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FINITE GROUPS WITH AT MOST NINE NON-CENTRAL CONJUGACY CLASSES

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Abstract

We classify all finite groups with at most nine non-central conjugacy classes.

1. Introduction

Let G be a finite group with $k(G)$ conjugacy classes and centre $Z(G)$, and let $k_0(G) = k(G) - |Z(G)|$ be the number of non-central conjugacy classes in G. Starting with Sigley in 1935 [17], many authors have considered the groups with $k_0(G)$ small (see [20], [13] and [14]). This analysis becomes progressively longer and more complicated as $k_0(G)$ increases. In this paper we derive a procedure for reading off all the groups with $k_0(G)$ small from the known classification of all finite groups with $k(G) < 15$ (see [7], [8], [19] and the references therein). We believe that some of our incidental results may be of independent interest.

For purposes of comparison, we begin with a naïve approach, based on the class equation of a group, and show that this method quickly runs out of steam and becomes intractable. We then prove a simple theorem which leads to a new and shorter solution to the problem.

We write $\mathrm{cl}(x)$ for the conjugacy class containing x in G, and $C_G(x)$ for the centralizer of x in G. For convenience, we will denote $k_0(G)$ by n in what follows.

We note that there is an extensive literature relating the size of $k(G)$ to the primes p which divide $|G|$. For example, Maróti [9] proves that if the prime p divides $|G|$, then $k(G) \geq 2\sqrt{p-1}$. In addition, Hethelyi and Kulshammer [6] show that if p^2 divides the order of a solvable group G, then $k(G) \geq (49p + 1)/60$.

2. The naïve approach

Case $n = 0$**:** Clearly this happens if and only if G is abelian, and no further analysis is needed. From now on we may assume that G is non-abelian.

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Case $n = 1$: Here we have $G = Z(G) \cup \mathrm{cl}(x)$ and G is the disjoint union of its centre and a single conjugacy class. Thus

$$|G| = |Z(G)| + |\,\mathrm{cl}(x)| = |Z(G)| + \frac{|G|}{|C_G(x)|}$$

which simplifies to

$$1 = \frac{1}{|G : Z(G)|} + \frac{1}{|C_G(x)|}$$

where both $|G : Z(G)|$ and $|C_G(x)|$ are positive integers. Now it is easy to see that the Diophantine equation $1/a + 1/b = 1$ has the unique solution $a = b = 2$. But then $|G : Z(G)| = 2$, a contradiction since $G/Z(G)$ cannot be non-trivial cyclic. So there are no groups corresponding to the case $n = 1$.

Case $n = 2$: Here $G = Z(G) \cup \mathrm{cl}(x) \cup \mathrm{cl}(y)$ where x and y are non-central, non-conjugate elements of G. This leads at once to

$$1 = \frac{1}{|G : Z(G)|} + \frac{1}{|C_G(x)|} + \frac{1}{|C_G(y)|}$$

where $|G : Z(G)|$, $|C_G(x)|$ and $|C_G(y)|$ are all positive integers. Easy Diophantine analysis shows that the solutions of $1/a + 1/b + 1/c = 1$ are given by $(a, b, c) = (3, 3, 3)$, $(2, 4, 4)$ and $(2, 3, 6)$ in some order. If $(a, b, c) = (3, 3, 3)$, then $|G : Z(G)| = 3$, again a contradiction. If $(a, b, c) = (2, 4, 4)$, then $|G : Z(G)| = 2$ leads to a contradiction. So, $|G : Z(G)| = 4$ and in this case it is well known that $k(G) = \frac{5}{8}|G|$. Now, $k(G) = |Z(G)| + 2$, so $\frac{5}{8}|G| = \frac{1}{4}|G| + 2$ which leads to $3|G| = 16$, a contradiction. Finally, consider $(a, b, c) = (2, 3, 6)$. As before, $|G : Z(G)| = 2$ or 3 is not possible, so $|G : Z(G)| = 6$. Again, it is well known that in this case $k(G) = \frac{1}{2}|G|$ and so $k(G) = |Z(G)| + 2$ gives $\frac{1}{2}|G| = \frac{1}{6}|G| + 2$ leading to $|G| = 6$. So, $G \cong S_3$, the symmetric group on three symbols, which does indeed have exactly two non-central conjugacy classes.

