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A STOCHASTIC STINESPRING THEOREM

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Abstract. Completely positive Markovian cocycles on a von Neumann algebra, adapted to a Fock filtration, are realised as conjugations of *-homomorphic Markovian cocycles. The conjugating processes are affiliated to the algebra, and are governed by quantum stochastic differential equations whose coefficients evolve according to the *-homomorphic process. Some perturbation theory for quantum stochastic flows is developed in order to achieve the above Stinespring decomposition.

0. Introduction

The classical Stinespring Theorem ([Sti]) states that every bounded completely positive linear map T from a unital C^*-algebra $\mathcal{A}$ into the algebra of bounded operators on a Hilbert space $\mathcal{H}$ may be decomposed as

$$Ta = V^*\pi(a)V \tag{0.1}$$

for a unital representation $(\pi, \mathcal{K})$ of $\mathcal{A}$ and a bounded operator V from $\mathcal{H}$ to $\mathcal{K}$. A decomposition of the same form holds for linear maps $T : \mathcal{A} \to \mathcal{L}(\mathcal{D}; \mathcal{H})$, in which $\mathcal{D}$ is a subspace of $\mathcal{H}$ which is not necessarily closed. In this case $V \in \mathcal{L}(\mathcal{D}; \mathcal{K})$ and is unbounded if T is. Complete positivity for such a map means that for any finite collection $\{(a_i, \xi_i)\}$ from $\mathcal{A} \times \mathcal{D}$,

$$\sum_{i,j} \langle \xi_i, T(a_i^* a_j)\xi_j\rangle \geq 0. \tag{0.2}$$

In this paper we consider completely positive Markovian cocycles on a von Neumann algebra which are adapted to a Fock filtration ([LW2]). We show that every such cocycle k which is sufficiently regular satisfies

$$\widetilde{k}_t(a) = W_t^* j_t(a) W_t$$

(in which $\widetilde{k}$ is a certain extension of k) where j is a *-homomorphic Markovian cocycle, and W is a process affiliated to the algebra and satisfying a quantum stochastic differential equation of the form

$$dW_t = j_t(l^\alpha_\beta) W_t \, d\Lambda^\beta_\alpha(t), \quad W_0 = 1 \tag{0.3}$$

([Par],[LW1]). To achieve this continuous time stochastic Stinespring decomposition it may be necessary to enlarge the dimension of the quantum noise driving the given process k.

Interest in completely positive stochastic flows and their structure comes from several sources. They arise naturally in the quantum theory of filtering ([Be1]); they will be required for any quantum theory of measure-valued diffusions ([GLSW]); but also their study allows quantum dynamical semigroups ([EvL]), quantum diffusions on an operator algebra ([Hud],[Eva]), and contraction processes on a Hilbert space ([Fag],[Moh]), to be viewed from a common vantage point ([LiP],[LW1]). In a sister

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paper ([GLSW]) completely positive Markovian contraction cocycles are dilated to *-homomorphic Markovian cocycles driven by a higher dimensional quantum noise, extending the dilation of quantum dynamical semigroups achieved in [GoS]. Perturbations of the form (0.3), in which W is unitary and j is unital, were studied originally in [EvH] and subsequently in [DaS].

The plan of the paper is as follows. Notations are developed in the first section. In the second section some elementary properties of algebra matrices, and maps of algebra matrices induced by C^*-maps, are established, and some quantum stochastic theory is recalled. The equation (0.3) is studied, for contractivity, isometry and coisometry of its solution, in the third section, and this is then applied to obtain completely positive perturbations of the *-homomorphic Markovian cocycle j, by conjugation. In the final section the advertised Stinespring decomposition is obtained, by combining the perturbation theory with characterisations of completely positive regular Markovian cocycles by the form of their generator. We also give a realisation in which j is replaced by ampliation:

$$k_t(a) = X_t^*(a \otimes 1)X_t.$$

Here the conjugating process need no longer be affiliated to the algebra. In both cases it may be necessary to enlarge the noise dimension space.

1. Preliminaries

In this section we establish notations, and various identifications to be used throughout the paper. *We fix once and for all* a Hilbert space $\mathfrak{h}$, called the *initial space.*

General notations. We use the symbol $\odot$ to denote algebraic tensor products, and reserve the symbol $\otimes$ for tensor products of Hilbert spaces, bounded Hilbert space operators, von Neumann algebras and normal completely bounded linear maps between von Neumann algebras. For Hilbert spaces $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2$, and a von Neumann algebra $\mathcal{M}$ acting on $\mathcal{K}$, $\mathcal{M} \otimes \mathcal{B}(\mathcal{K}_1; \mathcal{K}_2)$ denotes the Hilbert W^*-module

$$\{T \in \mathcal{B}(\mathcal{K} \otimes \mathcal{K}_1; \mathcal{K} \otimes \mathcal{K}_2) : T(a' \otimes 1_{\mathcal{K}_1}) = (a' \otimes 1_{\mathcal{K}_2})T \quad \forall a' \in \mathcal{M}'\};$$

it is abbreviated to $\mathcal{M} \otimes \mathcal{K}_2$ when $\mathcal{K}_1 = \mathbb{C}$.

Particular notations. Let k be a separable Hilbert space with orthonormal basis $\eta = (e_i)_{i \geq 1}$, extended to a basis $(e_\alpha)_{\alpha \geq 0}$ of $\widehat{\mathsf{k}} := \mathbb{C} \oplus \mathsf{k}$ by putting $e_0 = 1$. Let $D = \dim \widehat{\mathsf{k}}$. The isometric embeddings

$$E_\alpha : \mathfrak{h} \to \mathfrak{h} \otimes \widehat{\mathsf{k}}, \quad E_\alpha u = u \otimes e_\alpha$$

give rise to the identification

$$\oplus_{\gamma \geq 0} \mathfrak{h} = \mathfrak{h} \oplus (\mathfrak{h} \otimes \mathsf{k}) = \mathfrak{h} \otimes \widehat{\mathsf{k}}$$

via the mapping $(u^\gamma)_{\gamma \geq 0} \mapsto E_\gamma u^\gamma$ (summation over γ), with inverse $\xi \mapsto (E^\gamma \xi)_{\gamma \geq 0}$, where $E^\gamma := (E_\gamma)^*$. The following dense subspace is useful:

$$\mathfrak{h}_{00} := \{(u^\gamma) \in \mathfrak{h} \otimes \widehat{\mathsf{k}} : u^\alpha \neq 0 \text{ for only finitely many } \alpha\};$$

we write $\dim(u^\gamma)$ for $\max\{\alpha \geq 0 : u^\alpha \neq 0\}$.

For any complex vector space V, $\mathrm{M}_D(V)$ denotes the collection of $(D \times D)$ matrices $v = [v^\alpha_\beta]$ with entries in V, the indices ranging over the set $\{\gamma : 0 \leq \gamma < D\}$. $\mathrm{M}_D(V)$ is endowed with linear structure by pointwise defined operations, and when V has an involution $\dagger$, this gives rise to an involution $v \mapsto v^\dagger$ on $\mathrm{M}_D(V)$ defined by $(v^\dagger)^\alpha_\beta = (v^\beta_\alpha)^\dagger$. Throughout the paper Einstein's summation convention for repeated indices will apply, with greek indices running over the set $\{\gamma : 0 \leq \gamma < D\}$ and roman indices running over the set $\{i : 1 \leq i < D\}$ indexing the basis η (where k, and so D, will be evident from the context). For any $v \in \mathrm{M}_D(V)$ its *cutoff* matrices $\{v^{[N]}\}_{N<D}$ are defined by

$$v^{[N]}{}^{\alpha}_{\beta} = \begin{cases} v^{\alpha}_{\beta}, & \text{if } \alpha, \beta \leq N \\ 0, & \text{otherwise} \end{cases}.$$

Sometimes $v^{[N]}$ is viewed as an element of $\mathrm{M}_{N+1}(V)$. Define $c(D)$ to be $\mathbb{C}^{D-1}$ if $D < \infty$, and $c_{00}(\mathbb{N})$ otherwise, where $c_{00}(\mathbb{N})$ is the collection of complex sequences $(z^i)_{i \geq 1}$ whose terms are eventually zero. For $v \in \mathrm{M}_D(V)$ and $w, z \in c(D)$, we use the notation

$$v_{w,z} := w_\alpha v^{\alpha}_{\beta} z^{\beta}$$

where $w_\alpha = \overline{w^\alpha}$, and $w^0 = z^0 = 1$.

Fock space. For a Hilbert space $\mathcal{K}$ containing a vector f, $\Gamma(\mathcal{K}), \varepsilon(f)$ and $\Omega(\mathcal{K})$ denote respectively the symmetric Fock space over $\mathcal{K}$: $\bigoplus_{n \geq 0} \otimes^{(n)}_{\text{sym.}} \mathcal{K}$, the exponential vector of f: $((n!)^{-1/2} \otimes^{(n)} f) \in \Gamma(\mathcal{K})$, and the vacuum vector for $\mathcal{K}$: $\varepsilon(0) = (1, 0, 0, \ldots)$. The exponential property $\Gamma(\mathcal{K}_1 \oplus \mathcal{K}_2) = \Gamma(\mathcal{K}_1) \otimes \Gamma(\mathcal{K}_2)$ is best viewed via exponential vectors: $\varepsilon(f_1, f_2) \leftrightarrow \varepsilon(f_1) \otimes \varepsilon(f_2)$. When $\mathcal{K} = L^2(J; \mathsf{k})$ for some measurable subset J of $\mathbb{R}_+$ and Hilbert space k, we write $\mathcal{F}_{J,\mathsf{k}}$, $\Omega_{J,\mathsf{k}}$, $\mathcal{B}_{J,\mathsf{k}}$ and $1_{J,\mathsf{k}}$ for $\Gamma(\mathcal{K})$, $\Omega(\mathcal{K})$, $\mathcal{B}(\Gamma(\mathcal{K}))$ and the identity operator on $\Gamma(\mathcal{K})$ respectively, abbreviating further, to $\mathcal{F}_{\mathsf{k}}$, Ω_{k}, $\mathcal{B}_{\mathsf{k}}$ and 1_{k}, when $J = \mathbb{R}_+$.

