonable criterion.

Within this large class of extrinsic time foliations, there are universal properties exhibited by solutions of the constraints which are either independent of, or do not depend sensitively on the particular foliation. These properties divide naturally into those of globally regular geometries and those of singular geometries.

In [2], we examined these properties when the initial data was momentarily static. We first identified a geometrical bound on the spatial gradient of the areal radius R (the prime is a derivative with respect to proper radius),

$$-1\leqslant R'\leqslant 1, \tag{4}$$

independent of the source which was valid in all globally regular geometries. This bound was seen to operate at a more fundamental level than the positivity of the Arnowitt-Deser-Misner (ADM) mass. We then went on to investigate how the matter content of the slice can potentially force the appearance of either apparent horizons or singularities. We showed that when singularities occurred, they possessed a universal form and we could place bounds on the rate of divergence of geometrical scalars.

How do these results generalize? In globally regular geometries, the spacetime gradients of R are bounded. First, when the weak energy condition is satisfied and $\alpha\geqslant 0.5$ everywhere, the bound Eq. (4) on the spatial gradient continues to hold. Secondly, and perhaps more surprising, an analogous bound can be placed on the extrinsic curvature when the dominant energy condition is satisfied. We obtain the highly nontrivial result that

$$-1\leqslant RK_R\leqslant 1, \tag{5}$$

if $\alpha\geqslant 1$. The bound (5) can be interpreted as a bound on $\dot{R}$, the derivative of R with respect to normal proper time both of these bounds are independent of the source magnitude. They will play a central role in the establishment of sufficient conditions for the appearance of apparent horizons and singularities [5]. If $\alpha<1$, no such bound exists—indeed, a counterexample can be constructed.

Singular geometries can occur even though both ρ and J are finite. The only way that the geometry can become singular, however, is by pinching off at some finite proper radius from the center. Generically, at this radius (ℓ_S say), R will vanish nonanalyically

$$R\sim C(\ell_S-\ell)^{1/(\alpha+1)}. \tag{6}$$

Remarkably, the quasilocal mass (QLM) remains finite even when the geometry is singular. Indeed, we show that this is always true regardless of the gauge condition. Our ability to identify universal behavior of this form will be crucial for the establishment of necessary conditions for singular geometries in a subsequent publication [6].

Generally, the singularities of the three-geometry consistent with the constraints will be more severe than those which are admissable at a moment of time symmetry. If, however, the movement of matter is tuned so that the extrinsic curvature vanishes as the singularity is approached, the strength of the singularity will be determined entirely by the QLM, exactly as it is at a moment of time symmetry [2]. We show that this tuning corresponds to an integrability condition on the current. If, in addition, the tuning is refined so that the QLM also vanishes as we approach the singularity the curvature singularity disappears and the spatial geometry pinches off in a regular way. This latter integrability condition involving the QLM is completely analogous to the integrability condition encountered at a moment of time symmetry. Regularity at the singularity is, of course, precisely the condition that the interior be a regular closed universe. If the matter fields carry conserved charges these will, in their turn,

have integrability conditions associated with them. Viewed this way, regular closed universes appear to be very special universes [7].

The paper is organized as follows. We begin in Sec. II with a discussion of the solution of the momentum constraint. In Sec. III we provide a derivation of the bounds on R' and RK_R. In Sec. IV, we derive Eq. (6). In Sec. V, we discuss the integrability conditions and comment briefly on the regularity of Euclidean relativity. We conclude in Sec. VI with a brief discussion and outline of future work.

II. THE CONSTRAINTS

In this section we examine the analytical structure of the constraints when $K_{ab}\neq 0$. We recall that the constraints can be written as

$$K_R[K_R+2K_{\mathcal{L}}]-\frac{1}{R^2}[2(RR')'-R'^2-1]=8\pi\rho \qquad (7)$$

and

$$K_R'+\frac{R'}{R}(K_R-K_{\mathcal{L}})=4\pi J, \qquad (8)$$

where the line element on the spatial geometry is parametrized by

$$ds^2=d\ell^2+R^2d\Omega^2, \qquad (9)$$

ℓ is the proper radial distance on the hypersurface, R is the areal radius, and we have expanded the extrinsic curvature according to Eq. (1). All derivatives are with respect to the proper radius of the spherical geometry ℓ. The spatial geometries we consider consist of a single asymptotically flat region with a regular center $\ell=0$. We will subsequently refer to such geometries as regular. The appropriate boundary condition on the metric at $\ell=0$ is then

$$R(0)=0. \qquad (10)$$

We recall that $R'(0)=1$ if the geometry is regular at this point. We assume that both ρ and J are appropriately bounded functions of ℓ on some compact support. This compact support restriction could be easily relaxed so as to consider solutions where both ρ and J decay appropriately as one approaches infinity with little extra effort but also with little extra insight.