Case $n = 3$: Unfortunately, the number of solutions of the Diophantine equation

$$1 = \sum_{i=1}^{r} \frac{1}{x_i}$$

increases very rapidly with r. For example, for $r = 4$ there are 14 (ordered) solutions and there are 147 solutions for $r = 5$ and 3462 solutions for $r = 6$. (See [15]).

One solution for $n = 3$ is clearly $(4, 4, 4, 4)$, leading to $|G : Z(G)| = 4$ and $k(G) = \frac{5}{8}|G|$, which gives $\frac{5}{8}|G| = \frac{1}{4}|G| + 3$ leading to $|G| = 8$. There are two non-abelian groups of order 8, namely the dihedral group D_4 and the quaternion group Q_2, both of which have three non-central conjugacy classes. But there are many other cases to consider, some quite long and difficult, and while this approach is theoretically correct, it is not computationally feasible and clearly it is not the way to proceed. We adopt an alternative approach.

3. Some useful results

Let $\mathcal{G}_p$ be the set of all finite groups G such that $|G|$ is divisible by the prime p but by no smaller prime. Note that if $|G|$ is odd, that is if $G \in \mathcal{G}_p$ for $p > 2$, then the Feit-Thompson theorem tells us that G is solvable.

Lemma 1. *If G is a non-abelian group in $\mathcal{G}_p$, then $|G : Z(G)| \geq p^2$, and this is the best possible result.*

PROOF. Since G is a non-abelian group in $\mathcal{G}_p$, we have $|G : Z(G)| \geq p$. If $|G : Z(G)| = p$, then $G/Z(G)$ is cyclic and G is abelian, a contradiction. If $q > p$ is any other prime dividing $|G|$, then $|G : Z(G)| = q$ also gives a contradiction. We note that $pq > p^2$, so the next possible value of $|G : Z(G)|$ is p^2, so $|G : Z(G)| \geq p^2$, as claimed. This is the best possible value; for $p = 2$ we have $|D_4 : Z(D_4)| = 4 = 2^2$; for p an odd prime there is a group of order p^3 with $|G : Z(G)| = p^2$. ■

Theorem 1. *Suppose that the non-abelian group G in $\mathcal{G}_p$ has $k(G)$ conjugacy classes. Then*

$$k(G) \geq \frac{p^2 + p - 1}{p} |Z(G)|$$

This result is best possible.

PROOF. By Lemma 1

$$\frac{|Z(G)|}{|G|} \leq \frac{1}{p^2} \qquad (\star)$$

Pick $x \in G - Z(G)$. Then $C_G(x) \supseteq \{x, Z(G)\}$, so $C_G(x) \supseteq \langle x, Z(G) \rangle$ and thus $|C_G(x)| \geq p|Z(G)|$. Now $|\operatorname{cl}(x)| = |G|/|C_G(x)|$ and $1/|C_G(x)| \leq 1/p|Z(G)|$. So

$$|\operatorname{cl}(x)| \leq \frac{|G|}{p|Z(G)|}.$$

Now $G - Z(G)$ is partitioned into conjugacy classes of G, each of which has at most $\frac{|G|}{p|Z(G)|}$ elements. Thus there are at least

$$\frac{|G| - |Z(G)|}{\frac{|G|}{p|Z(G)|}}$$

such classes. So

$$\begin{aligned}
k(G) &\geq |Z(G)| + \frac{1}{|G|}\left(|G| - |Z(G)|\right) p|Z(G)| \\
&= |Z(G)|\left(\frac{1}{|G|}\left(1 + p|G| - p|Z(G)|\right)\right) \\
&= |Z(G)|\left(1 + p - \frac{p|Z(G)|}{|G|}\right) \\
&\geq |Z(G)|\left(1 + p - \frac{p}{p^2}\right) \text{ by equation } (\star) \\
&= |Z(G)|\left(\frac{p^2 + p - 1}{p}\right)
\end{aligned}$$

as claimed. ■

This result has some pretty consequences.

Corollary 1. *If G is a non-abelian p-group, $|Z(G)| \geq p$, so $k(G) \geq p^2 + p - 1$.*

Thus a non-abelian 2-group has at least 5 conjugacy classes and D_4 shows this result is best possible. A non-abelian 3-group has at least 11 conjugacy classes and both the non-abelian groups of order 27 show this is best possible. In general, a non-abelian group of order p^3 has $p^2 + p - 1$ conjugacy classes showing our result is best possible.