If J is a disjoint union of measurable sets J_0 and J_1, and k is an orthogonal sum of Hilbert spaces k_0 and k_1 then the following identifications result:

$$\mathcal{F}_{\mathsf{k}} = \mathcal{F}_{\mathsf{k}_0} \otimes \mathcal{F}_{\mathsf{k}_1}; \quad \mathcal{F}_{\mathsf{k}_0} = \mathcal{F}_{\mathsf{k}_0} \otimes \Omega_{\mathsf{k}_1}; \quad \mathcal{B}_{\mathsf{k}} = \mathcal{B}_{\mathsf{k}_0} \otimes \mathcal{B}_{\mathsf{k}_1}; \quad \mathcal{B}_{\mathsf{k}_0} = \mathcal{B}_{\mathsf{k}_0} \otimes 1_{\mathsf{k}_1};$$
$$\mathcal{F}_{J,\mathsf{k}} = \mathcal{F}_{J_0,\mathsf{k}} \otimes \mathcal{F}_{J_1,\mathsf{k}}; \quad \mathcal{F}_{J_0,\mathsf{k}} = \mathcal{F}_{J_0,\mathsf{k}} \otimes \Omega_{J_1,\mathsf{k}};$$
$$\mathcal{B}_{J,\mathsf{k}} = \mathcal{B}_{J_0,\mathsf{k}} \otimes \mathcal{B}_{J_1,\mathsf{k}}; \quad \mathcal{B}_{J_0,\mathsf{k}} = \mathcal{B}_{J_0,\mathsf{k}} \otimes 1_{J_1,\mathsf{k}}.$$

Now let $\eta = (e_i)$ be an orthonormal basis of k. Since

$$\mathbb{M}[\eta] := \mathrm{Lin}\{\varphi \otimes e_i : \varphi \in (L^2 \cap L^\infty_{\mathrm{loc}})(\mathbb{R}_+), i \geq 1\}$$

is dense in $L^2(\mathbb{R}_+) \otimes \mathsf{k} = L^2(\mathbb{R}_+; \mathsf{k})$,

$$\mathcal{E}[\eta] := \mathrm{Lin}\{\varepsilon(f) : f \in \mathbb{M}[\eta]\}$$

is a dense subspace of $\mathcal{F}_{\mathsf{k}}$. For $f \in \mathbb{M}[\eta]$ let

$$f^0(t) = 1; \quad f^i(t) = \langle e_i, f(t) \rangle, \quad i \geq 1,$$

let $\dim f = \max\{\gamma \geq 0 : f^\gamma \neq 0\}$, and write f_α for $\overline{f^\alpha}$.

2. Algebra matrix mappings and quantum processes

In this section we prove some elementary properties of algebra matrices and algebra matrix maps induced by C^*-algebra maps. We also recall some facts from quantum stochastic calculus, for which our basic references are [Par] and [LW1]. Fix a separable Hilbert space k with orthonormal basis η, and let $D = \dim \widehat{\mathsf{k}}$.

Algebra matrices. Let $\mathcal{A}$ be a C^*-algebra acting on $\mathfrak{h}$. Elements of $\mathrm{M}_D(\mathcal{B}(\mathfrak{h}))$ and $\mathrm{M}_D(\mathcal{A})$ are called *operator matrices* and *algebra matrices* respectively. These spaces are well understood when $D < \infty$, however for the purposes of this paper it is vital that we are able to cope with the case $D = \infty$. The basic properties of operator matrices (when $D = \infty$) are developed in [LW1], starting from the regularity conditions introduced in [MoS]. Here we continue this study since we need to consider the behaviour of algebra matrices under multiplication/composition, and under transformations induced by completely positive maps.

An operator matrix F determines (possibly unbounded) operators $F^\alpha : \oplus_{\gamma\geq 0}\mathfrak{h} \to \mathfrak{h}$, $\widetilde{F} : \oplus_{\gamma\geq 0}\mathfrak{h} \to \oplus_{\gamma\geq 0}\mathfrak{h}$ and $F_\beta : \mathfrak{h} \to \oplus_{\gamma\geq 0}\mathfrak{h}$ as follows:

$$\begin{aligned} &F^\alpha(u^\gamma) = F^\alpha_\beta u^\beta, && \operatorname{Dom} F^\alpha = \mathfrak{h}_{00} \\ &\widetilde{F}(u^\gamma) = (F^\alpha_\beta u^\beta)_{\alpha\geq 0}, && \operatorname{Dom} \widetilde{F} = \{(u^\gamma) \in \mathfrak{h}_{00} : \textstyle\sum_{\alpha\geq 0}\|F^\alpha_\beta u^\beta\|^2 < \infty\} \\ &F_\beta u = (F^\alpha_\beta u)_{\alpha\geq 0}, && \operatorname{Dom} F_\beta = \{u \in \mathfrak{h} : \textstyle\sum_{\alpha\geq 0}\|F^\alpha_\beta u\|^2 < \infty\}, \end{aligned}$$

An operator matrix F is *regular* if each F_β is everywhere defined and bounded, or, equivalently, $\widetilde{F}$ has domain $\mathfrak{h}_{00}$; and is *bounded* if $\widetilde{F}$ is bounded and has domain $\mathfrak{h}_{00}$. When F is bounded we identify it with the bounded operator on $\mathfrak{h}\otimes\widehat{\mathsf{k}}$ obtained by continuous linear extension of $\widetilde{F}$. The salient properties are as follows:

$$(F^\dagger)_\gamma = (F^\gamma)^*, \quad F_\beta = \widetilde{F}E_\beta, \quad F^\alpha_\beta \supset E^\alpha \widetilde{F} E_\beta,$$

in particular each F_β is a closed operator. When F is regular the above inclusion is an equality, and

$$(F^\alpha_\beta)^* F^\alpha_\gamma = (F_\beta)^* F_\gamma,$$

the convergence in the left hand side being in the strong operator sense.

When F is bounded,

$$F_\beta(F_\beta)^* = FF^*, \quad (F^\alpha)^* F^\alpha = F^* F$$

and

$$F^{[N]} \to F,$$

again in the strong operator sense. If F is only regular, then the strong convergence of the cutoffs $F^{[N]}$ to F still holds on $\mathfrak{h}_{00}$. We shall use the notation

$$\begin{aligned} \mathrm{M}_D(\mathcal{A})_{\mathrm{R}} &= \{l \in \mathrm{M}_D(\mathcal{A}) : l \text{ is regular}\} \\ \mathrm{M}_D(\mathcal{A})_{\mathrm{b}} &= \{l \in \mathrm{M}_D(\mathcal{A}) : l \text{ is bounded}\} \end{aligned}$$

Note that if $l \in \mathrm{M}_D(\mathcal{A})_{\mathrm{R}}$ then

$$l_\beta a' v = (a' l^\alpha_\beta v)_{\alpha\geq 0} = (a' \otimes 1) l_\beta v \quad \forall a' \in \mathcal{A}', v \in \mathfrak{h}, \tag{2.1}$$

so $l_\beta \in \mathcal{A}'' \otimes \widehat{\mathsf{k}}$; also

$$\mathcal{A} \otimes_\sigma \mathcal{B}(\widehat{\mathsf{k}}) \subset \mathrm{M}_D(\mathcal{A})_{\mathrm{b}} \subset \mathcal{A}'' \otimes \mathcal{B}(\widehat{\mathsf{k}}),$$

where $\otimes_\sigma$ denotes the spatial tensor product; the second inclusion being an equality if and only if $\mathcal{A}$ is a von Neumann algebra; the first inclusion being an equality when either $\mathcal{A}$ or k is finite dimensional.

An operator matrix F is *nonnegative*, written $F \geq 0$, if the corresponding densely defined quadratic form

$$\mathfrak{h}_{00} \to \mathbb{C}, \quad (u^\gamma) \mapsto \langle u^\alpha, F^\alpha_\beta u^\beta \rangle$$

is nonnegative, that is if and only if each of the cutoff matrices $F^{[N]}$ is nonnegative as an element of $\mathrm{M}_{N+1}(\mathcal{B}(\mathfrak{h}))$. This corresponds with the usual definition when F is bounded.

Lemma 2.1. *The map*

$$\begin{aligned} \mathrm{M}_D(\mathcal{B}(\mathfrak{h}))_{\mathrm{R}} \times \mathrm{M}_D(\mathcal{B}(\mathfrak{h}))_{\mathrm{R}} &\to \mathrm{M}_D(\mathcal{B}(\mathfrak{h})) \\ (F, G) &\mapsto F^\dagger G := [(F_\alpha)^* G_\beta] \end{aligned}$$

enjoys the following properties:

(a) $F^\dagger(G + \lambda H) = F^\dagger G + \lambda F^\dagger H$; $G^\dagger F = (F^\dagger G)^\dagger$; *and* $F^\dagger F \geq 0$ *with equality if and only if* $F = 0$.
(b) *If* F *and* G *are bounded, then* $F^\dagger G = F^* G$.
(c) *If* F *is bounded then* $F^\dagger G$ *is regular.*

(d) *For a C^*-algebra $\mathcal{A}$ acting on $\mathfrak{h}$, the map restricts to a map*

$$\mathrm{M}_D(\mathcal{A})_{\mathrm{R}} \times \mathrm{M}_D(\mathcal{A})_{\mathrm{R}} \to \mathrm{M}_D(\mathcal{A}'').$$

Proof. When F is bounded, $(F^\dagger G)_\beta = F^* G_\beta$, so $F^\dagger G$ is regular. If $l, m \in \mathrm{M}_D(\mathcal{A})_{\mathrm{R}}$ then $l_\alpha, m_\beta \in \mathcal{A}'' \otimes \widehat{\mathsf{k}}$ by (2.1), so $(l_\alpha)^* m_\beta \in \mathcal{A}''$. The rest of the lemma follows easily. □

Algebra matrix maps. Let $\mathcal{A}$ be a C^*-algebra acting on $\mathfrak{h}$ and let $\mathcal{C}$ be another C^*-algebra acting on a Hilbert space $\mathfrak{h}'$. For a linear map $T : \mathcal{A} \to \mathcal{C}$ we write $T^{(D)}$ for the induced map $\mathrm{M}_D(\mathcal{A}) \to \mathrm{M}_D(\mathcal{C})$ defined by $(T^{(D)}l)^\alpha_\beta = T(l^\alpha_\beta)$. This notation is consistent with the usual notation $T^{(n)}$ for the induced map between the C^*-algebras $\mathrm{M}_n(\mathcal{A})$ and $\mathrm{M}_n(\mathcal{C})$, for $n \in \mathbb{N}$.

