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# On the Convergence of the Chi Square and Noncentral Chi Square Distributions to the Normal Distribution

Donagh Horgan and Colin C. Murphy

***Abstract*—A simple and novel asymptotic bound for the maximum error resulting from the use of the central limit theorem to approximate the distribution of chi square and noncentral chi square random variables is derived. The bound enables the quick calculation of the number of degrees of freedom required to ensure a given approximation error, and is significantly tighter than bounds derived using the Berry-Esseen theorem. An application to widely-used approximations for the decision probabilities of energy detectors is also provided.**

***Index Terms*—Probability, statistics, random variables, closed-form solutions, upper bound, cognitive radio, spectrum sensing, energy detection.**

## I. Introduction

THE central limit theorem is a useful tool when dealing with sums of random variables, allowing the distribution of the result of the summation to be approximated using the well-known normal distribution [1]. However, the resulting error can be difficult to quantify accurately, and so the region of applicability of such approximations is often unclear.

Generally, this difficulty can be circumvented by using the *Berry-Esseen theorem* [2], which states an upper bound on the magnitude of the error resulting from the use of the central limit theorem. However, the theorem is a general one, applying to sums of random variables of *any* distribution, and so it can often significantly overestimate the actual error. Thus, one might be led to believe that the error resulting from the use of the central limit theorem is much larger than it actually is and avoid using approximations where it may have been convenient to do so. Consequently, rules of thumb for the minimum number of summands, above which the central limit theorem gives an accurate approximation, are often proposed. For instance, in the case of chi square random variables, Box et al. suggest that as few as fifty degrees of freedom are necessary [3, p. 118], while Urkowitz proposes that 250 are required [4]. However, such rules cannot be relied upon generally: while a certain number of degrees of freedom may give sufficient accuracy for one application, it may not for another. Without quantifying the approximation error resulting from a given rule, one can only make subjective assessments of its accuracy.

In spectrum sensing literature, Urkowitz's proposed rule has found widespread use in the approximation of the decision probabilities of energy detectors [5]–[8]. Typically, it is stated that such approximations are valid when the number of samples is large, but no specific guarantees are made about the magnitude of the error one should expect when using them. While Urkowitz argues that 250 samples are sufficient to give good accuracy, as will be shown in this letter, this can result in an absolute approximation error of as large as 0.012, which may be significant, depending on the application.

Thus, the aim of this letter is two-fold: firstly, we will quantify the approximation error resulting from the application of the central limit theorem to chi square and noncentral chi square distributed random variables and, secondly, we will apply this bound to derive further bounds for the normal approximations to the decision probabilities of energy detectors that are in widespread use in the literature.

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## II. Normal Distribution Approximations to the Chi Square and Noncentral Chi Square Distributions

Consider a noncentral chi square distributed random variable, $\chi_k^2(s)$, with $k$ degrees of freedom and noncentrality parameter $s$. We can write $\chi_k^2(s)$ as

$$\chi_k^2(s) = \sum_{i=1}^{k} X_i^2, \tag{1}$$

where $X_1, X_2, \ldots X_k$ are independent and identically distributed (i.i.d.) Gaussian random variables with finite common mean, $\mu = \sqrt{\frac{s}{k}}$, and unit variance [1, p. 46].

When the noncentrality parameter is equal to zero, the noncentral chi square distribution is equivalent to a chi square distribution, i.e. $\chi_k^2 \triangleq \chi_k^2(0)$, where $\chi_k^2$ is a chi square distributed random variable with $k$ degrees of freedom. Thus, we can represent both distributions using the notation $\chi_k^2(s)$.

Typically, the cumulative distribution function (CDF) of $\chi_k^2(s)$, $P[\chi_k^2(s) > x]$, is represented as

$$P[\chi_k^2(s) > x] = Q_\nu(\sqrt{s}, \sqrt{x}), \tag{2}$$

where $\nu = \frac{k}{2}$ and $Q_m(a, b)$ is the Marcum $Q_m$ function[1] [1, Eq. 2.3-36].