We define the quasilocal mass m as follows:

$$m=\frac{R}{2}(1-R'^2)+\frac{1}{2}K_R^2R^3. \qquad (11)$$

When the constraints (7) and (8) and the boundary condition at the origin, Eq. (10), are satisfied, m is determined by the sources as follows:

$$m=4\pi\int_0^{\ell}d\ell R^2[\rho R'+JRK_R]. \qquad (12)$$

This way, m arises as a first integral of the constraints. These equations are gauge invariant. In a globally regular geometry, m coincides at infinity with the ADM mass, m_∞. As we found in [2] in a simpler context, the introduction of m is extremely useful and will be exploited repeatedly in our analysis.

To solve the constraints classically, we need to specify some foliation. In this paper, we will focus on a gauge condition of the general form (3) where α is some specified function of the configuration variables R, K_R, and ℓ. It is possible to provide a geometrical interpretation for these gauges. To begin with, we know that when $\alpha=2$ this condition specifies a spacelike hypersurface with maximum volume in spacetime: the trace of the extrinsic curvature $K=K_{\mathcal{L}}+2K_R$ vanishes. It is simple to show that when $\alpha=\alpha(\ell)$, Eq. (3) is precisely the condition that the modified spatial volume of a closed ball,

$$V_\alpha=4\pi\int_0^{\ell}d\ell R^\alpha, \qquad (13)$$

be a maximum.

When α is a constant, the momentum constraint can be solved uniquely for K_R in terms of the radial flow of matter, J, as follows:

$$K_R=\frac{4\pi}{R^{1+\alpha}}\int_0^{\ell}d\ell R^{1+\alpha}J, \qquad (14)$$

where we have exploited the regularity of the geometry at the origin to set $K_R(0)=0$.

When Eqs. (3) and (14) are substituted into Eq. (7), we obtain a second order singular nonlinear integro-ordinary differential equation (ODE) for R [8]. Subject to the boundary condition, Eq. (10), the solution is uniquely determined. Not only is the extrinsic curvature completely determined by the material sources, so also is the spatial geometry. There are no independent gravitational degrees of freedom, exactly as expected. We note that in the gauge Eq. (3), the spatial geometry does not depend on the global sign of J. Of course, if we reverse the sign of J, the extrinsic curvature picks up a negative sign.

When α is not constant, it is still possible to mimic the solution when α is constant. To do this we recast the momentum constraint in the form

$$(R^{1+\alpha}K_R)'=4\pi R^{1+\alpha}J+\alpha'\ln(R/L)R^{1+\alpha}K_R, \qquad (15)$$

where L is any characteristic length scale. The spatial variation of α has been absorbed completely into the second term on the right-hand side (RHS). The solution is given by

$$K_R=\frac{4\pi}{R^{1+\alpha}}\int_0^{\ell}d\ell_1R^{1+\alpha}J\Delta(\ell_1,\ell), \qquad (16)$$

where

$$\Delta(\ell_1,\ell)=e^{\int_{\ell_1}^{\ell}d\ell_2\alpha'\ln(R/L)}. \qquad (17)$$

The constant α result is simply modulated by an exponential multiplicative correction Δ. We note in particular that $K_R=0$ when $J=0$. This provides a very strong justification for casting the gauge in the form of Eq. (3). If $\alpha=\alpha(R,\ell)$ alone, K_R will also be linear and homogeneous in J. If we

admit a K_R dependence explicitly into α the linear correlation of K_R and J no longer holds. K_R will nonetheless be positive when J is. If, however, α is an even function of K_R then K_R will echo the parity of J: $K_R[-J]=-K_R[J]$.

It is clear from inspection of the definition of Δ that spatial variations in α are antiscreened: remote source contributions get distorted more than nearby ones. This is potentially worrysome but, as we will see, it is not a serious obstacle.

We will now look more closely at the analytic structure of solutions. Let us first focus on regular geometries.

III. GLOBALLY REGULAR GEOMETRIES

We first comment on the behavior of the spatial geometry in the neighborhood of the origin. In fact, in the neighborhood of $\ell=0$, we have $R\sim\ell$, so that [9]

$$K_R\sim 4\pi\frac{J(0)}{2+\alpha(0)}\ell. \tag{18}$$

Thus if K_R is regular at the origin then it must also vanish there. This is not, however, surprising. Spherical symmetry is very restrictive leaving a regular geometry no freedom to evolve at the origin.