The following identities are useful: let $(v^\gamma) \in \mathfrak{h}_{00}$, and for each N define cutoffs $\mathbf{v}^{[N]} = (v^0, v^1, \dots, v^N) \in \oplus_{\alpha=0}^N \mathfrak{h}$, then

$$\langle v^\alpha, l^\alpha_\beta v^\beta \rangle = \langle \mathbf{v}^{[N]}, l^{[N]} \mathbf{v}^{[N]} \rangle \quad \text{for all} \quad N \geq \dim(v^\gamma); \tag{2.2}$$

for $T \in \mathcal{B}(\mathcal{A}; \mathcal{C})$ and $l \in \mathrm{M}_D(\mathcal{A})$,

$$(T^{(D)}l)^{[N]} = T^{(N+1)} l^{[N]}; \tag{2.3}$$

and for $l \in \mathrm{M}_D(\mathcal{A})_{\mathrm{R}}$,

$$\|l(v^\gamma)\| = \lim_{N \to \infty} \|l^{[N]} \mathbf{v}^{[N]}\|. \tag{2.4}$$

Lemma 2.2. *Let $T : \mathcal{A} \to \mathcal{C}$ be a completely bounded linear map, and let $l \in \mathrm{M}_D(\mathcal{A})$.*

(a) *If l is regular then $T^{(D)}l$ is regular, and*

$$\|(T^{(D)}l)_\beta\| \leq \|T\|_{\mathrm{cb}} \|l_\beta\| \quad \forall \beta.$$

(b) *If l is bounded then $T^{(D)}l$ is bounded, and*

$$\|T^{(D)}l\| \leq \|T\|_{\mathrm{cb}} \|l\|.$$

(c) *If l is nonnegative and T is completely positive, then $T^{(D)}l \geq 0$.*

(d) *If l is regular and T is completely positive, then*

$$(T^{(D)}l)^\dagger T^{(D)}l \leq \|T\| T^{(D)}(l^\dagger l),$$

provided $l^\dagger l \in \mathrm{M}_D(\mathcal{A})$.

Proof. Let $l \in \mathrm{M}_D(\mathcal{A})$ and put $m = T^{(D)}l$.

(a) By the Wittstock-Paulsen Theorem ([Pau], Theorem 7.4) there is a representation $(\pi, \mathcal{K})$ of $\mathcal{A}$ with bounded operators $V, W \in \mathcal{B}(\mathfrak{h}'; \mathcal{K})$ such that

$$Ta = V^* \pi(a) W, \quad \|V\| = 1, \quad \|W\| = \|T\|_{\mathrm{cb}}.$$

If l is regular then, for each N, β and v,

$$\begin{aligned}
\sum_{\alpha \leq N} \|m^\alpha_\beta v\|^2 &\leq \sum_{\alpha \leq N} \|\pi(l^\alpha_\beta) W v\|^2 \\
&= \langle Wv, \pi\big(\textstyle\sum_{\alpha \leq N} l^{\alpha *}_\beta l^\alpha_\beta\big) Wv \rangle \\
&\leq \|Wv\|^2 \|\textstyle\sum_{\alpha \leq N} (l^\alpha_\beta)^* l^\alpha_\beta\| \\
&\leq \|T\|^2_{\mathrm{cb}} \|v\|^2 \|l_\beta\|^2,
\end{aligned}$$

and the result follows.

(b) If l is bounded then, for $(v^\gamma) \in \mathfrak{h}'_{00}$, (2.2) and (2.3) give

$$\begin{aligned}
|\langle v^\alpha, m^\alpha_\beta v^\beta \rangle| &\leq \|\mathbf{v}^{[N]}\|^2 \|T^{(N+1)}\| \|l^{[N]}\| \\
&\leq \|(v^\gamma)\|^2 \|T\|_{\mathrm{cb}} \|l\|,
\end{aligned}$$

thus the form associated with m is bounded with norm at most $\|T\|_{\mathrm{cb}}\|l\|$, and so the result follows.

(c) This follows immediately from (2.2) and (2.3).

(d) Let $(v^\gamma) \in \mathfrak{h}'_{00}$ and $N \geq \dim(v^\gamma)$. Then $(l^\dagger l)^{[N]} \geq l^{\dagger[N]} l^{[N]}$, and so applying the operator Schwarz inequality to $T^{(N+1)}$ gives

$$\begin{aligned}\|T\|\langle v^\alpha, (T^{(D)}(l^\dagger l))^\alpha_\beta v^\beta\rangle &= \|T\|\langle \mathbf{v}^{[N]}, T^{(N+1)}((l^\dagger l)^{[N]})\mathbf{v}^{[N]}\rangle \\ &\geq \|T^{(N+1)}\|\langle \mathbf{v}^{[N]}, T^{(N+1)}(l^{\dagger[N]} l^{[N]})\mathbf{v}^{[N]}\rangle \\ &\geq \|(T^{(N+1)} l^{[N]})\mathbf{v}^{[N]}\|^2 \\ &= \|(T^{(D)} l)^{[N]}\mathbf{v}^{[N]}\|^2,\end{aligned}$$

since $\|T\|_{\mathrm{cb}} = \|T\|$ for completely positive operators. Letting $N \to \infty$ gives the result by (2.4). $\square$

Mapping matrices. Again let $\mathcal{A}$ be a C^*-algebra acting on $\mathfrak{h}$. Elements of $\mathrm{M}_D(\mathcal{B}(\mathcal{A}))$ are called *mapping matrices*. A mapping matrix may be viewed as a linear map $\theta : \mathcal{A} \to \mathrm{M}_D(\mathcal{A})$ by

$$\theta(a)^\alpha_\beta = \theta^\alpha_\beta(a).$$

The following mapping matrices are useful:

$$\Delta^\alpha_\beta = \begin{cases}\mathrm{id}, & \alpha = \beta \neq 0\\ 0, & \text{otherwise}\end{cases}, \quad \iota^\alpha_\beta = \begin{cases}\mathrm{id}, & \alpha = \beta\\ 0, & \text{otherwise}\end{cases}.$$

The first of these is used to define transforms on $\mathrm{M}_D(\mathcal{B}(\mathfrak{h}))$, $\mathrm{M}_D(\mathcal{B}(\mathcal{A}))$ and $\mathrm{M}_D(\mathcal{A})$ as follows:

$$\widehat{F} = F + \Delta(1), \quad \widehat{\theta} = \theta + \Delta, \quad \widehat{l} = l + \Delta(1).$$

A number of regularity conditions on mapping matrices are discussed in [LW1]. The most important for us here are (*complete*) *boundedness* and *M-S regularity*. A mapping matrix θ is *bounded* if $\theta(\mathcal{A}) \subset \mathrm{M}_D(\mathcal{A})_{\mathrm{b}}$, and so each $\theta(a)$ may be identified with a bounded operator in $\mathcal{A}'' \otimes \mathcal{B}(\widehat{\mathsf{k}})$. The Uniform Boundedness Principle applied to the sequence of cutoffs $(\theta^{[N]})$ implies that θ is then bounded as a linear operator: $\theta \in \mathcal{B}(\mathcal{A}; \mathcal{A}'' \otimes \mathcal{B}(\widehat{\mathsf{k}}))$. The space of *completely bounded* mapping matrices are thus defined in the obvious manner.

A mapping matrix θ is *M-S regular* if for each a the algebra matrix $\theta(a)$ is regular, and if there exists a Hilbert space $\mathcal{K}$, and a sequence $\{B_\beta\} \subset \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathcal{K})$ such that

$$\theta(a)^*_\beta \theta(a)_\beta \leq B^*(a^*a \otimes 1_{\mathcal{K}})B \quad \forall a \in \mathcal{A}.$$

This is a sufficient condition to ensure the existence of a solution to the quantum stochastic differential equation (2.10) below. A weaker sufficient condition on θ for the existence of a solution is *strong regularity* ([LW1]). However, if $\mathcal{A}$ is a von Neumann algebra and θ is completely bounded and normal then it is necessarily M-S regular ([LW1], Proposition 1.3).

Quantum stochastic processes. A $\mathcal{E}[\eta]$-process on the initial space $\mathfrak{h}$, with *noise dimension space* k and fixed basis η, is a time-indexed family of operators $(X_t)_{t\geq 0}$ on $\mathfrak{h} \otimes \mathcal{F}_{\mathsf{k}}$ such that

(i) $\operatorname{Dom} X_t = \mathfrak{h} \odot \mathcal{E}[\eta]$
(ii) $t \mapsto X_t u\varepsilon(f)$ is weakly measurable
(iii) $X_t u\varepsilon(f_{[0,t[}) \in \mathfrak{h} \otimes \mathcal{F}_{[0,t[,\mathsf{k}}$ and $X_t u\varepsilon(f) = [X_t u\varepsilon(f_{[0,t[})] \otimes \varepsilon(f_{[t,\infty[})$

where $f_I := f\mathbf{1}_I$ and $\mathbf{1}_I$ denotes the indicator function of the set I.