However, using the central limit theorem [1, p. 63], we can approximate the CDF as

$$P[\chi_k^2(s) > x] \approx Q\left(\frac{x - (k+s)}{\sqrt{2(k+2s)}}\right), \text{ as } k \to \infty, \tag{3}$$

where $Q(x)$ is the Gaussian $Q$ function [1, Eq. 2.3-10].

## III. Previous Work

As $k$ is usually finite, there is some error, $\epsilon(k, s, x)$, resulting from the use of (3), which we can write as

$$\epsilon(k, s, x) = Q_\nu(\sqrt{s}, \sqrt{x}) - Q\left(\frac{x - (k+s)}{\sqrt{2(k+2s)}}\right). \tag{4}$$

[1]As the Marcum $Q_m$ function is defined for $m \in \mathbb{N}^+$ only [1], $\nu$ must be a positive integer.

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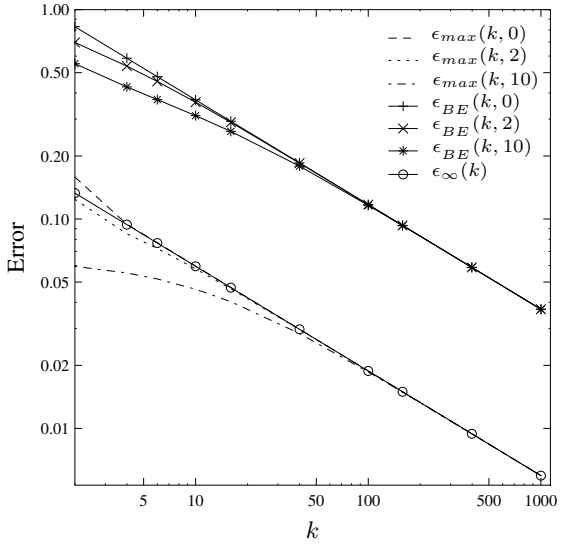



Fig. 1. A log-log plot of the maximum absolute error, the Berry-Esseen error bound and the proposed asymptotic error bound, as functions of $k$, for different values of $s$.

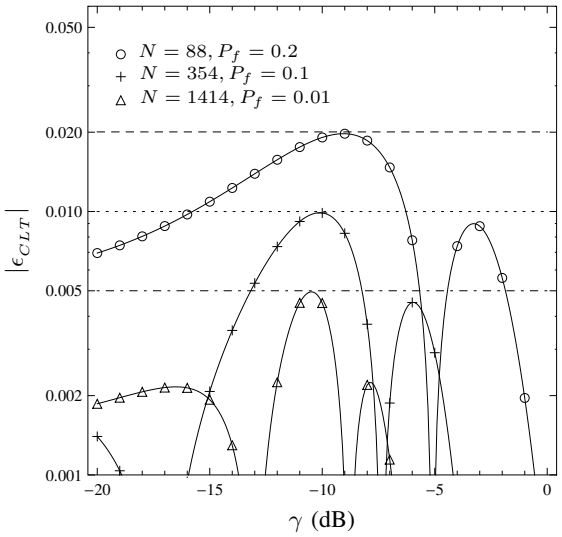



Fig. 2. Approximation error resulting from the use of (10), for different values of $N$ and $P_f$. The dashed line represents the bound given in (11) for $N = 88$, while the dotted line represents it for $N = 354$ and the dot-dashed line for $N = 1414$.

While (4) precisely describes the error resulting from the use of the central limit theorem, both the exact and approximate CDFs must be calculated in order to evaluate it, and deeper insight into the behaviour of the error with varying $k$, $s$ and $x$ is not readily apparent.