We can expand $R(\ell)$ in a power series in the neighborhood of $\ell=0$ and substitute into Eq. (7) to get

$$R(\ell)=\ell+\frac{4\pi}{9}\rho(0)\ell^3+\cdots. \tag{19}$$

A consequence of the vanishing of $K_R(0)$ is that J will only show up at order five in this expansion—two orders behind ρ. The metric at the origin clearly is not sensitively dependent on the current flowing there.

The other region we need to check is outside the source. What constraints does asymptotic flatness place on α? To recover an asymptotically flat spatial geometry we require that $R(\ell)\sim\ell$ to leading order. We have that $K_R\sim\text{constant}\Delta(\ell_0,\ell)/R^{1+\alpha}$. Now for an appropriate falloff (faster than ℓ^{-1}) on α', $\Delta(\ell_0,\ell)$ will always saturate so that we can absorb it into the constant. On one hand, we note that outside the source, the integral identity (12) implies m is a constant. However, the contribution of extrinsic curvature to m [Eq. (11)] $\sim 1/R^{2\alpha_\infty-1}$. There represents an inconsistency if α tends asymptotically to any value, α_∞, lower than 0.5. If $\alpha_\infty>0.5$, not only K_R but also its contribution to m vanish asymptotically. Such a choice is simultaneously regular at the origin.

We note that with strict inequality, $\alpha_\infty>0.5$, m be dominated asymptotically by the first term in Eq. (11) so that m_∞ is encoded completely in the intrinsic geometry. In the limiting case, $\alpha=0.5$, the intrinsic and the extrinsic geometries share the burden. However, such a falloff invalidates the traditional expressions for the ADM mass.

What can we say in general about globally regular geometries? We will demonstrate that they possess the remarkable property that for an appropriate dense subset of extrinsic time foliations both the spatial and temporal gradients of R are bounded numerically in a way which is entirely independent of the material sources and of α.

Suppose ρ satisfies the weak energy condition $\rho\geqslant 0$. Consider any foliation satisfying Eq. (3) with $\alpha\geqslant 0.5$ everywhere. Then, if the geometry is regular, $R'^2\leqslant 1$ everywhere. The proof is very simple and was given in Ref. [1]. We repeat it here to emphasize that the spatial variation of α does not enter: R' must be bounded in any regular geometry. We note that $R'(0)=1$ and $R'\to 1$ at infinity. Thus R' must possess an interior critical point. At this point $R''=0$. In the gauge (3), the Hamiltonian constraint, Eq. (7), now reads, at this point,

$$R'^2=1-8\pi\rho R^2-(2\alpha-1)R^2K_R^2. \tag{20}$$

Both the second and the third terms on the RHS are negative. The result follows immediately.

A simple corollary of this result is the positivity of m. We can obtain analogous bounds on K_R in the gauge Eq. (3). However, it is important to stress that without some control over J we should not expect K_R to be bounded. Let us therefore suppose that the dominant energy condition (DEC) $|J|\leqslant\rho$ is satisfied everywhere. Our experience in [1] suggests that when the DEC holds, the appropriate variables are the optical scalars defined by [10]

$$\omega_\pm=2(R'\pm RK_R), \tag{21}$$

which are (R times) the null expansions in the out-future and out-past directions. They are a useful set of variables to exploit when we are interested in identifying apparent horizons [1,10,5]. The optical scalar which marks the presence of a future trapped surface is ω_+: $\omega_+=0$ at a future apparent horizon.

It was shown in [10] (and again in [1]) that when the dominant energy condition $|J|\leqslant\rho$ is satisfied,

$$|\omega_\pm|\leqslant\kappa+\sqrt{|\kappa|^2+4}, \tag{22}$$

where $\kappa=\max|RK|$ and K is the trace of the extrinsic curvature. These bounds depend on the sources only through K. When $K=0$ ($\alpha=2$) they become numerical bounds which are independent of the sources:

$$-2\leqslant\omega_\pm\leqslant 2. \tag{23}$$

It then follows that

$$RK_R=(\omega_+-\omega_-)/4, \tag{24}$$

and $R'=(\omega_++\omega_-)/4$ are bounded, $|RK_R|\leqslant 1$ and $|R'|\leqslant 1$.

