PHYSICAL REVIEW D VOLUME 36, NUMBER 8 15 OCTOBER 1987

Average energy density and the size of the Universe

Niall O'Murchadha
Physics Department, University College, Cork, Ireland
(Received 7 May 1987)

This paper shows that there is a relationship between the maximum volume of any cosmological solution of the Einstein equations and its average energy density. An inequality is derived which shows that the average energy density multiplied by the volume to the $\frac{2}{3}$ power must be greater than a fixed constant. Equivalently, the total energy content, divided by the volume to the $\frac{1}{3}$ power, is bounded below.

Cosmologies are usually solutions of the Einstein field equations which are spatially compact, emerge from a big bang, expand to a maximum volume and then recontract to a big crunch. Given that the spatial cross sections are all compact, they must have sufficient curvature to close them up. Thus if the spatial volume is large then the average curvature can be small, conversely small volume implies large curvature.

The field equations tell us that the curvature measures the strength of the gravitational field, and is generated by the stress energy of the sources. Therefore if the volume of the closed spatial manifold is small then the average energy density (gravitational+sources) must be large, but if we have a large manifold, the energy density can be small. There is a tradeoff between the volume and the average energy density.

This tradeoff will be made explicit by the following inequality:

$$\text{(average energy density)(volume of three-manifold)}^{2/3} \geq C_0/2\pi \,, \tag{1}$$

where C_0 is given constant. This can also be written as

$$\text{(total energy content)(volume of Universe)}^{-1/3} \geq C_0/2\pi \,. \tag{2}$$

The trick is in how one defines the average energy density and the total energy of a closed universe.

The basis for these physics statements (1) and (2) is a purely geometrical inequality which holds for a large set of compact Riemannian three-manifolds. This is of the form

$$\text{(average scalar curvature)(volume of manifold)}^{2/3} \geq 8C_0 \,. \tag{3}$$

If we have a three-dimensional spacelike slice through a four-manifold which is a solution of the Einstein equations, there are two natural geometrical objects defined on the three-slice. One is a three-metric g_{ij} and the other is a three-symmetric-tensor K^{ij}, which is essentially the time-derivative of g_{ij} (Ref. 1 and 2). One of the Einstein equations, the so-called Hamiltonian constraint, connects g_{ij} and K^{ij}. This is of the form

$${}^{(3)}R - K^{ij}K_{ij} + (\mathrm{tr}K)^2 = 16\pi\rho \,, \tag{4}$$

where ${}^{(3)}R$ is the three scalar curvature of g_{ij} and ρ is the source energy density. It is natural to identify $K^{ij}K_{ij} - (\mathrm{tr}K)^2$ as the kinetic energy of the gravitational field. From (4) we have that

$$\text{(average scalar curvature)} = \{\text{average of } [16\pi\rho + K^{ij}K_{ij} - (\mathrm{tr}K)^2]\} \tag{5}$$

and it is this quantity that can be used in inequalities (1) and (2).

To derive inequality (3), let us first do it for a very special case. Consider flat space and a positive function ϕ which goes to zero at infinity. I now wish to use ϕ as a conformal factor so as to transform the flat space into a closed compact manifold M. To do this, let me define a metric

$$g_{ij} = \phi^4 \delta_{ij} \tag{6}$$

and, with respect to this metric, the whole of infinity is compactified into a single point. (This is the three-dimensional analogue of the projective mapping from a sphere to the plane.) The scalar curvature ${}^{(3)}R$ of M can now be calculated as

$${}^{(3)}R = -8\phi^{-5}\nabla^2\phi \,, \tag{7}$$

where ∇^2 is the flat-space Laplacian.