Corollary 2. *If G is a non-abelian group of odd order and p is the least prime dividing $|G|$, then $k(G) \geq p+2$. Equality is possible if p is a Sophie Germain prime.*

We have

$$k(G) \geq \left(\frac{p^2 + p - 1}{p}\right)|Z(G)| \geq \frac{p^2 + p - 1}{p} = p + 1 - \frac{1}{p}.$$

Since $k(G)$ is an integer, $k(G) \geq p + 1$. But since $|G|$ is odd, $k(G)$ is odd, so $k(G) \geq p + 2$.

This result is best possible, since there is a non-abelian group of order 21 in $\mathcal{G}_3$ with $k(G) = 3 + 2 = 5$.

In general, if p and $2p + 1$ are both prime, there is a non-abelian group G of order $p(2p + 1)$ with $p + \frac{2p+1-1}{p} = p + 2$ conjugacy classes.

The ratio $\Pr(G) = \frac{k(G)}{|G|}$ is the probability that two elements of the finite group G, selected at random with replacement, commute.

Corollary 3.

$$\Pr(G) \geq \frac{\frac{p^2+p-1}{p}}{|G:Z(G)|}$$

for G in $\mathcal{G}_p$, and in general,

$$\Pr(G) \geq \frac{5/2}{|G:Z(G)|}.$$

Theorem 2. *We have n satisfies $\frac{3}{5}k(G) \leq n$.*

PROOF. We have $k(G) \geq \frac{5}{2}|Z(G)|$, so $\frac{2}{5}k(G) \geq |Z(G)|$. Thus

$$k(G) - \frac{2}{5}k(G) \leq k(G) - |Z(G)| = n$$

and $k(G)\left(1 - \frac{2}{5}\right) \leq n$ or $\frac{3}{5}k(G) \leq n$, as claimed. ∎

Theorem 3. *There are only finitely many groups G with a given value of n.*

PROOF. Given n, Theorem 2 implies that $k(G)$ is bounded, which in turn implies that $|G|$ is bounded and the result follows. ∎

When considering groups with a given n, we can reduce the number of groups that need to be considered by bounding $|Z(G)|$ in terms of the given value of n.

Theorem 4. *We have n satisfies $n \geq \frac{3}{2}|Z(G)|$.*

PROOF. We have $k(G) \geq \frac{5}{2}|Z(G)|$, so

$$n = k(G) - |Z(G)| \geq \frac{5}{2}|Z(G)| - |Z(G)| = \frac{3}{2}|Z(G)|.$$

∎

Lemma 2 ([16]). *Let $m = |G:Z(G)|$. Then*

$$\frac{3m-2}{m^2} \leq \Pr(G).$$

This result is easily seen by considering the Cayley table of $G/Z(G)$.

Lemma 3. *If G is a finite non-abelian group, then $k(G) = \frac{5}{2}|Z(G)|$ if and only if $|G:Z(G)| = 4$.*

PROOF. Suppose that $|G:Z(G)| = 4$. Then $\Pr(G) = \frac{5}{8}$ and so $k(G) = \frac{5}{8}|G| = \frac{5}{8}(4|Z|) = 5/2|Z(G)|$.

Now, suppose that $k(G) = \frac{5}{2}|Z(G)|$. Then, writing $m = |G : Z(G)|$,

$$\Pr(G) = \frac{5}{2}\frac{|Z(G)|}{|G|} = \frac{5}{2}\frac{1}{m} = \frac{5}{2m}.$$

So,

$$\frac{3m-2}{m^2} \leq \frac{5}{2m}$$

and so $m \leq 4$. Since $m \leq 3$ implies that G is abelian, we have $m = 4$. This result can also be deduced from the proof of Theorem 1. ■

4. Groups with less than 15 conjugacy classes

Finite groups having less than fifteen conjugacy classes are classified in [10], [1], [12], [5], [11], [7], [8] and [19].

The following table lists all groups with $k(G) \leq 14$ together with the order of their centres. If the group in question is listed in the library of small groups [2] as accessed through GAP [3], then the group is listed as n/i to indicate that it is group i of order n in this library. In some instances, these groups have more familiar names (for example, $6/1 \cong S_3$) but, in the interests of brevity, we do not give these. Of the groups in the table that are not listed in the small groups library, some are well-known simple groups (e.g. M_{11}) and these are given by name without further explanation. A presentation or other identifying information is given below for each of those groups in the table that are neither simple nor listed in the library of small groups. These groups are denoted by $G_{k,i}$ where k is the number of conjugacy classes. In some cases we identify groups as belonging to the library of primitive groups as accessed through GAP [3]. In these cases, the group is denoted by i/j to indicate that is group number j of degree i in the library of primitive groups.