Here, we would like to show that even when $\alpha\neq 2$ in the gauge Eq. (3) it is still possible to bound $\omega_\pm$ by Eq. (23). One way to do this is to bootstrap on Eq. (22). This way, one can bound $|\omega_\pm|$ when α lies within the range $1<\alpha<3$. However, the bound does depend on α and diverges at the points, $\alpha=1$ and $\alpha=3$. However, one can do better.

We showed in [1] that it is possible to add and subtract the two constraints (7) and (8) to obtain simple equations for the spatial derivative of ω_+ and ω_-

$$(\omega_\pm)'=-8\pi R(\rho\mp J)-\frac{1}{4R}(\omega_+\omega_--4)\pm\omega_+K_{\mathcal{L}}, \tag{25}$$

We now exploit the gauge conditions (3) and (24) to recast these equations in the form

$$(\omega_\pm)' = -8\pi R(\rho \mp J) + \frac{1}{4R}[(\alpha-1)\omega_+\omega_- + 4 - \alpha\omega_\pm^2]. \tag{26}$$

Let us establish the inequality (23) for ω_+. The argument is very similar to the one we used above to derive the bound $R'^2 \leqslant 1$. We note that ω_+ must be bounded in a regular geometry and that $\omega_+(0)=2$ and $\omega_+ \to 2$ at infinity. Thus ω_+ must possess an interior critial point. At this point $\omega_+'=0$ so that

$$(\alpha-1)\omega_+\omega_- + 4 - \alpha\omega_+^2 = 32\pi R^2(\rho - J), \tag{27}$$

the right-hand side of which is positive by hypothesis. Thus

$$\alpha\omega_+^2 \leqslant (\alpha-1)\omega_+\omega_- + 4. \tag{28}$$

However, the quasilocal mass is positive or, equivalently, [1,10]

$$\omega_+\omega_- \leqslant 4, \tag{29}$$

so that when $\alpha \geqslant 1$, ω_+ satisfies Eq. (23).

In [1], we pointed out that the ''binding energy'' $M-m$ of a regular spherically symetric system satisfying the dominant energy condition is positive when the slicing is maximal. In general, we have

$$M - m = 4\pi \int_0^\ell d\ell R^2[(\rho+J)(2-\omega_+) + (\rho-J)(2+\omega_-)], \tag{30}$$

which is manifestly positive whenever $|\omega_\pm| \leqslant 2$. Thus this result is also true for all values of $\alpha \geqslant 1$.

It is clear from Eq. (28) that Eq. (23) cannot be extended to $\alpha<1$—$\omega_-\omega_+$ is not bounded from below. To obtain a bound we need to exclude both future and past trapped surfaces so that $\omega_+\omega_-$ is positive. We then have, for all positive α,

$$|\omega_+| \leqslant 2\max(2/\sqrt{\alpha}_{\min}, 1). \tag{31}$$

The results for ω_- are identical.

This implies the bound on K_R:

$$|RK_R| \leqslant \max(1/\sqrt{\alpha_{\min}}, 1). \tag{32}$$

We already have $R'^2 \leqslant 1$ for all α when the weak energy condition is satisfied.

One can, in fact, easily construct a counterexample demonstrating explicitly that we should not expect to do better than Eq. (31) when $\alpha<1$. We do this by examining the values assumed by the optical scalars in the neighborhood of $\ell=0$. We can combine Eqs. (18) and (19) to obtain

$$\omega_\pm \sim 2 - \frac{8\pi}{3}\left(\rho(0) \mp \frac{3}{2+\alpha}J(0)\right)\ell^2. \tag{33}$$

If the dominant energy condition is satisfied, then when $\alpha \geqslant 1$, we have $\omega_\pm \leqslant 2$ near the origin which is consistent with our result. If, however, $\alpha<1$ this is not necessarily the case. If $J(0)$ exceeds $(2+\alpha)\rho(0)/3$ and $\alpha<1$, then $\omega_+ \geqslant 2$ in the neighborhood of the origin. We note also that

$$\omega_+\omega_- \leqslant \left(2 - \frac{8\pi}{3}[\rho(0)\ell^2]\right)^2 - \left(\frac{8}{2+\alpha}\right)^2 \pi^2 J(0)^2 \ell^4 \leqslant 4. \tag{34}$$

This is consistent with the inequality Eq. (29). Note also that the absolute maximum of the product $\omega_+\omega_-$ obtains at the boundary values $\ell=0$ and $\ell=\infty$ and it is also the flat space value. When $K=0$, this is also true of both ω_+ and ω_-. In general, the absolute maximum of neither need occur at these points.