An $\mathcal{E}[\eta]$-process X is *measurable*/*continuous* if it satisfies

(ii)′ $t \mapsto X_t u\varepsilon(f)$ is strongly measurable/continuous, and *strongly regular* if for all f and $T > 0$

$$\sup\{\|X_t u\varepsilon(f)\| : \|u\| = 1, t \in [0,T]\} < \infty.$$

Processes X and Y are identified when for each $\xi \in \mathfrak{h} \otimes \mathcal{F}_{\mathsf{k}}$ and $\zeta \in \mathfrak{h} \odot \mathcal{E}[\eta]$,

$$\langle \xi, X_t \zeta \rangle = \langle \xi, Y_t \zeta \rangle \quad \text{for a.a. } t.$$

The collection of (equivalence classes of) such processes is denoted $\mathbb{P}(\mathfrak{h}, \mathcal{E}[\eta])$. Quantum stochastic integrals, written $\int_0^t X_s \, d\Lambda^\alpha_\beta(s)$, are defined for any process X such that $t \mapsto X_t \xi$ is locally square integrable for all $\xi \in \mathfrak{h} \odot \mathcal{E}[\eta]$, where $\Lambda = [\Lambda^\alpha_\beta]$ is the matrix of fundamental integrator processes defined with respect to the basis η of k ([HuP],[Par]). A process matrix $Z = [Z^\alpha_\beta] \in \mathrm{M}_D(\mathbb{P}(\mathfrak{h}, \mathcal{E}[\eta]))$ is *stochastically integrable* if

$$\int_0^t \sum_{\alpha \geq 0} \|Z^\alpha_\beta(s) u\varepsilon(f)\|^2 \, ds < \infty$$

for all t, β, u and f. It is for such matrices that the process $\int_0^t Z^\alpha_\beta(s) \, d\Lambda^\beta_\alpha(s)$ is defined as a strong limit (on $\mathfrak{h} \odot \mathcal{E}[\eta]$) of partial sums over α, β, and satisfies the *fundamental estimate*

$$\| \int_0^t Z^\alpha_\beta(s) \, d\Lambda^\beta_\alpha(s) u\varepsilon(f)\|^2 \leq 2e^{\nu_f(t)} \sum_{\beta=0}^{\dim f} \int_0^t \sum_{\alpha \geq 0} \|Z^\alpha_\beta(s) u\varepsilon(f)\|^2 \, d\nu_f(s) \tag{2.5}$$

where $\nu_f(t) = \int_0^t (1 + \|f(s)\|^2) \, ds$ (see [Par],[LW1]). It follows that the process $(\int_0^t Z^\alpha_\beta(s) \, d\Lambda^\beta_\alpha(s))_{t \geq 0}$ is continuous. Moreover it satisfies the identity

$$\langle u\varepsilon(f), \int_0^t Z^\alpha_\beta(s) \, d\Lambda^\beta_\alpha(s) v\varepsilon(g) \rangle = \int_0^t f_\alpha(s) g^\beta(s) \langle u\varepsilon(f), Z^\alpha_\beta(s) v\varepsilon(g) \rangle \, ds, \tag{2.6}$$

which serves to determine its action on $\mathfrak{h} \odot \mathcal{E}[\eta]$. If $Y = [Y^\alpha_\beta] \in \mathrm{M}_D(\mathbb{P}(\mathfrak{h}, \mathcal{E}[\eta]))$ is another stochastically integrable process matrix then we have

$$\begin{aligned} \langle I_t(Y) u\varepsilon(f), I_t(Z) v\varepsilon(g) \rangle = \int_0^t & f_\alpha(s) g^\beta(s) \big\{ \langle I_s(Y) u\varepsilon(f), Z^\alpha_\beta(s) v\varepsilon(g) \rangle \\ & + \langle Y^\beta_\alpha(s) u\varepsilon(f), I_s(Z) v\varepsilon(g) \rangle + \langle Y^i_\alpha(s) u\varepsilon(f), Z^i_\beta(s) v\varepsilon(g) \rangle \big\} \, ds, \end{aligned} \tag{2.7}$$

where $I_t(Y) = \int_0^t Y^\alpha_\beta(s) \, d\Lambda^\beta_\alpha(s)$, and similarly for Z. This is a rigorous formulation of the quantum Itô formula. Thus we see that the formal expression $dX_t = Z^\alpha_\beta(t) \, d\Lambda^\alpha_\beta(t)$ can have different levels of meaning for $X, Z^\alpha_\beta \in \mathbb{P}(\mathfrak{h}, \mathcal{E}[\eta])$. X has a *weak stochastic integral representation* in terms of the processes Z^α_β if

$$t \mapsto f_\alpha(t) g^\beta(t) \langle u\varepsilon(f), Z^\alpha_\beta(t) v\varepsilon(g) \rangle \quad \text{is locally integrable}$$

and

$$\langle u\varepsilon(f), (X_t - X_0) v\varepsilon(g) \rangle = \int_0^t f_\alpha(s) g^\beta(s) \langle u\varepsilon(f), Z^\alpha_\beta(s) v\varepsilon(g) \rangle \, ds \tag{2.8}$$

for all u, v, f, g and t. If also the process matrix $[Z^\alpha_\beta]$ is stochastically integrable, then we say that X has a *strong stochastic integral representation* in terms of Z, since the following equation holds:

$$X_t - X_0 = \int_0^t Z^\alpha_\beta(s) \, d\Lambda^\beta_\alpha(s).$$

It is in this sense that we distinguish between weak and strong solutions of quantum stochastic differential equations in what follows, noting that the identity (2.7) is only available to us when we have strong representations.

For a C^*-algebra $\mathcal{A}$ acting on the initial space $\mathfrak{h}$, $\mathbb{P}(\mathcal{A}, \mathcal{E}[\eta])$ denotes the space of linear maps $k : \mathcal{A} \to \mathbb{P}(\mathfrak{h}, \mathcal{E}[\eta])$ satisfying the affiliation condition

$$k_t(a)a'u\varepsilon(f) = (a' \otimes 1_{\mathsf{k}})k_t(a)u\varepsilon(f) \tag{2.9}$$

for all $a' \in \mathcal{A}'$. Such a process is called *bounded, completely bounded, contractive, completely positive* or, when $\mathcal{A}$ is a von Neumann algebra, *normal*, if each k_t has that property. Recall the meaning of complete positivity for unbounded maps (0.2). It is *weakly regular* if for all choices of $f, g \in \mathbb{M}[\eta]$ and $T > 0$

$$\sup\{|\langle u\varepsilon(f), k_t(a)v\varepsilon(g)\rangle| : \|u\| = \|a\| = \|v\| = 1, t \in [0, T]\} < \infty,$$

and *measurable* if each $k.(a) \in \mathbb{P}(\mathfrak{h}, \mathcal{E}[\eta])$ is measurable. We will identify bounded densely defined operators with their continuous extension, and consequently write $\mathbb{P}_{\mathrm{b}}(\mathfrak{h}, \mathsf{k})$, $\mathbb{P}_{\mathrm{b}}(\mathcal{A}, \mathsf{k})$ and $\mathbb{P}_{\mathrm{cb}}(\mathcal{A}, \mathsf{k})$ for the space of bounded processes on $\mathfrak{h}$, bounded processes on $\mathcal{A}$ and completely bounded processes on $\mathcal{A}$ respectively.

For a von Neumann algebra $\mathcal{M}$ acting on $\mathfrak{h}$, if $k \in \mathbb{P}_{\mathrm{cb}}(\mathcal{M}, \mathsf{k})$ is normal then it is a *Markovian cocycle* if it satisfies

$$k_{s+t} = \widehat{k}_s \circ \sigma_s \circ k_t$$

([Bra]) where $\sigma_s = \mathrm{id} \otimes \check{\sigma}_s$ and $\widehat{k}_s = k_s \otimes \mathrm{id}_{[s,\infty[}$, $\check{\sigma}_s : \mathcal{B}_{\mathsf{k}} \to \mathcal{B}_{[s,\infty[,\mathsf{k}}$ being the right shift on Fock space operators and $\mathrm{id}_{[s,\infty[}$ being the identity operator on $\mathcal{B}_{[s,\infty[,\mathsf{k}}$, the algebra of Fock space operators "beyond s". By means of a Feller-type condition Bradshaw's definition of Markovian cocycle can be extended to the much broader context of C^*-algebras where they may even be unbounded ([LW2]). The following result is proved there.

Theorem 2.3 ([LW2]). *Let $k \in \mathbb{P}_{\mathrm{b}}(\mathcal{A}, \mathsf{k})$ be a completely positive contraction process on a unital C^*-algebra $\mathcal{A}$ acting on $\mathfrak{h}$. Then the following are equivalent*:

(i) *k is a regular Markovian cocycle*
(ii) *k strongly satisfies a quantum stochastic differential equation of the form*

$$dk_t = k_t \circ \theta^\alpha_\beta \, d\Lambda^\beta_\alpha(t), \quad k_0(a) = a \otimes 1 \tag{2.10}$$

for some $\theta \in \mathrm{M}_D(\mathcal{B}(\mathcal{A}))$.

In this case the mapping matrix θ is completely bounded and has the form

$$\widehat{\theta}(a) = \psi(a) + J^*aE_0 + E^0aJ$$

where $\psi \in \mathcal{B}(\mathcal{A}; \mathcal{A}'' \otimes \mathcal{B}(\widehat{\mathsf{k}}))$ is completely positive, $J \in \mathcal{A}'' \otimes \mathcal{B}(\widehat{\mathsf{k}}; \mathbb{C})$ and $\psi(1) + J^*E_0 + E^0J \leq \Delta(1)$.

Regularity for a Markovian cocycle means that the associated Markov semigroup $(\mathbb{E}_0 \circ k_t)_{t\geq 0}$ is norm continuous.

3. Perturbations of processes

For this section let $\mathcal{A}$ be a unital C^*-algebra acting on the initial Hilbert space $\mathfrak{h}$, let $\mathcal{H} = \mathfrak{h} \otimes \mathcal{F}_{\mathsf{k}}$ where k is a separable noise dimension space with orthonormal basis η, and let $D = \dim \widehat{\mathsf{k}}$. We are interested in the following class of QSDE's:

$$dW_t = j_t(l^\alpha_\beta)W_t \, d\Lambda^\beta_\alpha(t), \quad W_0 = 1 \tag{3.1}$$

in which $l \in \mathrm{M}_D(\mathcal{A})$ and $j \in \mathbb{P}(\mathcal{A}, \mathcal{E}[\eta])$. Such equations were first considered in [EvH], for finite dimensional k, and extended to countably infinite dimensional k in [DaS]. In those works, j was assumed to be a unital *-homomorphic flow governed by a quantum stochastic differential equation of the form (2.10) and l was assumed to be an algebra matrix satisfying the conditions which are sufficient ([HuP],[MoS]) and necessary ([LW1]) for unitarity of the solution of the equation obtained from (3.1) when j is taken to be the identity map. In particular, by [LW1], Proposition 7.5, such l is necessarily bounded, even for infinite dimensional k.