$k(G)$	GAP	$\|Z(G)\|$	$k(G)$	GAP	$\|Z(G)\|$
1	1/1	1	2	2/1	2
3	3/1	3	3	6/1	1
4	4/1	4	4	4/2	4
4	10/1	1	4	12/3	1
5	5/1	5	5	8/3	2
5	8/4	2	5	14/1	1
5	20/3	1	5	21/1	1
5	24/12	1	5	60/5	1
6	6/2	6	6	12/4	2
6	12/1	2	6	18/1	1
6	18/4	1	6	36/9	1
6	72/41	1	6	168/42	1
7	7/1	7	7	16/7	2
7	16/8	2	7	16/9	2
7	22/1	1	7	24/3	2
7	39/1	1	7	42/1	1

$k(G)$	GAP	$\|Z(G)\|$	$k(G)$	GAP	$\|Z(G)\|$
7	52/3	1	7	55/1	1
7	120/34	1	7	360/118	1
8	8/1	8	8	8/2	8
8	8/5	8	8	20/1	2
8	20/4	2	8	24/13	2
8	26/1	1	8	48/3	1
8	48/28	2	8	48/29	2
8	48/50	1	8	56/11	1
8	68/3	1	8	78/1	1
8	80/49	1	8	168/43	1
8	200/44	1	8	300/23	1
8	600/150	1	8	660/13	1
8	720/765	1	9	9/1	9
9	9/2	9	9	18/3	3
9	24/4	2	9	24/6	2
9	24/8	2	9	30/3	1
9	36/10	1	9	57/1	1
9	60/7	1	9	72/15	1
9	72/39	1	9	72/40	1
9	72/43	1	9	114/1	1
9	120/5	2	9	144/182	1
9	192/1023	1	9	192/1025	1
9	336/208	1	9	504/156	1
9	960/11357	1	9	1092/25	1
9	1176/215	1	9	$G_{9,1}$	1
9	A_7	1	10	10/2	10
10	16/3	4	10	16/4	4
10	16/6	4	10	16/11	4
10	16/12	4	10	16/13	4
10	28/1	2	10	28/3	2
10	34/1	1	10	40/3	2
10	40/12	2	10	42/2	2
10	48/30	2	10	48/48	2
10	54/5	1	10	54/6	1
10	54/8	3	10	96/64	1
10	96/70	1	10	96/71	1
10	96/72	1	10	96/227	1
10	100/3	1	10	100/11	1
10	100/12	1	10	120/35	2
10	136/12	1	10	150/6	1
10	160/234	1	10	216/153	1
10	448/179	1	10	588/33	1
10	784/162	1	10	M_{11}	1
10	$G_{10,1}$	1	10	$\mathrm{PSL}_3(4)$	1
11	11/1	11	11	27/3	3

$k(G)$	GAP	$\|Z(G)\|$	$k(G)$	GAP	$\|Z(G)\|$
11	27/4	3	11	32/6	2
11	32/7	2	11	32/8	2
11	32/18	2	11	32/19	2
11	32/20	2	11	32/43	2
11	32/44	2	11	38/1	1
11	75/2	1	11	96/204	2
11	108/17	1	11	110/1	1
11	116/3	1	11	155/1	1
11	171/3	1	11	186/1	1
11	192/184	1	11	192/185	1
11	200/40	1	11	203/1	1
11	320/1635	1	11	336/114	2
11	392/38	1	11	432/734	1
11	720/763	1	11	720/764	1
11	1344/814	1	11	1344/11686	1
11	1512/779	1	11	$\mathrm{PSL}_2(17)$	1
11	$Sz(8)$	1	12	12/2	12
12	12/5	12	12	24/1	4
12	24/5	4	12	24/7	4
12	24/14	4	12	30/2	3
12	36/1	2	12	36/3	3
12	36/4	2	12	36/7	2
12	36/11	3	12	36/13	2
12	42/5	1	12	48/15	2
12	48/16	2	12	48/17	2
12	48/18	2	12	60/8	1
12	72/19	2	12	72/44	1
12	72/45	2	12	84/11	1
12	96/3	2	12	96/203	2
12	108/37	1	12	126/9	1
12	144/120	2	12	144/187	2
12	168/49	1	12	216/161	1
12	222/1	1	12	240/89	2
12	240/90	2	12	336/209	2
12	360/120	1	12	384/591	1
12	384/592	1	12	384/18134	1
12	384/18135	1	12	960/11358	1
12	1296/3523	1	12	1620/419	1
12	1920/240993	1	12	$G_{12,1}$	1
12	$G_{12,2}$	1	12	$\mathrm{PSL}_2(19)$	1
12	$\mathrm{PSL}_3(3)$	1	12	$G_{12,3}$	1
12	$G_{12,4}$	1	12	M_{22}	1
13	13/1	13	13	40/4	2
13	40/6	2	13	40/8	2
13	46/1	1	13	64/32	2