This gives

$${}^{(3)}R\,\phi^6 = -8\phi\nabla^2\phi \,.$$

Now integrate over flat space to give

$$\int {}^{(3)}R\phi^6 d^3x = -8\oint_\infty \phi\nabla\phi\cdot ds + 8\int (\nabla\phi)^2 d^3x \ . \quad (8)$$

Since $g_{ij}=\phi^4\delta_{ij}$, we have

$$\sqrt{\det g}=\phi^6$$

and therefore $\phi^6 d^3x$ is the volume element in M, and the volume of M, V, is defined as

$$V=\int \phi^6 d^3x \ . \quad (9)$$

Therefore the left-hand side of (8) is nothing more than the integral of the scalar curvature of M over M. If we define $\bar{R}$ as the average of ${}^{(3)}R$ over M, we obviously get

$$\int {}^{(3)}R\phi^6 d^3x=\bar{R}V \ . \quad (10)$$

If ϕ goes to zero faster than $r^{-1/2}$ the surface integral on the right of (8) will vanish, and so we will get

$$\bar{R}V=8\int (\nabla\phi)^2 d^3x \ . \quad (11)$$

Since ϕ is a function which vanishes at infinity it must satisfy the Sobolev inequality, which states[3]

$$\int (\nabla\phi)^2 d^3x \geq C_0 \left[\int \phi^6 d^3x\right]^{1/3} , \quad (12)$$

where C is a constant, $3(\pi^2/4)^{2/3}$. This, when substituted into (11) gives

$$\bar{R}V\geq 8C_0V^{1/3}$$

or

$$\bar{R}V^{2/3}\geq 8C_0 \quad (13)$$

which is exactly inequality (3).

Of course, we have only shown inequality (13) to be valid for conformally flat compact manifolds. However, it is very easy to extend it. Let us now consider the asymptotically flat base manifold not to be flat space, but to have a nontrivial metric g'_{ij}. However, we demand that the scalar curvature of g'_{ij} vanish everywhere. The argument above goes through exactly as before. With an arbitrary positive ϕ (which vanishes at ∞) we define the metric

$$g_{ij}=\phi^4 g'_{ij} \quad (14)$$

which gives us a compact manifold without boundary M. Its scalar curvature R is still given by

$$R=-8\phi^{-5}\nabla^2\phi \ , \quad (15)$$

where ∇^2 is now the Laplacian with respect to g'_{ij}. We get

$$\int R\phi^6\sqrt{g'}d^3x=+8\int (\nabla\phi)^2\sqrt{g'}d^3x \ . \quad (16)$$

Of course $\phi^6\sqrt{g'}=\sqrt{g}$ is the volume element in M and

$$\int R\phi^6\sqrt{g'}d^3x=\bar{R}V \ . \quad (17)$$

The Sobolev inequality is still valid, and gives

$$\int (\nabla\phi)^2\sqrt{g'}d^3x\geq C_0\left[\int \phi^6\sqrt{g'}d^3x\right]^{1/3} = C_0V^{1/3} \ . \quad (18)$$

The only difference is that the Sobolev constant C_0 may depend weakly on g'_{ij}. Thus we recover (3).

This is not yet the end of the story. Not all compact manifolds can be obtained by conformally compactifying asymptotically flat spaces with vanishing scalar curvature. There is a long-standing conjecture in geometry called the Yamabe theorem.[4] This divides compact Riemannian manifolds without boundary into three classes: those with essentially positive scalar curvature, those with essentially zero, and those with essentially negative scalar curvature. In a recent completion of the proof of the Yamabe theorem, Schoen[5] showed that all manifolds in the positive Yamabe class could be conformally decompactified into asymptotically flat manifolds with vanishing scalar curvature. Therefore, we can only claim that inequality (13) is valid for manifolds in the positive Yamabe class. Happily, these turn out to be the physically interesting ones.