IV. SINGULAR GEOMETRIES

So far we have assumed that the geometry is regular everywhere. A nonsingular asymptotically flat solution defined for all $\ell \geqslant 0$ will not, however, always exist for every specification of ρ and J. In this section, our task will be to understand what can go wrong and to identify the mechanism driving the geometry into a singularity.

In Sec. III we showed that $R'^2 \leqslant 1$ in any globally regular geometry. Thus if $R'^2 > 1$ anywhere the geometry must be singular.

Let us suppose that $R'^2 > 1$ at some point. Then, when K_{ab} satisfies Eq. (3), Eq. (7) implies

$$RR'' = \frac{1}{2}[1 - R'^2] + \frac{R^2}{2}(1-2\alpha)K_R^2 - 4\pi\rho, \tag{35}$$

so that $R''<0$ and R' is decreasing there. This can only occur by R' falling through $R'=-1$. Once R' falls below this value it will continue decreasing monotonically thereafter. The surface with $R'=-1$ in the configuration space therefore acts as a oneway membrane. Suppose that the areal radius is R_0 when $R'=-1$. We know now that the solution must crash, i.e., $R \to 0$ within a finite proper distance which is less than or equal to R_0 from that point.

Since $R'' \leqslant 0$ whenever $R'=1$ we see that the surface $R'=1$ in the configuration space also acts as a oneway membrane and the solution can only pass downwards through it. However, since at a regular center we have $R'=1$ and R' starts to reduce as soon as we enter matter, it is clear that the region defined by $R'>1$ is completely forbidden.

We conclude that crashing through $R'=-1$ is the generic way the spatial geometry can become singular. Singularities with $R'=-1$ at $R=0$ are also possible. They result, however, only for special finely tuned matter distributions. We will discuss them more fully below in Sec. V. Putting the regular and singular results together, we have the following: if the geometry is globally regular, then $-1 \leqslant R' \leqslant 1$ everywhere; if $-1 < R' \leqslant 1$ everywhere, then the geometry is globally regular.

One possible way that this method of constructing initial data for a spherically symmetric gravitational field can break down is that the slicing turns null. Since R is a four-dimensional scalar, nothing will go bad with it. On the other hand, ℓ is the spacelike proper distance along the slice and so $d\ell$ will become small and thus R' will become unbound-

edly large if the slice turns null. But we have shown that this cannot happen if we assume $\alpha \geq 0.5$ and $\rho \geq 0$. Note that we do not have to assume that $\alpha =$ constant. Therefore, if we stay inside the light cone of the supermetric, we stay outside the light cone of the spacetime. For any slice satisfying $\alpha \geq 0.5$ the only possible singularity is when $R \to 0$.

Let us now examine more carefully the approach towards a singularity. In the neighborhood of the point $\ell = \ell_S > 0$ at which $R = 0$, Eq. (16) implies that

$$K_R \sim \frac{C_\alpha(\ell_S)}{R^{1+\alpha}}, \tag{36}$$

where

$$C_\alpha(\ell) = 4\pi \int_0^\ell d\ell_1 J R^{1+\alpha} \Delta(\ell, \ell_1) \tag{37}$$

is finite if Δ is. K_R will therefore be singular (for physically acceptable values of α) if the geometry pinches off unless the current is tuned such that

$$C_\alpha(\ell_S) = 0. \tag{38}$$

To examine the structure of singularities it is extremely useful to exploit the definition of the quasilocal mass introduced earlier. From a functional point of view, Eq. (11) is identical to the energy integral in classical mechanics. To exploit this analogy, we therefore recast this equation as follows:

$$R'^2 = 1 - \frac{2m}{R} + K_R^2 R^2, \tag{39}$$

where m is given by Eq. (12) and K_R by Eq. (14).

Let us suppose that m remains finite. Now, if C_α does not vanish and $\alpha > 0.5$, the most singular term in Eq. (39) is the quadratic in K_R. This implies that

$$R'^2 \sim R^2 K_R^2 \tag{40}$$

in the neighborhood of $R = 0$, or $R'^2 \sim C_\alpha^2 / R^{2\alpha}$. Generically, therefore, R'^2 diverges. The solution is

$$R \sim \left(\frac{C_\alpha}{\alpha + 1} \right)^{1/(\alpha+1)} (\ell_S - \ell)^{1/(\alpha+1)}. \tag{41}$$