$k(G)$	GAP	$\|Z(G)\|$	$k(G)$	GAP	$\|Z(G)\|$
13	64/33	2	13	64/34	2
13	64/35	2	13	64/36	2
13	64/37	2	13	93/1	1
13	96/190	2	13	96/191	2
13	96/193	2	13	100/10	1
13	120/38	1	13	148/3	1
13	150/5	1	13	156/7	1
13	160/199	2	13	162/11	1
13	162/13	1	13	162/15	1
13	162/19	1	13	162/20	1
13	162/21	1	13	162/22	1
13	192/1491	2	13	192/1492	2
13	192/1493	2	13	192/1494	2
13	205/1	1	13	216/86	1
13	216/87	1	13	253/1	1
13	258/1	1	13	301/1	1
13	310/1	1	13	324/160	1
13	328/12	1	13	333/3	1
13	400/206	1	13	576/8652	1
13	600/148	1	13	720/409	2
13	960/11359	1	13	1000/86	1
13	1053/51	1	13	1320/133	1
13	1440/5841	1	13	1944/2289	1
13	1944/2290	1	14	14/2	14
14	32/9	4	14	32/10	4
14	32/11	4	14	32/13	4
14	32/14	4	14	32/15	4
14	32/27	4	14	32/28	4
14	32/29	4	14	32/30	4
14	32/31	4	14	32/32	4
14	32/33	4	14	32/34	4
14	32/35	4	14	32/39	4
14	32/40	4	14	32/41	4
14	32/42	4	14	44/1	2
14	44/3	2	14	48/32	4
14	48/33	4	14	50/1	1
14	50/4	1	14	78/2	2
14	80/29	2	14	80/33	2
14	80/34	2	14	80/31	2
14	84/1	2	14	84/7	2
14	96/185	2	14	96/187	2
14	96/195	2	14	104/3	2
14	104/12	2	14	108/15	3
14	110/2	2	14	128/145	2
14	128/138	2	14	128/139	2

$k(G)$	GAP	$\lvert Z(G)\rvert$	$k(G)$	GAP	$\lvert Z(G)\rvert$
14	128/144	2	14	164/3	1
14	192/956	1	14	200/42	1
14	200/43	1	14	240/91	2
14	240/189	2	14	240/192	1
14	288/1025	1	14	288/1026	1
14	294/1	1	14	294/13	1
14	294/14	1	14	300/24	1
14	300/25	1	14	320/1581	2
14	320/1582	2	14	384/4	1
14	384/5	1	14	384/6	1
14	384/5868	2	14	384/5870	2
14	384/5871	2	14	392/36	1
14	410/1	1	14	432/520	1
14	480/1188	1	14	500/21	1
14	500/23	1	14	648/703	1
14	720/766	2	14	1280/1116310	1
14	1280/1116311	1	14	1280/1116312	1
14	1344/816	1	14	1920/241000	2
14	1920/241001	1	14	$G_{14,1}$	1
14	$G_{14,2}$	1	14	$G_{14,3}$	1
14	$G_{14,4}$	1	14	$\mathrm{PSU}_3(3)$	1
14	$\mathrm{PSL}_2(23)$	1	14	$G_{14,5}$	1
14	$G_{14,6}$	1	14	$G_{14,7}$	1
14	$G_{14,8}$	1	14	A_8	1
14	$\mathrm{PSU}_3(5)$	1			

The group $G_{9,1}$ has order 2352 and may be identified as group 49/25 in the library of primitive groups.

The group $G_{10,1}$ has order 14520 and may be identified as group 121/56 in the library of primitive groups.