A class of QSDE's. We first discuss quantum stochastic differential equations of the form

$$dX_t = F^\alpha_\beta(t) X_t \, d\Lambda^\beta_\alpha(t), \quad X_0 = 1 \tag{3.2}$$

in which the coefficients F^α_β are time-dependent and adapted, but bounded. In order to do this we extend the regularity condition introduced by Mohari and Sinha from operator matrices to process matrices: $F \in \mathrm{M}_D(\mathbb{P}_\mathrm{b}(\mathfrak{h}, \mathsf{k}))$ is a *regular process matrix* if it satisfies:

(i) each F^α_β is a measurable process;
(ii) $F(t) \in \mathrm{M}_D(\mathcal{B}(\mathcal{H}))_\mathrm{R}$ for all t;
(iii) $F_\beta : t \to F(t)_\beta$ has locally uniform bounds, for all β.

Thus F is a regular process matrix if and only if (i) holds and for each $\beta \geq 0$ and $T \geq 0$ there is a constant $c_{\beta,T}$ such that

$$\sup_{t \in [0,T]} \sum_{\alpha \geq 0} \|F^\alpha_\beta(t)\xi\|^2 \leq c_{\beta,T} \|\xi\|^2 \tag{3.3}$$

for all $\xi \in \mathcal{H}$. The collection of such process matrices is denoted $\mathrm{M}_D(\mathbb{P}_\mathrm{b}(\mathfrak{h}, \mathsf{k}))_\mathrm{R}$.

Proposition 3.1. *Let $F \in \mathrm{M}_D(\mathbb{P}_\mathrm{b}(\mathfrak{h}, \mathsf{k}))_\mathrm{R}$. Then the quantum stochastic differential equation* (3.2) *has a unique strong solution $X \in \mathbb{P}(\mathfrak{h}, \mathcal{E}[\eta])$, which furthermore is strongly regular.*

Proof. Let $c_{\beta,T}$ be regularity constants for F, according to (3.3), and consider the Picard iteration scheme:

$$X^{\{0\}} = 1; \quad X^{\{n+1\}}_t = 1 + \int_0^t F^\alpha_\beta(s) X^{\{n\}}_s \, d\Lambda^\beta_\alpha(s), \ n \geq 0.$$

Suppose that the process $X^{\{k\}} \in \mathbb{P}(\mathfrak{h}, \mathcal{E}[\eta])$ has been shown to exist, then $F^\alpha_\beta X^{\{k\}}$ inherits measurability from $X^{\{k\}}$ and (3.3) implies that $[F^\alpha_\beta X^{\{k\}}]$ is stochastically integrable. The fundamental estimate (2.5) together with (3.3) therefore gives

$$\begin{aligned}
\|[X^{\{k+1\}}_t - &X^{\{k\}}_t] u\varepsilon(f)\|^2 \\
&\leq 2\exp(\nu_f(t)) \sum_{\beta=0}^{\dim f} \int_0^t \sum_{\alpha \geq 0} \|F^\alpha_\beta(s)[X^{\{k\}}_s - X^{\{k-1\}}_s] u\varepsilon(f)\|^2 \, d\nu_f(s) \\
&\leq C(f,t) \int_0^t \|[X^{\{k\}}_s - X^{\{k-1\}}_s] u\varepsilon(f)\|^2 \, d\nu_f(s)
\end{aligned}$$

for $t \leq T$, where $C(f,t) = 2\exp \nu_f(t) \sum_{\beta=0}^{\dim f} c_{\beta,T}$. By induction we see that the iterative scheme is well-defined, and that the iterates satisfy

$$\|[X^{\{n+1\}}_t - X^{\{n\}}_t] u\varepsilon(f)\|^2 \leq ((n+1)!)^{-1} D(f,t)^{n+1} \|u\varepsilon(f)\|^2$$

where $D(f,t) = \nu_f(t) C(f,t)$. It follows that the sequence of processes $(X^{\{n\}})$ is strongly convergent on $\mathfrak{h} \odot \mathcal{E}[\eta]$, that the convergence is uniform on bounded time intervals, and that the process obtained in the limit is both a strong solution of (3.2) and strongly regular. If $X' \in \mathbb{P}(\mathfrak{h}, \mathcal{E}[\eta])$ is another strong solution, then for $u \in \mathfrak{h}$ and $f \in \mathbb{M}\,[\eta]$,

$$\|[X_t - X'_t] u\varepsilon(f)\|^2 \leq C(f,t) \int_0^t \|[X_s - X'_s] u\varepsilon(f)\|^2 \, d\nu_f(s),$$

by the same reasoning as above. Iterating this relation, and using the continuity of strong solutions, therefore gives equality of X and X'. $\square$

The method of proof is standard for quantum stochastic lore ([HuP],[Par],[DaS]). It is included here merely to highlight natural conditions for its validity in the context of time-dependent coefficient process matrices whose components are bounded operator-valued. In any case a direct proof of Proposition 3.2 below would be no simpler.

Perturbations of processes. We begin with existence and uniqueness for the class of equations of interest here.

Proposition 3.2. *Let $l \in \mathrm{M}_D(\mathcal{A})_{\mathrm{R}}$ and let $j \in \mathbb{P}_{\mathrm{cb}}(\mathcal{A}, \mathsf{k})$ be measurable, and have locally uniform complete bounds. Then the quantum stochastic differential equation* (3.1) *has a unique strongly regular strong solution $W \in \mathbb{P}(\mathfrak{h}, \mathcal{E}[\eta])$. Moreover W is affiliated to $\mathcal{A}'' \otimes \mathcal{B}_{\mathsf{k}}$ in the sense that $(a' \otimes 1)W_t = W_t(a' \otimes 1)$ for all $a' \in \mathcal{A}'$.*

Proof. As in the proof of Lemma 2.2(a), we have

$$\sum_{\alpha \geq 0} \|j_t(l^\alpha_\beta)\xi\|^2 \leq \|j_t\|^2_{\mathrm{cb}} \|l_\beta\|^2 \|\xi\|^2 \tag{(3.3)$'$}$$

for all ξ in $\mathcal{H}$, thus the process matrix $[j.(l^\alpha_\beta)]$ is regular. The existence, uniqueness and strong regularity of W therefore follow from Proposition 3.1. For $a' \in \mathcal{A}'$ put $Y_t = (a' \otimes 1)W_t - W_t(a' \otimes 1)$ and let $T > 0$. Then since j is assumed to satisfy the affiliation condition (2.9) we have

$$\begin{aligned} \|Y_t u\varepsilon(f)\|^2 &= \| \int_0^t j_s(l^\alpha_\beta) Y_s \, d\Lambda^\beta_\alpha(s) u\varepsilon(f)\|^2 \\ &\leq E \int_0^t \|Y_s u\varepsilon(f)\|^2 \, d\nu_f(s). \end{aligned}$$

for all $t \leq T$, by (2.5) and (3.3)$'$, where E is a constant depending on f, j, l and T. Iterating this identity, and using the continuity of the process Y, shows that $Y = 0$ as required. □

We now determine necessary and sufficient conditions on l for the process W obtained above to be a contraction, isometry or coisometry process. Relevant to these discussions are the following operator matrices constructed from l:

$$q(l) = l + l^\dagger + l^\dagger \Delta(1) l \quad \text{and} \quad r(l) = l + l^\dagger + l\Delta(1) l^\dagger.$$

By Lemma 2.1 $q(l)$ is defined for all $l \in \mathrm{M}_D(\mathcal{A})_{\mathrm{R}}$, and $r(l)$ is defined for all $l \in \mathrm{M}_D(\mathcal{A})$ such that $l^\dagger \in \mathrm{M}_D(\mathcal{A})_{\mathrm{R}}$; they are both elements of $\mathrm{M}_D(\mathcal{A}'')$. If $l \in \mathrm{M}_D(\mathcal{A})_{\mathrm{R}}$ and $q(l) \leq 0$ (as a form on $\mathfrak{h}_{00}$) then l is necessarily bounded ([LW1], Theorem 5.2 and Proposition 7.3). Thus in this case it follows that $r(l)$ is well-defined, and also that $r(l) \leq 0$ ([LW1], Theorem 7.5).

Proposition 3.3. *Let $l \in \mathrm{M}_D(\mathcal{A})_{\mathrm{R}}$, let $j \in \mathbb{P}_{\mathrm{b}}(\mathcal{A}, \mathsf{k})$ be a measurable CP contraction process, and let $W \in \mathbb{P}(\mathfrak{h}, \mathcal{E}[\eta])$ be the strong solution of* (3.1).