However, we can use the Berry-Esseen theorem to simplify the problem. While the theorem can often overestimate the actual error by a large amount, recent refinements by Korolev and Shevtsova have led to increased accuracy and so, to the best of the authors' knowledge at the time of writing, the tightest Berry-Esseen type bound on $\epsilon(k,s,x)$, $\epsilon_{_{BE}}(k,s)$, is given by

$$\epsilon_{max}(k,s) \leq \epsilon_{_{BE}}(k,s) \triangleq \frac{0.33477(\beta+0.429)}{\sqrt{k}}, \qquad (5)$$

where $\epsilon_{max}(k,s) \triangleq \max_x |\epsilon(k,s,x)|$ and $\beta$ is a function of the distribution of $X_i^2$, and is given by

$$\beta = \mathbb{E}\left[\left|\frac{X_i^2 - (1+\frac{s}{k})}{\sqrt{2(1+2\frac{s}{k})}}\right|^3\right], \qquad (6)$$

where $\mathbb{E}[\cdot]$ is the expectation operator. To the best of the authors' knowledge, (6) does not have a closed form and so must be evaluated numerically.

In Fig. 1, we have plotted both $\epsilon_{max}(k,s)$ and $\epsilon_{_{BE}}(k,s)$ for various values of $k$ and $s$. As can be seen, $\epsilon_{_{BE}}(k,s) > \epsilon_{max}(k,s)$ in each case, but the bounds are by no means tight, and consistently overestimate the magnitude of the error. For $s = 0$, the Berry-Esseen bounds are approximately 6.23 times larger than the actual error across the entire range of values of $k$.

## IV. Proposed Asymptotic Error Bound

To avoid the use of the Berry-Esseen bound, we propose a novel, asymptotic error bound, which we will state here as a theorem, and is proved in Appendix A.

**Theorem 1:** For noncentral chi square random variables, the maximum absolute error resulting from the use of the central limit theorem, $\epsilon_\infty(k)$, is given by

$$\epsilon_\infty(k) \approx \frac{1}{\sqrt{9\pi k}}, \text{ as } k \to \infty, \qquad (7)$$

where $\epsilon_\infty(k) \triangleq \max_{s,x} |\epsilon(k,s,x)| \geq \epsilon_{max}(k,s)$.

In Fig. 1, we have also plotted $\epsilon_\infty(k)$ across the entire range of values of $k$. As can be seen, when $k$ is small (e.g. $k \leq 4$), the relation in (7) is approximate; however, it becomes more accurate as $k$ becomes larger, and is a much more accurate estimate of the actual error than any of the Berry-Esseen bounds, even for small values of $k$. Consequently, henceforth, we will use (7) to describe the maximum error resulting from the use of (3).

## V. Application: Energy Detection

One common application of (3) is to approximate the distribution of the received energy in energy detector-based spectrum sensors operating on additive white Gaussian noise (AWGN) channels [4]. This, in turn, enables the derivation of simple approximations for the probabilities of false alarm and detection.

Consider the scenario where the energy detector must decide between one of two hypotheses:

$$\begin{aligned} \mathcal{H}_0 : r[n] &= w[n] && n = 1,2,\ldots N \\ \mathcal{H}_1 : r[n] &= s[n] + w[n] && n = 1,2,\ldots N, \end{aligned} \qquad (8)$$

where $r[n]$, $w[n]$ and $s[n]$ represent the discrete samples of the received, noise only and transmitted signals, respectively, $N$ is the total number of samples, $\mathcal{H}_0$ is the null hypothesis, and corresponds to the channel being unoccupied, and $\mathcal{H}_1$ is the alternative hypothesis and corresponds to the channel being occupied.

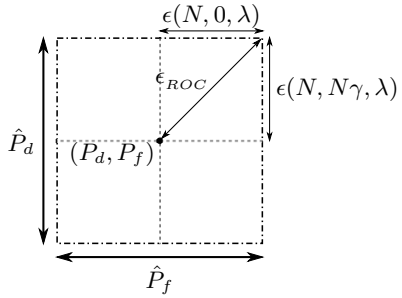



Fig. 3. Two dimensional error resulting from the use of approximations for both decision probabilities.

Usually, the energy detector computes a test statistic, $T$, from the samples of the received signal, that is

$$T = \frac{1}{\sigma^2}\sum_{n=0}^{N-1} |r[n]|^2, \tag{9}$$

where $\sigma^2$ is the power of the noise only signal.