The group $G_{12,1}$ has order 2904 and may be given by the following presentation:

$$G_{12,1} \cong \left\langle a,b,c,d \;\middle|\; \begin{array}{l} a^{11}=b^{11}=[a,b]=1=c^4=d^3=[c^2,d]=(cd)^3, \\ a^c=b^3a^{-1}, a^d=a^{-5}b^{-4}, b^c=ba^3, b^d=a^{-3}b^{-7} \end{array} \right\rangle.$$

The group $G_{12,2}$ has order 3240 and may be given by the following presentation:

$$G_{12,2} \cong \left\langle a,b,c \;\middle|\; \begin{array}{l} a^3=b^5=c^8=(c^{-4}a)^2=1, b^c=b^{-2}, (b^{-2})^c=b^{-1}, \\ (b^{-1})^a=bcab^{-1}c^{-1}, (a^{-1})^b=a^{-1}b^{-1}a^{-1}ba, ca^{-1}cb^{-1}=abc^2a, \\ (ba)^2=c^{-1}ab^{-1}c^2ac^{-1}, (a^{-1}c)^2=b^{-1}c^{-1}a^{-1}c^3b^{-1}a, \\ (ab^{-1})^2=cab^{-2}a^{-1}bc^{-1}a^{-1}, (b^2)^a=b^2ab^{-2}a^{-1}b^2, \\ (ca)^2=b^{-1}a^{-1}ba^{-1}ca^{-1}cbab^{-1} \end{array} \right\rangle.$$

The group $G_{12,3}$ has order 5670 and may be given by the following presentation:

$$G_{12,3} \cong \left\langle a,b,c,d,e,f \;\middle|\; \begin{array}{l} a^2=b^2=c^2=d^2=1 \\ [a,b]=[a,c]=[a,d]=[b,c]=[b,d]=[c,d]=1, \\ e^2=f^4=(ef)^5=(ef^2)^5=1, \\ a^e=b^{-1}c^{-1}d^{-1}, b^e=a^{-1}c^{-1}, c^e=c^{-1}d^{-1}, d^e=d^{-1}, \\ a^f=b^{-1}, b^f=c^{-1}, c^f=a^{-1}d^{-1}, d^f=a^{-1}b^{-1} \end{array} \right\rangle.$$

The group $G_{12,4}$ has order 43320 and may be given by the following presentation:

$$G_{12,4} \cong \left\langle a,b,c,d \;\middle|\; \begin{array}{l} a^{19}=b^{19}=[a,b]=1=c^4=d^3=[c^2,d]=(cd)^5, \\ a^c=b^9a^4, a^d=b^3a^{-3}, b^c=a^{-4}b^{-4}, b^d=b^2a^4 \end{array} \right\rangle.$$

The group $G_{14,1}$ has order 2420 and may be given by the following presentation:

$$G_{14,1} \cong \left\langle o_1,o_2,a,b \;\middle|\; \begin{array}{l} o_1^{11}=o_2^{11}=[o_1,o_2]=a^5=b^4=1 \\ a^b=a^{-1}, o_1^a=o_1^4, o_2^a=o_2^3, o_1^b=o_2, o_2^b=o_1^{-1} \end{array} \right\rangle.$$

The group $G_{14,2}$ has order 2904 and may be given by the following presentation:

$$G_{14,2} \cong \left\langle o_1,o_2,a,b,c \;\middle|\; \begin{array}{l} o_1^{11}=o_2^{11}=[o_1,o_2]=c^3=a^4=1 \\ b^2=a^2, a^b=a^{-1}, c^a=c, c^b=c^{-1}, o_1^c=o_2, \\ o_2^c=o_1^{-1}o_2^{-1}, o_1^a=o_1^2o_2^4, o_2^a=o_1^{-4}o_2^{-2}, \\ o_1^b=o_1^{-4}o_2, o_2^b=o_1^5o_2^4 \end{array} \right\rangle.$$

The group $G_{14,3}$ has order 3000 and may be given by the following presentation:

$$G_{14,3} \cong \left\langle o_1,o_2,a,b \;\middle|\; \begin{array}{l} o_1^5=o_2^5=[o_1,o_2]=a^5=b^3=(ab)^4=[a,(ab)^2]=1 \\ o_1^a=o_1, o_2^a=o_1o_2, o_1^b=o_2, o_2^b=o_1^{-1}o_2^{-1} \end{array} \right\rangle.$$