(i) *If $q(l) \leq 0$ then W is a contraction process.*
(ii) *If j is pointwise weakly continuous at 0, with $j_0(a) = a \otimes 1$, and if W is a contraction process, then $q(l) \leq 0$.*

Proof. Let $\xi = \sum_p u_p \varepsilon(f_p) \in \mathfrak{h} \odot \mathcal{E}[\eta]$. Since W is a strong solution of (3.1) the quantum Itô formula (2.7) along with the Dominated Convergence Theorem may be applied, giving

$$\|W_t\xi\|^2 - \|\xi\|^2 = \lim_{N \to \infty} \int_0^t \langle x^\alpha(s), [j_s(l^\alpha_\beta + l^{\beta *}_\alpha) + \sum_{i \leq N} j_s(l^i_\alpha)^* j_s(l^i_\beta)] x^\beta(s)\rangle \, ds \tag{3.4}$$

where $x^\alpha(s) = \sum_p f_p^\alpha(s) W_s u_p \varepsilon(f_p) \in \mathcal{H}$. Let $L_i \in \mathrm{M}_D(\mathcal{A})_\mathrm{R}$ be the operator matrix whose only nonzero entries lie in the first row, which is $[l_0^i \ l_1^i \ \ldots]$, the ith row of l. Then, for $(\xi^\gamma) \in \mathcal{H}_{00} \subset \mathcal{H} \otimes \widehat{\mathsf{k}}$ and $N \in \mathbb{N}$,

$$\begin{aligned}\sum_{i \le N} \langle \xi^\alpha, j_s(l_\alpha^i)^* j_s(l_\beta^i) \xi^\beta \rangle &= \sum_{i \le N} \langle (\xi^\gamma), j_s^{(D)}(L_i)^\dagger j_s^{(D)}(L_i)(\xi^\gamma) \rangle \\ &\le \sum_{i \le N} \langle (\xi^\gamma), j_s^{(D)}(L_i^\dagger L_i)(\xi^\gamma) \rangle \\ &= \langle (\xi^\gamma), j_s^{(D)}(l^\dagger \Delta^{[N]}(1) l)(\xi^\gamma) \rangle,\end{aligned} \tag{3.5}$$

by Lemma 2.2(d), since $L_i^\dagger L_i \in \mathrm{M}_D(\mathcal{A})$ (no summation). Feeding this inequality back into (3.4) gives

$$\|W_t \xi\|^2 - \|\xi\|^2 \le \limsup_{N \to \infty} \int_0^t \langle x(s), j_s^{(D)}(l + l^\dagger + l^\dagger \Delta^{[N]}(1) l) x(s) \rangle \, ds.$$

So now if $q(l) \le 0$ then the *algebra* matrix inequality $l + l^\dagger + l^\dagger \Delta^{[N]}(1) l \le 0$ holds for all N. Thus, by Lemma 2.2(c), W is a contraction process.

Suppose conversely that W is a contraction process and that j is pointwise weakly continuous at 0 with $j_0(a) = a \otimes 1$. Then

$$0 \ge \|W_t \xi\|^2 - \|\xi\|^2 \ge \int_0^t \langle x^\alpha(s), [j_s(l_\beta^\alpha + l_\alpha^{\beta *}) + \sum_{i \le N} j_s(l_\alpha^i)^* j_s(l_\beta^i)] x^\beta(s) \rangle \, ds$$

for each N. The operator Schwarz inequality implies that $t \mapsto j_t(a)$ is strongly continuous at 0, so if we choose the f_p to be continuous then the integrand above is continuous at 0. Thus we can differentiate and then vary the u_p and f_p to get the form inequality $l + l^\dagger + l^\dagger \Delta^{[N]}(1) l \le 0$. The result now follows by letting $N \to \infty$. □

The pointwise weak continuity condition is satisfied by weak solutions of quantum stochastic differential equations of the form (2.10). Our next result specialises the above to the case where W is an isometry or coisometry process. Note that by the proposition above and the comment preceding it we know that the algebra matrix l must be bounded if W is to be an isometry or coisometry process.

Proposition 3.4. *Let l, j and W be as in* Proposition 3.3*, with l bounded.*

(a) *Suppose that $q(l) \in \mathrm{M}_D(\mathcal{A})$.*
   (i) *If j is pointwise weakly continuous at* 0 *with $j_0(a) = a \otimes 1$, and if W is an isometry process, then $q(l) = 0$.*
   (ii) *If j is *-homomorphic,*

$$j_t(l_\alpha^i)^* j_t(l_\beta^i) = j_t(l_\alpha^{i*} l_\beta^i) \quad \forall \alpha, \beta, \tag{3.6}$$

   and $q(l) = 0$, then W is an isometry process.

(b) *Suppose that $r(l) \in \mathrm{M}_D(\mathcal{A})$, and that j is pointwise weakly continuous at* 0 *with $j_0(a) = a \otimes 1$. If W is a coisometry process then $r(l) = 0$.*

Proof. (a) Using Lemma 2.2, the equation (3.4) becomes

$$\|W_t \xi\|^2 - \|\xi\|^2 = \int_0^t \langle x(s), \big[j_s^{(D)}(l + l^*) + j_s^{(D)}(l)^* \Delta(1) j_s^{(D)}(l) \big] x(s) \rangle \, ds, \tag{(3.4)$'$}$$

with $\xi \in \mathfrak{h} \odot \mathcal{E}[\eta]$ and x defined as in the proof of Proposition 3.3.

(i) By Lemma 2.2(b) it follows that $t \mapsto j_t^{(D)}(l)$ is weakly continuous at 0, and then part (d) of the same result implies that it is in fact strongly continuous at 0. Thus if W is an isometry process then we can differentiate (3.4)$'$ at 0 to show that $q(l) = 0$.

(ii) By (3.6) we have $j_s^{(D)}(l)^*\Delta(1)j_s^{(D)}(l) = j_s^{(D)}(l^*\Delta(1)l)$, and so if $q(l) = 0$ then the integrand in $(3.4)'$ is zero, hence W is an isometry process.

(b) Since W is coisometric and continuous,

$$\|(W_t^* - W_s^*)\xi\|^2 = 2\|\xi\|^2 - 2\,\mathrm{Re}\langle \xi, W_t W_s^* \xi\rangle \to 0$$

as $t \to s$. That is, the adjoint process W^* is continuous, and it follows that

$$W_t^* = 1 + \int_0^t W_s^* j_s(l_\alpha^{\beta *})\, d\Lambda_\alpha^\beta(s).$$

Applying (2.7) then gives

$$\|W_t^*\xi\|^2 - \|\xi\|^2 = \int_0^t \langle y(s), \big[j_s^{(D)}(l)\iota(W_s W_s^*) + \iota(W_s W_s^*) j_s^{(D)}(l^*) \\ + j_s^{(D)}(l)\Delta(W_s W_s^*) j_s^{(D)}(l^*)\big] y(s)\rangle\, ds$$

with ξ as before and $y^\alpha(s) = \sum_p f_p^\alpha(s) u_p \varepsilon(f_p)$. The arguments of (a(i)) show that the integrand is continuous at 0 when the f_p are continuous, and so once more we can differentiate to get the required result. □

Remark. The condition $q(l) \in \mathrm{M}_D(\mathcal{A})$, equivalent to $(l_\alpha^i)^* l_\beta^i \in \mathcal{A}$ for all α and β, is automatically satisfied if k is finite dimensional or $\mathcal{A}$ is a von Neumann algebra.

The identity in (3.6) is another instance of a condition that is automatically true when k is finite dimensional, but which must be inserted by hand when k is infinite dimensional. We will need to carry out similar manipulations below, and for this reason the following definition introduced in [LW1] is relevant. A process $k \in \mathbb{P}(\mathcal{A}, \mathcal{E}[\eta])$ is *sequentially strongly continuous* if it satisfies

$$a_n \to 0 \text{ strongly in } \mathcal{A} \Rightarrow k_t(a_n)\xi \to 0 \quad \forall t, \xi.$$

Processes enjoying this property include:

(i) Weakly regular processes satisfying an equation of the form $dk_t = k_t \circ \theta_\beta^\alpha d\Lambda_\alpha^\beta(t)$ in which the mapping matrix θ is M-S regular ([LW1], Proposition 3.4).

(ii) Normal completely bounded processes on a von Neumann algebra. To see this, let (a_n) be a sequence strongly converging to 0. If k_t is normal and completely bounded then it has a Wittstock-Paulsen decomposition of the form $k_t(a) = V^*(a \otimes 1_{\mathcal{L}})W$, for a Hilbert space $\mathcal{L}$ and bounded operators V and W ([Haa]). By the Uniform Boundedness Principle the weakly convergent sequence $(a_n^* a_n)$ is bounded. It follows that also $a_n^* a_n \otimes 1_{\mathcal{L}} \to 0$ weakly, thus

$$\|k_t(a_n)\xi\|^2 \le \|V\|^2 \langle W\xi, (a_n^* a_n \otimes 1_{\mathcal{L}})W\xi\rangle \to 0.$$

Lemma 3.5. *Let $l \in \mathrm{M}_D(\mathcal{A})_{\mathrm{R}}$, let $j \in \mathbb{P}_{\mathrm{b}}(\mathcal{A}, \mathsf{k})$ be a *-homomorphic regular Markovian cocycle with generator θ, and let $W \in \mathbb{P}(\mathfrak{h}, \mathcal{E}[\eta])$ be the strong solution of* (3.1) *for this l and j. Under the assumptions*

$$\theta_i^\alpha(a) l_\beta^i \in \mathcal{A}, \quad j_t(\theta_i^\alpha(a) l_\beta^i) = j_t(\theta_i^\alpha(a)) j_t(l_\beta^i) \quad \forall \alpha, \beta, a, \tag{3.7}$$

the process $\bar{\jmath} \in \mathbb{P}(\mathcal{A}, \mathcal{E}[\eta])$ defined by $\bar{\jmath}_t(a) = j_t(a)W_t$ strongly satisfies the QSDE

$$d\bar{\jmath}_t = \bar{\jmath}_t \circ \chi_\beta^\alpha\, d\Lambda_\alpha^\beta(t), \quad \bar{\jmath}_0(a) = a \otimes 1 \tag{3.8}$$

where $\chi \in \mathrm{M}_D(\mathcal{B}(\mathcal{A}))$ is given by

$$\chi(a) = \theta(a) + \iota(a) l + \theta(a)\Delta(1) l \tag{3.9}$$

Proof. Since θ is bounded each $\chi(a)$ is a regular algebra matrix by Lemma 2.1(c), and so it follows from Lemma 2.2(a) and strong regularity of W that each process matrix $[\bar{\jmath}.(\chi^\alpha_\beta(a))]$ is stochastically integrable. It therefore suffices to show that $\bar{\jmath}$ weakly satisfies (3.8). By (2.7) we have

$$\langle j_t(a^*)u\varepsilon(f), W_t v\varepsilon(g)\rangle = \langle u\varepsilon(f), av\varepsilon(g)\rangle + \int_0^t f_\alpha(s) g^\beta(s) h^\alpha_\beta(s)\, ds$$

where

$$\begin{aligned} h^\alpha_\beta(s) = &\langle j_s(a^*)u\varepsilon(f), j_s(l^\alpha_\beta)W_s v\varepsilon(g)\rangle + \langle j_s(\theta^\beta_\alpha(a^*))u\varepsilon(f), W_s v\varepsilon(g)\rangle \\ &+ \langle j_s(\theta^i_\alpha(a^*))u\varepsilon(f), j_s(l^i_\beta)W_s v\varepsilon(g)\rangle. \end{aligned}$$

Rearranging this expression using the reality of θ (which is a consequence of the reality of j), the *-homomorphic property of j, and sequential continuity assumptions (3.7), we obtain

$$h^\alpha_\beta(s) = \langle u\varepsilon(f), \bar{\jmath}_s(\chi^\alpha_\beta(a))v\varepsilon(g)\rangle,$$

showing that $\bar{\jmath}$ satisfies (3.8) weakly. □

Using the above lemma we can now give some sufficient conditions for W to be a coisometry process.