Under $\mathcal{H}_0$, $T$ follows a chi square distribution with $N$ degrees of freedom while, under $\mathcal{H}_1$, it follows a noncentral chi square distribution with $N$ degrees of freedom and noncentrality parameter $N\gamma$, where $\gamma$ represents the signal to noise ratio [8]. Consequently, we can approximate the probability of detection using (3) as

$$P_d(\gamma) = P[T > \lambda|\mathcal{H}_1] \approx Q\left(\frac{\lambda - N(1+\gamma)}{\sqrt{2N(1+2\gamma)}}\right), \tag{10}$$

where $\lambda$ is the decision threshold. An approximation for the probability of false alarm, $P_f$, can be obtained in a similar manner by letting $\gamma = 0$ in (10), i.e. $P_f = P[T > \lambda|\mathcal{H}_0] = P_d(0)$.

Urkowitz states that $N = 250$ is sufficient for the approximation error resulting from the use of (10), which we will denote by $\epsilon_{_{CLT}}$, to be considered negligible. However, as the magnitude of the error is unclear, this is a subjective statement. Now, using Theorem 1, we can write

$$\max|\epsilon_{_{CLT}}| \approx \frac{1}{\sqrt{9\pi N}}. \tag{11}$$

Thus, the maximum error can be bounded quite simply, as shown in Fig. 2. As can be seen, the bound in (11) describes the maximum error quite well. Furthermore, for $N = 250$, as proposed by Urkowitz, $|\epsilon_{_{CLT}}| \leq 0.012$. Whether this error is negligible or not depends on the system designer's willingness or freedom to tolerate error, and so the decision to use (10) or not should be made on a case by case basis.

If the approximations for both the probability of false alarm and the probability of detection are used, e.g. in a receiver operating characteristic (ROC) plot, then further caution should be exercised. Fig. 3 illustrates the problem in more detail: given a decision probability pair, $(P_f, P_d)$, and an approximate decision probability pair, $(\hat{P}_f, \hat{P}_d)$, the Euclidean distance between the two, $\epsilon_{_{ROC}}$, is given by

$$\epsilon_{_{ROC}} = \sqrt{\epsilon^2(N,0,\lambda) + \epsilon^2(N,N\gamma,\lambda)}. \tag{12}$$

Typically, $\epsilon_{_{ROC}}$ must be calculated numerically. However, using Theorem 1, it is not difficult to show that

$$\max_{\gamma,\lambda}|\epsilon_{_{ROC}}| \leq \sqrt{2}\ \max|\epsilon_{_{CLT}}|. \tag{13}$$

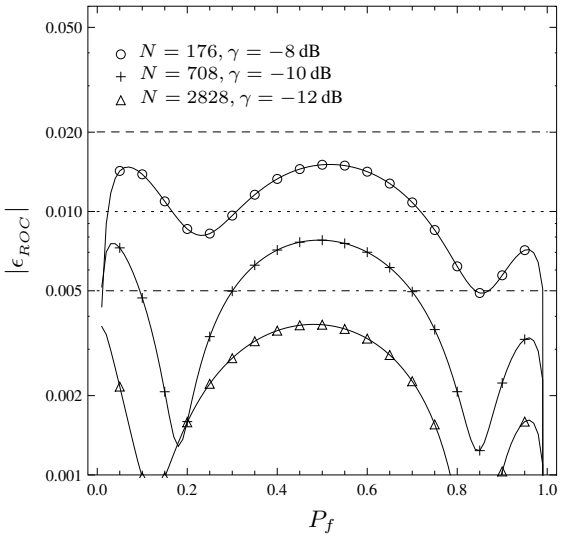



Fig. 4. Approximation error resulting from the simultaneous use of approximations for both decision probabilities, for different values of $\gamma$. The dashed line represents the bound given in (13) for $N = 176$, while the dotted line represents it for $N = 708$ and the dot-dashed line for $N = 2828$.

The bound in (13) is illustrated in Fig. 4 for several different scenarios. As can be seen, $\epsilon_{_{ROC}}$ can be bounded both simply and accurately using (13).