If $\alpha > 0.5$, such spatial singularities are more severe than the strong singularities discussed in [2] which are consistent with the Hamiltonian constraint at a moment of time symmetry. We will refer to the generic kind of singularity driven by extrinsic curvature as a strong J-type singularity. As α increases, the power law determining the strength of the singularity increases. Note that the limit $\alpha \to \infty$ (the polar gauge discussed in Ref. [1]) is extremely singular. This is, however, a gauge artifact reflecting how poor the polar gauge really is. Unlike the strong singularities occurring when $K_{ab} = 0$, at which the scalar curvature $\mathcal{R}$ remained finite, $\mathcal{R}$ will generally blow up [just like $K_R^2 \sim 1/(\ell_S - \ell)^2$]. On dimensional grounds, we expect all curvature scalars to blow up as $1/(\ell_S - \ell)^2$ as we approach a singularity unless there is some constraint obstructing them from doing so.

To show that the above analysis is self-consistent, we need to demonstrate that, for finite α', (i) the form factor Δ defined by Eq. (17) remains finite; (ii) m remains finite.

To this end, we note that

$$\int_0^\ell d\ell\, \alpha' \ln R/L \leq |\alpha'|_{\max} \int_0^\ell d\ell |\ln R/L|. \tag{42}$$

The integrated logarithm is bounded. While the integrand diverges at $R = 0$, the integral is nevertheless well behaved. We note that

$$\int_0^s ds\ \ln s = s\ \ln s - s. \tag{43}$$

Thus in particular, the form factor is well behaved on the flat solution, $R = \ell$. This is essentially all we need to check because as we have just seen $R(\ell) \sim (\ell_0 - \ell)^{1/1+\alpha_0}$ at a singularity so that logarithm is multiplicatively identical to the flat space value.

We now confirm that m remains finite as we approach a strong singularity. We do this by demonstrating that the volume integral (12) is always finite. We note that, for suitably bounded ρ and J,

$$(\rho R^2 R', J R^3 K_R) \sim (\rho, J)(\ell_S - \ell)^{-[(\alpha-2)/(\alpha+1)]}. \tag{44}$$

If $\alpha \leq 2$, the integrand itself remains finite. In general, the integral will be finite if the exponent of $(\ell_S - \ell)^{-1}$ is bounded by 1. But $(\alpha - 2)/(\alpha + 1) < 1$ for all finite values of α thus guaranteeing that the integrals over $R^2 \rho R'$ and $R^3 J K_R$ converge. The only possible gap in this argument is the assumption that m remains finite. It is possible that m diverges fast enough when the singularity is approached so that m/R dominates $R^2 K_R^2$. This would require m to diverge faster than $R^{1-2\alpha}$. The first term in Eq. (12) cannot give a divergent m as $R^2 \rho$ obviously remains finite and $\int R' d\ell = R$ is also bounded as we approach the singularity. Therefore we need only consider the $J R^3 K_R$ term. This will diverge as $R^{2-\alpha}$. Let us assume that $R \sim (\ell_S - \ell)^\beta$ for some $\beta > 0$. Then, from Eq. (12), m will, at worst, diverge as $m \sim (\ell_S - \ell)^{(2-\alpha)\beta + 1}$. Now the requirement that the m term in Eq. (39) dominates implies $(2 - \alpha)\beta + 1 < (2\alpha - 1)\beta < 0$. This in turn gives $-(\alpha + 1)\beta > 1$, a negative β. Therefore it is clear that $m(\ell_S)$ is always finite.

The sign of m will, however, depend on the details of the current flow. This is obvious from the definition Eq. (11). Even if $R'^2 > 1$, a sufficiently large value of K_R can render m positive. In particular, unlike the value of m assumed at strong ρ singularities when $K_{ab} = 0$ (discussed in [2]) which is always negative, the sign generally can assume either value. Indeed m need never even be negative in a singular geometry. Though R' decreases monotonically, m nonetheless can remain positive. There is no conflict with the positive QLM theorem. In our examination of momentarily static configurations in [2], we found that m is positive everywhere except at the origin or in a neighborhood of it if and only if the geometry is nonsingular. This is a consequence of the coincidence of the converse of the bounded R'^2 lemma and

the converse of the positive QLM theorem when $K_{ab}=0$. In the general case, when $K_{ab}\neq 0$, no such coincidence occurs.