Proposition 3.6. *Let $l \in \mathrm{M}_D(\mathcal{A})_\mathrm{b}$, let $j \in \mathbb{P}_\mathrm{b}(\mathcal{A}, \mathsf{k})$ be a *-homomorphic regular Markovian cocycle, and let $W \in \mathbb{P}(\mathfrak{h}, \mathcal{E}[\eta])$ be the strong solution of* (3.1) *for this l and j. If $r(l) = 0$, if j is unital, and if either*

(i) *k is finite dimensional, or*
(ii) *$\mathcal{A}$ is a von Neumann algebra and j is sequentially strongly continuous,*

then W is a coisometry process.

Proof. First note that W is a contraction process, by Proposition 3.3. Also θ, the generator of j, is completely bounded, and l is bounded by hypothesis. So putting $\bar{\jmath}_t(a) = j_t(a)W_t$ as in Lemma 3.5, $\bar{\jmath}$ is a strong solution to $d\bar{\jmath}_t = \bar{\jmath}_t \circ \chi^\alpha_\beta\, d\Lambda^\beta_\alpha(t)$ where χ, defined by (3.9), is a completely bounded mapping matrix. Thus the adjoint matrix $\chi^\dagger$ is completely bounded, and so by (2.6) the adjoint process $\bar{\jmath}^\dagger_t(a) := \bar{\jmath}_t(a^*)^*$ strongly satisfies the equation $d\bar{\jmath}^\dagger_t = \bar{\jmath}^\dagger_t \circ {\chi^\dagger}^\alpha_\beta\, d\Lambda^\beta_\alpha(t)$ ([LW3]). In particular $t \mapsto \bar{\jmath}^\dagger_t(a)$ is strongly continuous, so taking $a = 1$ implies that $t \mapsto W^*_t$ is strongly continuous, hence the contraction process $(F_t := W_t W^*_t - 1)_{t \geq 0}$ is (continuous and so) measurable.

Put $\lambda = \chi^\dagger$, then (2.7), Lemma 2.2 and either of the conditions (i) or (ii), combine to show that

$$F_t j_t(a) = \int_0^t \left[j_s(l^\alpha_\beta)F_s j_s(a) + F_s j_s(\lambda^\alpha_\beta(a)) + j_s(l^\alpha_i)F_s j_s(\lambda^i_\beta(a))\right] d\Lambda^\beta_\alpha(s). \tag{3.10}$$

Since

$$\begin{aligned} \sum_{\alpha \geq 0} \|j_s(l^\alpha_i)F_s j_s(\lambda^i_\beta(a))u\varepsilon(f)\|^2 &= \|j^{(D)}_s(l)\Delta(F_s)j^{(D)}_s(\lambda(a))_\beta u\varepsilon(f)\|^2 \\ &\leq \|l\|^2 \sum_{\alpha \geq 0} \|F_s j_s(\lambda^\alpha_\beta(a))u\varepsilon(f)\|^2, \end{aligned}$$

we can apply the fundamental estimate (2.5) to (3.10) and iterate to get

$$\|F_t j_t(a)u\varepsilon(f)\|^2 \leq G_0^n \int_0^t d\nu_f(t_1)\cdots\int_0^{t_{n-1}} d\nu_f(t_n)\Big\{\|F_{t_n} j_{t_n}(a)u\varepsilon(f)\|^2$$
$$+\binom{n}{1}\sum_{\substack{\alpha\geq 0\\ \beta\leq d}}\|F_{t_n} j_{t_n}(\lambda^\alpha_\beta(a))u\varepsilon(f)\|^2+\binom{n}{2}\sum_{\substack{\alpha_i\geq 0\\ \beta_i\leq d}}\|F_{t_n} j_{t_n}(\lambda^{\alpha_2}_{\beta_2}\circ\lambda^{\alpha_1}_{\beta_1}(a))u\varepsilon(f)\|^2$$
$$+\cdots+\sum_{\substack{\alpha_i\geq 0\\ \beta_i\leq d}}\|F_{t_n} j_{t_n}(\lambda^{\alpha_n}_{\beta_n}\circ\cdots\circ\lambda^{\alpha_1}_{\beta_1}(a))u\varepsilon(f)\|^2\Big\}.$$

for some constant G_0 depending on f, l and T, and where $d = \dim f$. It is a straightforward matter to find constants G_1 and G_2, depending only on f, l, θ and T, such that

$$\sup_{s\in[0,T]}\sum_{\substack{\alpha_i\geq 0\\ \beta_i\leq d}}\|F_s j_s(\lambda^{\alpha_n}_{\beta_n}\circ\cdots\circ\lambda^{\alpha_1}_{\beta_1}(a))u\varepsilon(f)\|^2 \leq G_1 G_2^n\|a\|^2\|u\|^2, \tag{3.11}$$

when k is finite dimensional. It is also possible to do this when k is infinite dimensional. If j is normal then θ is normal, hence λ is completely bounded and normal and thus M-S regular, in which case the original estimates in [DaS] can be employed to obtain (3.11). In general the techniques of [LW3] are necessary. This inequality then implies that $F_t j_t(a) = 0$, from which the result follows by taking $a = 1$. □

Now let j be a *-homomorphic regular Markovian cocycle with generator θ, let $l \in \mathrm{M}_D(\mathcal{A})_\mathrm{R}$, and let W be the solution of (3.1) for this j and l. If $\operatorname{Dom} W_t^* \supset \operatorname{Ran}(j_t(a)W_t)$ for all a and t then we can define $k_t(a) = W_t^* j_t(a)W_t$ to generate a CP process on $\mathcal{A}$. The next result tells us that in fact this k is a regular Markovian cocycle, and shows how its generator is related to θ and l. The next section is devoted to proving the converse of this result.

Theorem 3.7. *Let $l \in \mathrm{M}_D(\mathcal{A})_\mathrm{R}$, let $j \in \mathbb{P}_\mathrm{b}(\mathcal{A}, \mathsf{k})$ be a *-homomorphic regular Markovian cocycle with generator θ, and let $W \in \mathbb{P}(\mathfrak{h}, \mathcal{E}[\eta])$ be the strong solution of* (3.1) *for this l and j. Assuming that either k is finite dimensional, or that $\mathcal{A}$ is a von Neumann algebra and j is sequentially strongly continuous,*

$$\begin{aligned}\psi(a) = \theta(a) + \iota(a)l + l^\dagger\iota(a) + l^\dagger\Delta(a)l\\ + l^\dagger\Delta(1)\theta(a) + \theta(a)\Delta(1)l + l^\dagger\Delta(1)\theta(a)\Delta(1)l.\end{aligned} \tag{3.12}$$

defines a mapping matrix $\psi \in \mathrm{M}_D(\mathcal{B}(\mathcal{A}))$, and the following are equivalent:

(i) *The equation*
$$dk_t = k_t\circ\psi^\alpha_\beta\, d\Lambda^\beta_\alpha(t), \quad k_0(a) = a\otimes 1 \tag{3.13}$$
has a weakly regular weak solution $k \in \mathbb{P}(\mathcal{A}, \mathcal{E}[\eta])$;

(ii) $\operatorname{Ran}(j_t(a)W_t) \subset \operatorname{Dom} W_t^* \quad \forall a, t.$

If these conditions hold then $k_t(\cdot) = W_t^* j_t(\cdot)W_t$.

Proof. Since l is regular, θ is bounded, and either $\mathcal{A}$ is a von Neumann algebra or k is finite dimensional, (3.12) defines a mapping matrix ψ by Lemma 2.1. By Lemma 3.5 we may apply (2.7) to $\langle W_t u\varepsilon(f), j_t(a)W_t v\varepsilon(g)\rangle$. Since j is *-homomorphic and because we have made sufficient continuity assumptions on $(\mathsf{k}, \mathcal{A}, j)$, we can rearrange the resulting expression to give the following:

$$\langle u\varepsilon(f), av\varepsilon(g)\rangle + \int_0^t f_\alpha(s)g^\beta(s)\langle W_s u\varepsilon(f), j_s(\psi^\alpha_\beta(a))W_s v\varepsilon(g)\rangle. \tag{3.14}$$

Thus if (ii) holds then $k_t(a) = W_t^* j_t(a)W_t$ clearly defines a weak solution of (3.13) in $\mathbb{P}(\mathcal{A}, \mathcal{E}[\eta])$. Moreover, since W is strongly regular, k is weakly regular. Conversely, suppose that (i) holds. Fix $f, g \in \mathbb{M}[\eta]$. By the weak regularity of k and the strong regularity of W,

$$\langle u, \Gamma_t(a)v\rangle = \langle u\varepsilon(f), k_t(a)v\varepsilon(g)\rangle - \langle W_t u\varepsilon(f), j_t(a)W_t v\varepsilon(g)\rangle$$

defines bounded operators $\Gamma_t : \mathcal{A} \to \mathcal{B}(\mathfrak{h})$ with locally uniform bounds. By (3.14)

$$\langle u, \Gamma_t(a)v\rangle = \int_0^t f_\alpha(s)g^\beta(s)\langle u, \Gamma_s(\psi^\alpha_\beta(a))v\rangle\, ds;$$

iterating this shows that Γ is identically zero. It follows that (ii) holds. □

Remark. If W is a bounded operator process, then (ii) trivially holds, so that $W_t^* j_t(\cdot)W_t$ satisfies (3.13) weakly. On the other hand, if l is bounded then ψ is completely bounded and so a strongly regular mapping matrix ([LW3]), hence there *is* a strongly regular strong solution to (3.13) by [LW1], Theorem 3.3, and so (ii) holds. The above result extends Theorem 7.4 of [LW1], where j is the identity process.