## VI. Conclusion

In this letter, we derived a novel asymptotic bound for the error resulting from the use of the central limit theorem to approximate the distribution of chi square and noncentral chi square random variables. The bound is in a simple form and describes the resulting error much more accurately than existing Berry-Esseen type bounds. Thus, it is possible to quickly and accurately quantify the number of degrees of freedom required for the normal approximation to meet a specified accuracy, and so relying on rules of thumb can be avoided.

Using the new bound, a further simple, accurate bound for central limit theorem approximations for the decision probabilities of energy detectors operating on AWGN channels was derived. Issues arising from the simultaneous use of approximations for both decision probabilities were also discussed, and a simple, accurate bound on the maximum Euclidean distance between the exact and approximate decision probabilities was derived.

## Appendix A
## Proof of the asymptotic bound

In order to prove Theorem 1, we must find the critical points, $(s, x) = (s_0, x_0)$, of $\epsilon(k, s, x)$ as $k \to \infty$. For a given value of $k$, these may be found using the first partial derivative test [9, Eq. 1.5.19], that is

$$\left.\frac{\delta\epsilon}{\delta s}\right|_{s=s_0,x=x_0} = 0, \quad \left.\frac{\delta\epsilon}{\delta x}\right|_{s=s_0,x=x_0} = 0. \tag{14}$$

Letting $k = 2\nu$ in (4), it can be shown, without loss of generality, that (14) is satisfied by

$$2(\nu + s_0)\sqrt{x_0}\ I_\nu(\sqrt{s_0 x_0}) = (s_0 + x_0)\sqrt{s_0}\ I_{\nu-1}(\sqrt{s_0 x_0}), \tag{15}$$

where $I_n(z)$ represents the modified Bessel function of the first kind.

Letting $s_0 = 0$ is a satisfactory solution of (15), but is by no means guaranteed to be the only solution[2], and so may not always maximise $|\epsilon(k, s, x)|$. However, for large values of $\nu$ or, equivalently, for large values of $k$, the problem can be simplified.

For large orders, the modified Bessel function of the first kind may be approximated [9, Eq. 10.41.1] as

$$I_n(z) \approx \hat{I}_n(z) = \frac{1}{\sqrt{2\pi n}} \left(\frac{ez}{2n}\right)^n, \text{ as } n \to \infty. \tag{16}$$

We also note the following useful identity [9, Eq. 4.4.17]

$$\left(\frac{n-1}{n}\right)^n \approx \frac{1}{e}, \text{ as } n \to \infty. \tag{17}$$

Using (16) and (17), (15) can be simplified to

$$(\nu + s_0)(\sqrt{s_0})^\nu(\sqrt{x_0})^{\nu+1} = \sqrt{\nu(\nu-1)}(s_0 + x_0)(\sqrt{s_0})^\nu(\sqrt{x_0})^{\nu-1}, \text{ as } \nu \to \infty. \tag{18}$$

Thus, for large values of $\nu$, (18) is equivalent to (15).

Assuming that $\nu \geq 2$, (18) admits three unique solutions. Two of these are immediately clear: $s_0 = 0$ and $x_0 = 0$; the third can be found by cancelling the common terms (which lead to the solutions $s_0 = 0$ and $x_0 = 0$) on both sides of (18) to give

$$x_0 = \frac{\sqrt{\nu(\nu-1)}s_0}{\nu + s_0 - \sqrt{\nu(\nu-1)}} \approx \nu, \text{ as } \nu \to \infty. \tag{19}$$

Using (14), it can be shown that, for the solution $x_0 = 0$, the only value of $s_0$ that satisfies both first partial derivative tests, as $\nu$ becomes large, is $s_0 \to \infty$ which, as we noted previously, is a trivial solution. Similarly, it can be shown that, if $x_0 = \nu$, then (14) also requires that $s_0 \to \infty$ when $\nu$ is large, and so $x_0 = \nu$ is a further trivial solution.