The mass is finite at $R=0$ independent of the gauge. We have shown above that the quasi-local mass m is finite even if the spatial slice is strongly singular with $R\to 0$ so long as (ρ,J) remain finite. This result was derived on the assumption that the slice was chosen to satisfy the gauge condition (3) with $\alpha\geqslant 0.5$. It turns out that the finiteness of m holds on any slice. To see this we need only consider Eq. (12), which is slicing independent, and Eq. (39), which is effectively the definition of m. As we argued above, the first term in Eq. (12) remains finite as we approach the singularity. We know that $R^2\rho$ is bounded and $\int d\ell R'=R$ also is well behaved. Thus we need only focus on the $\int d\ell R^2 JRK_R$ term. We know that R^2J is bounded so we only need to control the RK_R term. The necessary control is given by Eq. (39). If $R^2K_R^2$ is the dominant term on the right-hand side of Eq. (39), we get that $|RK_R|\sim -R'$ so the integral in Eq. (12) is finite as $R\to 0$. If m becomes large and negative so that the term $-2m/R$ dominates, we get $|RK_R|<-R'$ so again the integral converges. The only case left to consider is the possibility that $2m/R$ is positive and diverges at the same rate as $R^2K_R^2$ and some cancellation occurs so that R' is uncorrelated with RK_R. Let us assume $R\sim(\ell_S-\ell)^\beta$ for some $\beta>0$ and $|RK_R|\sim(\ell_S-\ell)^{-\gamma}$ for $\gamma>0$. We then get $m\sim(\ell_S-\ell)^{\beta-2\gamma}$. However, the argument in Eq. (12) goes as $R^3K_R\sim(\ell_S-\ell)^{2\beta-\gamma}$. Hence we get m diverging at worst as $m\sim(\ell_S-\ell)^{2\beta-\gamma+1}$. If this is self-consistent, we require $2\beta-\gamma+1=\beta-2\gamma$. This implies $\beta+\gamma+1=0$, which makes no sense. Thus such cancellation cannot take place.

V. INTEGRABILITY CONDITIONS

What are the implications of the integrability condition, Eq. (38)? If Eq. (38) is satisfied the strong J singularity is moderated to one which is only strong a la ρ. The behavior in the vicinity of the singularity will then be determined by the m/R term in Eq. (39) even if the system was originally ''driven'' towards the singularity by extrinsic curvature. If, in addition,

$$m(\ell_S)=4\pi\int_0^{\ell_S} d\ell[\rho R^2R'+JR^3K_R]=0, \qquad (45)$$

the singularity will be a weak one with $R'(\ell_S)=-1$. We note that $R''=0$ at this point. The corresponding bag of gold will be a regular closed universe. These integrability conditions do depend on α. If a given function J satisfies Eq. (38) with one function α, generally it will not satisfy that condition with any other function. There is no spacetime diffeomorphism invariant statement of the integration. The integrability condition need not be preserved by the evolution.

If J is positive (or negative) everywhere, $C_\alpha(\ell)$ defined by Eq. (37) cannot vanish. Thus, if matter is collapsing or exploding everywhere, all singularities must be strong J-type singularities. This contrasts with the obstruction, $\rho'<0$, discussed in [2], *prohibiting* the formation of any singularity when $K_{ab}=0$. In general, we note that on performing an integration by parts on the first term, m can be rewritten

$$m(\ell)=\frac{4\pi}{3}\rho R^3+4\pi\int_0^{\ell} d\ell R^3[JK_R-\rho']. \qquad (46)$$

The first term is manifestly positive. So is the third if $\rho'\leqslant 0$. If J is positive (negative) everywhere then so is m in any α gauge. However, the third term appearing on the RHS of Eq. (39) may still pull the geometry into a singularity if J is sufficiently large. The peculiarity of momentarily static configurations with $\rho'<0$ discussed in [2] can clearly be destabilized by the motion of matter. All regular closed cosmologies simultaneously satisfy two integrability conditions, Eqs. (38) and (45). There can be no net flow of material from one pole to the other. In particular, J must change sign between the poles. In addition, Eq. (46) tells us that

$$m(\ell_S)=4\pi\int_0^{\ell_S} d\ell R^3[JK_R-\rho']=0. \qquad (47)$$

In particular, $JK_R-\rho'$ must change sign between the poles. These conditions will be examined in the closed cosmological context in a subsequent publication [4].