The group $G_{14,4}$ has order 4056 and may be given by the following presentation:

$$G_{14,4} \cong \left\langle o_1,o_2,a,b,c \;\middle|\; \begin{array}{l} o_1^{13}=o_2^{13}=[o_1,o_2]=a^4=c^3=1, \\ b^2=a^2, a^b=a^{-1}, a^c=b, b^c=ab, o_1^a=o_1^5, \\ o_2^a=o_2^{-5}, o_1^b=o_2, o_2^b=o_1^{-1}, o_1^c=o_1^2o_2^3, o_2^c=o_1^2o_2^{-3} \end{array} \right\rangle.$$

The group $G_{14,5}$ has order 13872 and may be given by the following presentation:

$$G_{14,5} \cong \left\langle o_1,o_2,a,b,c,d \;\middle|\; \begin{array}{l} o_1^{17}=o_2^{17}=[o_1,o_2]=a^4=c^3=1, \\ b^2=a^2, a^b=a^{-1}, a^c=b, b^c=ab, \\ d^2=a^2, a^d=b^{-1}, b^d=a^{-1}, c^d=c^{-1}, o_1^a=o_1^4, \\ o_2^a=o_2^{-4}, o_1^b=o_2, o_2^b=o_1^{-1}, o_1^c=o_1^{-7}o_2^{-6}, \\ o_2^c=o_1^{-7}o_2^6, o_1^d=o_1^5o_2^3, o_2^d=o_1^{-3}o_2^{-5} \end{array} \right\rangle.$$

Finally, groups $G_{14,6}$, $G_{14,7}$ and $G_{14,8}$ are each of order 40320 and are non-split extensions of $\mathrm{PSL}_3(4)$ by C_2. These three groups may be identified as groups 56/4, 56/2 and 56/3 in the library of primitive groups.

We note that Vera-López and Sangroniz [18] kindly provided the presentations for the groups $G_{14,1}$, $G_{14,2}$, $G_{14,3}$, $G_{14,4}$ and $G_{14,5}$ that are given above. The information in this section was drawn from [4].

5. The classification

We are now in a position to discuss groups with a given number n of non-central conjugacy classes for $n \leq 9$. We assume, throughout, that G is a non-abelian finite group with n non-central conjugacy classes.

To begin, note that if $n = 1$, then $k(G) \leq \frac{5}{3}$ so $k(G) = 1$ and $|G| = 1$ which means that G is abelian, a contradiction. Thus there are no groups with $n = 1$.

Now, if we fix n, Theorem 4 gives us an upper bound on $|Z(G)|$. Moreover, since $k(G) = n + |Z(G)|$, each possibility for $Z(G)$ gives rise to a fixed value of $k(G)$. For example, if $n = 4$, then Theorem 4 gives $\frac{8}{3} \geq |Z(G)|$ and so $|Z(G)| = 1$ or 2. Then, since $k(G) = 4 + |Z(G)|$ we have $k(G) = 5$ or 6. Thus, to find all groups with $n = 4$ non-central conjugacy classes, we need only find those groups with $k(G) = 5$ and $|Z(G)| = 1$ and those groups with $k(G) = 6$ and $|Z(G)| = 2$. Consulting the table in Section 4, the groups with $k(G) = 5$ and $|Z(G)| = 1$ are 14/1, 20/3, 21/1, 24/12 and 60/5 and the groups with $k(G) = 6$ and $|Z(G)| = 2$ are 12/1 and 12/4.

Continuing this procedure, the following Lemma lists the cases that need to be considered for $n \leq 9$.