The value of $x_0$ corresponding to $s_0 = 0$ can be found using the identity [9, Eq. 10.30.1]

$$\lim_{s_0 \to 0} \left( \left(\frac{x_0}{s_0}\right)^{\frac{\nu-1}{2}} I_{\nu-1}\left(\sqrt{s_0 x_0}\right) \right) = \frac{\left(\frac{x_0}{2}\right)^{\nu-1}}{\Gamma(\nu)}, \tag{20}$$

where $\Gamma(n)$ represents the gamma function. Using (20), it can be shown that the first partial derivative test, with respect to $x$, reduces to the condition

$$\frac{e^{-\frac{x_0}{2}}\left(\frac{x_0}{2}\right)^{\nu-1}}{\Gamma(\nu)} = \frac{e^{-\frac{(x_0-2\nu)^2}{8\nu}}}{\sqrt{2\pi\nu}}, \tag{21}$$

which can be solved for $x_0$ using a numerical method, but admits the solution $x_0 = 2\nu$ for large values of $\nu$. This can easily be shown by letting $x_0 = 2\nu$ in (21) to give the condition

$$\frac{\nu!}{\sqrt{2\pi\nu}\left(\frac{\nu}{e}\right)^\nu} = 1, \tag{22}$$

which is simply the limit of Stirling's approximation [9, Eq. 5.11.3], and so is satisfied for large values of $\nu$.

[2]For instance, (15) is also satisfied when $s_0 \to \infty$ for arbitrary $0 \leq x_0 < \infty$ and when $x_0 \to \infty$ for arbitrary $0 \leq s_0 < \infty$. However these solutions are trivial as $\epsilon(k, s_0, x_0) = 0$ in both cases.

Substituting $(s, x) = (0, 2\nu)$ into (4), it can be shown that

$$\epsilon(2\nu, 0, 2\nu) = \frac{\Gamma(\nu, \nu)}{\Gamma(\nu)} - \frac{1}{2}, \tag{23}$$

where $\frac{\Gamma(k,z)}{\Gamma(k)}$ represents the regularised incomplete gamma function of order $k$.

For large values of $n$, the following identity [9, Eq. 8.11.12] holds

$$\frac{\Gamma(n, n)}{\Gamma(n)} \approx \frac{1}{2} - \frac{1}{\sqrt{18\pi n}}, \text{ as } n \to \infty, \tag{24}$$

and so we can write (23) as

$$\epsilon(2\nu, 0, 2\nu) \approx -\frac{1}{\sqrt{18\pi\nu}}, \text{ as } \nu \to \infty, \tag{25}$$

which suggests that the sign of the error with maximum magnitude is negative and so $(s, x) = (0, 2\nu)$ results in a minimum.

To prove that this intuition is correct, we can perform the second partial derivative test [9, Eqs. 1.5.20 and 1.5.21]. Using (4), it can be shown that

$$\left.\frac{\delta^2\epsilon}{\delta x^2}\right|_{s=0,x=2\nu} = \frac{1}{4\nu}\frac{\left(\frac{\nu}{e}\right)^\nu}{\nu!}, \tag{26}$$

$$\left.\frac{\delta^2\epsilon}{\delta x^2}\right|_{s=0,x=2\nu} \left.\frac{\delta^2\epsilon}{\delta s^2}\right|_{s=0,x=2\nu} - \left.\left(\frac{\delta^2\epsilon}{\delta x \delta s}\right)^2\right|_{s=0,x=2\nu} = \frac{1}{8\nu^2}\frac{\left(\frac{\nu}{e}\right)^\nu}{\nu!}\left(\frac{1}{\sqrt{2\pi\nu}} - \frac{\left(\frac{\nu}{e}\right)^\nu}{2(n+1)\Gamma(\nu)}\right) - \frac{1}{32\pi\nu^3}, \tag{27}$$

and both of which can be shown to be positive for $\nu \geq 1$. Consequently, $(s, x) = (0, 2\nu)$ results in a local minimum for large $\nu$ ($\nu \geq 1$, at least). However, as $(s, x) = (0, 2\nu)$ are the only non-trivial critical points for large $\nu$ ($\nu \geq 2$, at least), this local minimum must be a global minimum under these conditions, and so we can bound the maximum absolute error as

$$\max_{s,x} |\epsilon(k, s, x)| \approx \frac{1}{\sqrt{9\pi k}}, \text{ as } k \to \infty. \qquad \square \tag{28}$$

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