To see this, let us return to the Einstein equations, in particular to the Hamiltonian constraint (4)

$${}^{(3)}R=16\pi\rho+K^{ij}K_{ij}-(\mathrm{tr}K)^2 \ .$$

We cannot deduce from this that the scalar curvature is positive everywhere, because of the $(\mathrm{tr}K)^2$ term. However, since K_{ij} is essentially the rate of change of g_{ij} we have

$$\frac{\partial}{\partial t}\sqrt{g}\sim -\mathrm{tr}K\sqrt{g} \quad (19)$$

and the cross section of largest volume is identified by $\mathrm{tr}K\equiv 0$. On this slice we have

$${}^{(3)}R=16\pi\rho+K^{ij}K_{ij} \quad (20)$$

and if $\rho\geq 0$ (as it should be) we have ${}^{(3)}R\geq 0$ and so that the largest volume slice must belong to the positive Yamabe class and satisfy inequality (3). In turn we get

$$[\text{average value of } (\rho+K^{ij}K_{ij}/16\pi)](\text{largest volume})^{2/3}\geq C_0/2\pi \ . \quad (21)$$

I do not claim that all slices of all cosmologies belong to the positive Yamabe class. However, it is clear that any slice in a neighborhood of the largest slice belongs to this Yamabe class. Hence for all such slices we have

$$(\text{average value of } \{\rho+(1/16\pi)[K^{ij}K_{ij}-(\mathrm{tr}K)^2]\})(\text{volume of slice})^{2/3}\geq C_0/2\pi \ .$$

For a subset of cosmologies, this slice of largest volume defines a global instant of turnaround, an instant of unstable equilibrium. Such an instant is called a moment of time symmetry and is signaled by the total vanishing of K_{ij}. At a moment of time symmetry we get

$$^{(3)}R = 16\pi\rho$$

and the inequality reads

$$(\text{average value of } \rho)(\text{largest volume})^{2/3} \geq C_0/2\pi$$

or equivalently

$$(\text{total mass in Universe})(\text{largest volume})^{-1/3} \geq C_0/2\pi \; . \tag{22}$$

The easiest test of the inequality (3) is in the closed Friedmann cosmology. There we get a sequence of closed three-manifolds with constant curvature. The metric of each hypersurface is given by[6]

$$ds^2 = a^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)] \; .$$

The volume $V = 2\pi^2 a^3$, and the scalar curvature is $^{(3)}R = 6/a^2$. It is straightforward to show that these values of V and $^{(3)}R$ just satisfy (3). This implies that the inequality is sharp, that we cannot reduce the constant and still have it always true.

Another inequality has been recently arrived at which is similar in form but in the opposite sense to (3) (Refs. 7 and 8). This one says that one cannot have large positive scalar curvature everywhere in a large volume, because the volume would curl up too quickly. Given any set Ω on which the scalar curvature is bounded below by a positive constant R_0 we can show

$$R_0 L^2 \leq \frac{8\pi^2}{3} \; ,$$

where L is a linear measure of the size of Ω, and $L^2 \sim (\text{volume of } \Omega)^{2/3}$ if Ω is more or less spherical. This is not really very surprising, because dimensional analysis gives us that the scalar curvature has dimensions $(\text{length})^{-2}$ and so we have to multiply it by $(\text{length})^2$ or area or $(\text{volume})^{2/3}$ to convert it to a dimensionless quantity.

It is interesting to note that inequality (3) holds in any number of dimensions. More precisely, we can show, in n dimensions,

$$^{(n)}R\, V^{2/n} \geq \frac{4(n-1)}{n-2} C_n \; , \tag{23}$$

where C_n is the appropriate Sobolev constant satisfying

$$\int (\nabla\phi)^2 \geq C_n \left[\int \phi^{2n/n-2} \right]^{(n-2)/n} \; .$$

[1] R. Arnowitt, S. Deser, and C. Misner, in *Gravitation: An Introduction to Current Research,* edited by L. Witten (Wiley, New York, 1962).

[2] C. Misner, K. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Chap. 21.

[3] See, for example, D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order* (Springer, Berlin, 1977), p. 148.

[4] H. Yamabe, Osaka J. Math. **12**, 21 (1960).

[5] R. Schoen, J. Diff. Geom. **20**, 479 (1984).

[6] Misner, Thorne, and Wheeler, *Gravitation* (Ref. 2), Chap. 27.

[7] R. Schoen and S. T. Yau, Commun. Math. Phys. **90**, 575 (1983).

[8] N. O'Murchadha, Phys. Rev. Lett. **57**, 2466 (1986).