Lemma 4. $n = 2$**:** *$k(G) = 3$ and $|Z(G)| = 1$;*
$n = 3$**:** *$k(G) = 4$ and $|Z(G)| = 1$; $k(G) = 5$ and $|Z(G)| = 2$;*
$n = 4$**:** *$k(G) = 5$ and $|Z(G)| = 1$; $k(G) = 6$ and $|Z(G)| = 2$;*
$n = 5$**:** *$k(G) = 6$ and $|Z(G)| = 1$; $k(G) = 7$ and $|Z(G)| = 2$; $k(G) = 8$ and $|Z(G)| = 3$;*
$n = 6$**:** *$k(G) = 7$ and $|Z(G)| = 1$; $k(G) = 8$ and $|Z(G)| = 2$; $k(G) = 9$ and $|Z(G)| = 3$; $k(G) = 10$ and $|Z(G)| = 4$;*
$n = 7$**:** *$k(G) = 8$ and $|Z(G)| = 1$; $k(G) = 9$ and $|Z(G)| = 2$; $k(G) = 10$ and $|Z(G)| = 3$; $k(G) = 11$ and $|Z(G)| = 4$;*
$n = 8$**:** *$k(G) = 9$ and $|Z(G)| = 1$; $k(G) = 10$ and $|Z(G)| = 2$; $k(G) = 11$ and $|Z(G)| = 3$; $k(G) = 12$ and $|Z(G)| = 4$; $k(G) = 13$ and $|Z(G)| = 5$;*
$n = 9$**:** *$k(G) = 10$ and $|Z(G)| = 1$; $k(G) = 11$ and $|Z(G)| = 2$; $k(G) = 12$ and $|Z(G)| = 3$; $k(G) = 13$ and $|Z(G)| = 4$; $k(G) = 14$ and $|Z(G)| = 5$; $k(G) = 15$ and $|Z(G)| = 6$;*

The final case for $n = 9$ involves looking for groups with $k(G) = 15$ and $|Z(G)| = 6$. The groups with 15 conjugacy classes have not yet been classified. However, in this case, $k(G) = \frac{5}{2}|Z(G)|$ and so Lemma 3 says that $|G : Z(G)| = 4$ which implies that $|G| = 24$. There are two groups of order 24 that have 15 conjugacy classes, namely small groups 24/10 and 24/11, and each of these has centre of order 6.

Therefore, consulting the table in Section 4 for each of the cases given in Lemma 4, we have

Theorem 5.
- *There are no groups with exactly one non-central conjugacy class.*

- *The only group with exactly two non-central conjugacy classes is* 6/1*.*
- *The groups with exactly three non-central conjugacy classes are* 8/3*,* 8/4*,* 10/1 *and* 12/3*.*
- *The groups with exactly four non-central conjugacy classes are* 12/1*,* 12/4*,* 14/1*,* 20/3*,* 21/1*,* 24/12 *and* 60/5*.*
- *The groups with exactly five non-central conjugacy classes are* 16/7*,* 16/8*,* 16/9*,* 18/1*,* 18/4*,* 24/3*,* 36/9*,* 72/41 *and* 168/42*.*
- *The groups with exactly six non-central conjugacy classes are* 16/3*,* 16/4*,* 16/6*,* 16/11*,* 16/12*,* 16/13*,* 18/3*,* 20/1*,* 20/4*,* 22/1*,* 24/13*,* 39/1*,* 42/1*,* 48/28*,* 48/29*,* 52/3*,* 55/1*,* 120/34 *and* 360/118*.*
- *The groups with exactly seven non-central conjugacy classes are* 24/4*,* 24/6*,* 24/8*,* 26/1*,* 48/3*,* 48/50*,* 54/8*,* 56/11*,* 68/3*,* 78/1*,* 80/49*,* 120/5*,* 168/43*,* 200/44*,* 300/23*,* 600/150*,* 660/13 *and* 720/765*.*
- *The groups with exactly eight non-central conjugacy classes are* 24/14*,* 24/7*,* 24/5*,* 24/1*,* 27/4*,* 27/3*,* 28/3*,* 28/1*,* 30/3*,* 36/10*,* 40/3*,* 40/12*,* 42/2*,* 48/30*,* 48/48*,* 57/1*,* 60/7*,* 72/43*,* 72/40*,* 72/39*,* 72/15*,* 114/1*,* 120/35*,* 144/182*,* 192/1023*,* 192/1025*,* 336/208*,* 504/156*,* 960/11357*,* 1092/25*,* 1176/215*,* $G_{9,1}$ *and* A_7*.*
- *The groups with exactly nine non-central conjugacy classes are* 24/10*,* 24/11*,* 30/2*,* 32/44*,* 32/43*,* 32/20*,* 32/19*,* 32/18*,* 32/8*,* 32/7*,* 32/6*,* 34/1*,* 36/3*,* 36/11*,* 54/6*,* 54/5*,* 96/227*,* 96/72*,* 96/71*,* 96/70*,* 96/64*,* 96/204*,* 100/12*,* 100/11*,* 100/3*,* 136/12*,* 150/6*,* 160/234*,* 216/153*,* 336/114*,* 448/179*,* 588/33*,* 784/162*,* M_{11}*,* $G_{10,1}$ *and* $\mathrm{PSL}_3(4)$*.*

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