There are no strong J singularities in the Euclidean theory. The singularity structure we have investigated has one important consequence for Euclidean general relativity. If the sign of the quadratic term in K_R appearing in Eq. (39) had been negative, *instead* of facilitating the occurrence of singularities it would have presented an obstacle to their occurrance. Any nonvanishing extrinsic curvature would therefore tend to stabilize the spatial geometry against singularity formation. We note that there is precisely such a sign switch in the Hamiltonian constraint of Euclidean general relativity. The Bianchi identities there tell us that the solutions of the constraints represent all possible configurations the system may assume as it is evolved with respect to Euclidean time. This suggests that gravitational instantons will tend to be more regular than their Lorenzian counterparts. In fact, the most singular Euclidean geometries will occur when the geometry is momentarily static. In a tunneling Euclidean four-geometry, such three-geometries correspond to the initial and final hypersurfaces of the Lorentzian spacetimes between which it interpolates. If these hypersurfaces are themselves nonsingular, i.e., do not involve Planck scale structures, then Planck Scale physics does not enter the semiclassical description of tunneling between them. This would appear to validate the application of the semiclassical approximation.

VI. CONCLUSIONS

We have identified a dense subset of extrinsic time foliations with respect to which there exist universal bounds on certain geometrical invariants. When the geometry is regular, we have described how the spacetime gradients of R in this dense subset are bounded numerically, independent both of the gauge and of the sources. Near a singularity, these gradients diverge in a way we can quantify.

These results can be applied to address the question of identifying necessary and sufficient conditions for the presence of apparent horizons and singularities in the initial data [5,6] extending the work of [11,12], [2], and [10]. In the analysis of sufficient conditions for the appearance of trapped surfaces and singularities, first the moment-of-time-

symmetry case was examined [11,2] followed by maximal slices [12,10]. We find that, not only can we extend this work to constant α but to the large class of extrinsic time foliations described by Eq. (3) for variable α within the range $0.5 \leqslant \alpha < \infty$. We also find that we can provide very powerful generalizations of the necessary conditions introduced in [2] for moment of time symmetry initial data to general initial data.

There are a number of interesting spherically symmetric problems we intend to pursue. A very satisfactory representation of regular closed solutions of the constraints can be given as closed bounded trjectories in the (ω_+, ω_-) plane. In this representation R plays a subsidiary role. These variables suggest a novel approach to the canonical quantization of spherically symmetric general relativity [13]. Indeed, constant α foliations can be exploited to provide a new description of the Schwarzschild solution [14].

The next stage is the examination of the classical evolution. Write down the Einstein equations with respect to the optical scalar variables. Can we cast the theory in Hamiltonian form? If the value of these variables in the analysis of the constraints is anything to go by, one has every reason to expect that they will throw light on the solution of the dynamical Einstein equations, both analytically and numerically. Indeed these variables have recently been exploited to establish global existence results [15].

ACKNOWLEDGMENTS

We gratefully acknowledge support from CONACyT Grant No. 211085-5-0118PE to J. G. and Forbairt Grant No. SC/96/750 to N.Ó M.

[1] J. Guven and N. Ó Murchadha, Phys. Rev. D **52**, 758 (1995). An extensive list of references is provided here.
[2] J. Guven and N. Ó Murchadha, Phys. Rev. D **52**, 776 (1995).
[3] Solutions of the constraints when $K_{ab}=0$ are also often referred to as moment of time symmetry configurations.
[4] See [1] for relevant references.
[5] J. Guven and N. Ó Murchadha, the following paper, Phys. Rev. D **56**, 7658 (1997).
[6] J. Guven and N. Ó Murchadha, this issue, Phys. Rev. D **56**, 7666 (1997).
[7] J. Guven and N. Ó Murchadha (unpublished).
[8] It is possible to write this equation so that when differentiated once it is transformed into a *local* third order singular ODE. However, it is not particularly illuminating to cast it this way.
[9] The constraint equations place no restriction on the behavior of J at the origin. They are not disturbed by $J(0)\neq 0$. However, the continuity equation $d\rho/dt + div J = 0$ may force the time derivative of the density to diverge and a conical singularity to emerge on evolving this data.
[10] E. Malec and N.Ó Murchadha, Phys. Rev. D **50**, R6033 (1994).
[11] P. Bizoń, E. Malec, and N. Ó Murchadha, Phys. Rev. Lett. **61**, 1147 (1988); Class. Quantum Grav. **6**, 961 (1989).
[12] P. Bizoń, E. Malec, and N.Ó Murchadha, Class. Quantum Grav. **7**, 1953 (1990).
[13] The canonical quantization of vacuum spherically symmetric general relativity with Schwarzschild topology has been examined by K. Kuchar, Phys. Rev. D **50**, 3961 (1994).
[14] J. Guven and N. Ó Murchadha (in preparation).
[15] A. Rendall, Class. Quantum Grav. **12**, 1517 (1995); G. Burnett and A. Rendall, *ibid.* **13**, 111 (1996).