4. Continuous time stochastic Stinespring Theorem

In this section we present the main result of the paper, showing that all sufficiently well-behaved completely positive Markovian cocycles can be obtained by conjugating a *-homomorphic Markovian cocycle by a process on the initial Hilbert space which is affiliated to the algebra, and is governed by a quantum stochastic differential equation. To do this we use the characterisation of CP Markovian cocycle generators obtained in [LiP] and [LW1], as refined in [GLSW].

For this section $\mathcal{M}$ is a unital von Neumann algebra acting on the initial space $\mathfrak{h}$, and k_0 is a separable noise dimension space with orthonormal basis η_0. We shall use the notations

$$\delta_{R,\pi} : \mathcal{M} \to \mathcal{B}(\mathfrak{h}; \mathcal{K}), \quad a \mapsto \pi(a)R - Ra$$
$$\mathcal{L}_{R,\pi,H} : \mathcal{M} \to \mathcal{B}(\mathfrak{h}), \quad a \mapsto R^*\pi(a)R - \tfrac{1}{2}\{R^*R, a\} + i[a, H],$$

for a representation $(\pi, \mathcal{K})$ of $\mathcal{M}$, operators $R \in \mathcal{B}(\mathfrak{h}; \mathcal{K})$ and $H = H^* \in \mathcal{B}(\mathfrak{h})$, where $\{\cdot,\cdot\}$ and $[\cdot,\cdot]$ denote anticommutator and commutator respectively.

Thus $\delta_{R,\pi}$ is a π-derivation:

$$\delta_{R,\pi}(ab) = \delta_{R,\pi}(a)b + \pi(a)\delta_{R,\pi}(b).$$

The characterisation of CP flow generators that we need here is the following

Theorem 4.1. *Let $k \in \mathbb{P}(\mathcal{M}, \mathcal{E}[\eta_0])$ be a weakly regular process weakly satisfying the equation $dk_t = k_t \circ \phi^\alpha_\beta\, d\Lambda^\beta_\alpha(t), k_0(a) = a \otimes 1$, where $\phi \in \mathrm{M}_D(\mathcal{B}(\mathcal{M}))$ and each ϕ^α_β is ultraweakly continuous. Then the following are equivalent:*

(i) *k is completely positive*

(ii) *There is a quintuple $\mathcal{Q} = (\mathsf{k}_1, \pi, h, d, \{w_j\})$ consisting of a Hilbert space k_1, a normal *-homomorphism $\pi : \mathcal{M} \to \mathcal{M} \otimes \mathcal{B}(\mathsf{k}_1)$ and operators $h = h^* \in \mathcal{M}, d \in \mathcal{M} \otimes \mathsf{k}_1$ and $w_j \in \mathcal{M} \otimes \mathsf{k}_1 (j \geq 1)$ such that*

$$\phi^0_0(a) = \mathcal{L}_{d,\pi,h}(a) + \tfrac{1}{2}\{t, a\}$$
$$\phi^i_0(a) = \phi^0_i(a^*)^* = w_i^*\delta_{d,\pi}(a) + c^i a$$
$$\phi^i_j(a) = w_i^*\pi(a)w_j - \delta^i_j a$$
$$\pi(1)d = d \quad \text{and} \quad \pi(1)w_j = w_j$$

where $t = \phi^0_0(1)$ and $c^i = \phi^i_0(1)$.

If $\mathfrak{h}$ is separable then $\mathcal{Q}$ may be chosen so that k_1 is separable too.

Remark. If ϕ is bounded then it may be written in block matrix form

$$\widehat{\phi}(a) = \begin{bmatrix} \mathcal{L}_{d,\pi,h}(a) + \frac{1}{2}\{t, a\} & \delta^{\dagger}_{d,\pi}(a)w + ac^* \\ w^*\delta_{d,\pi}(a) + ca & w^*\pi(a)w \end{bmatrix}$$

where $w \in \mathcal{M} \otimes \mathcal{B}(\mathsf{k}_0; \mathsf{k}_1)$ has columns $w_1, w_2, \ldots$ with respect to the basis η_0, and similarly $c \in \mathcal{M} \otimes \mathsf{k}_0$. We shall sometimes use this notation even when ϕ is not bounded, using Lemma 2.1 to justify manipulating these expressions as if they were indeed (bounded) block matrices.

The processes appearing in the Stinespring decomposition below have noise dimension space $\mathsf{k} = \mathsf{k}_0 \oplus \mathsf{k}_1$, where the supplementary Hilbert space k_1 is that arising in the previous theorem.

Lemma 4.2. *Let $\mathsf{k} = \mathsf{k}_0 \oplus \mathsf{k}_1$ for a separable Hilbert space k_1, and let η be an orthonomal basis for k including η_0. If $k \in \mathbb{P}(\mathcal{M}, \mathcal{E}[\eta_0])$ is a weakly regular weak solution of the equation $dk_t = k_t \circ \phi^{\alpha}_{\beta}\, d\Lambda^{\beta}_{\alpha}(t)$, and $P^{(1)}_t$ is the orthogonal projection $\mathcal{F}_{\mathsf{k}_1} \to \mathcal{F}_{[t,\infty[,\mathsf{k}_1}$, then $\widetilde{k}_t(a) = k_t(a) \odot P^{(1)}_t$ defines a weakly regular process $\widetilde{k} \in \mathbb{P}(\mathcal{M}, \mathcal{E}[\eta])$ weakly satisfying the equation $d\widetilde{k}_t = \widetilde{k}_t \circ \widetilde{\phi}^{\mu}_{\nu}\, d\Lambda^{\nu}_{\mu}(t)$ where*

$$\widetilde{\phi} = \begin{bmatrix} \phi & 0 \\ 0 & -\iota \end{bmatrix}. \tag{4.1}$$

Proof. Let $f = (f_0, f_1), g = (g_0, g_1) \in \mathbb{M}[\eta]$. Then, for a.a. t,

$$\begin{aligned}
&\frac{d}{dt}\langle u\varepsilon(f), \widetilde{k}_t(a)v\varepsilon(g)\rangle \\
&\quad= \frac{d}{dt}\left\{\langle u\varepsilon(f_0), k_t(a)v\varepsilon(g_0)\rangle \exp\int_t^{\infty}\langle f_1(s), g_1(s)\rangle\, ds\right\} \\
&\quad= \left\{\overline{f^{\alpha}_0(t)}g^{\beta}_0(t)\langle u\varepsilon(f_0), k_t(\phi^{\alpha}_{\beta}(a))v\varepsilon(g_0)\rangle - \langle u\varepsilon(f_0), k_t(a)v\varepsilon(g_0)\rangle\langle f_1(t), g_1(t)\rangle\right\} \\
&\qquad\times \exp\int_t^{\infty}\langle f_1(s), g_1(s)\rangle\, ds \\
&\quad= \overline{f^{\alpha}_0(t)}g^{\beta}_0(t)\langle u\varepsilon(f), \widetilde{k}_t(\phi^{\alpha}_{\beta}(a))v\varepsilon(g)\rangle - \overline{f^{i}_1(t)}g^{j}_1(t)\langle u\varepsilon(f), \widetilde{k}_t(\iota^{i}_{j}(a))v\varepsilon(g)\rangle \\
&\quad= \overline{f^{\mu}(t)}g^{\nu}(t)\langle u\varepsilon(f), \widetilde{k}_t(\widetilde{\phi}^{\mu}_{\nu}(a))v\varepsilon(g)\rangle,
\end{aligned}$$

and so the result follows by integration. □

Theorem 4.3. *Let $\phi \in \mathrm{M}_D(\mathcal{B}(\mathcal{M}))$. Suppose that $k \in \mathbb{P}(\mathcal{M}, \mathcal{E}[\eta_0])$ is a completely positive weakly regular process weakly satisfying the equation $dk_t = k_t \circ \phi^{\alpha}_{\beta} d\Lambda^{\beta}_{\alpha}(t)$, each ϕ^{α}_{β} is ultraweakly continuous, and the initial space $\mathfrak{h}$ is separable. Then there is a separable Hilbert space k_1, a normal *-homomorphic process $j \in \mathbb{P}(\mathcal{M}, \mathcal{E}[\eta])$ and a process $W \in \mathbb{P}(\mathfrak{h}, \mathcal{E}[\eta])$ such that*

$$k_t(a) \odot P^{(1)}_t = W^*_t j_t(a) W_t$$

where $P^{(1)}_t$ is the orthogonal projection $\mathcal{F}_{\mathsf{k}_1} \to \mathcal{F}_{[t,\infty[,\mathsf{k}_1}$ and η is an orthonormal basis for $\mathsf{k} = \mathsf{k}_0 \oplus \mathsf{k}_1$ containing η_0. Moreover (k_1, j, W) may be chosen so that W is affiliated to the von Neumann algebra $\mathcal{M} \otimes \mathcal{B}_{\mathsf{k}}$:

$$W_t a' u\varepsilon(f) = (a' \otimes 1)W_t u\varepsilon(f) \quad \forall a' \in \mathcal{M}'.$$

Proof. Let $(\mathsf{k}_1, \pi, \mathfrak{h}, d, \{w_j\})$ be a quintuple as delivered by Theorem 4.1, with k_1 separable, and define a normal bounded mapping matrix $\theta \in \mathcal{B}(\mathcal{M}; \mathcal{M} \otimes \mathcal{B}(\widehat{\mathsf{k}}))$ where $\mathsf{k} = \mathsf{k}_0 \oplus \mathsf{k}_1$ through

$$\widehat{\theta} = \begin{bmatrix} \mathcal{L}_{d,\pi,h} & 0 & \delta^{\dagger}_{d,\pi} \\ 0 & 0 & 0 \\ \delta_{d,\pi} & 0 & \pi \end{bmatrix}.$$

By [LW1] Theorem 6.5, θ strongly generates a normal *-homomorphic flow j which is thus sequentially strongly continuous. Now define an algebra matrix l through

$$\widehat{l} = \begin{bmatrix} \frac{1}{2}t & c^* & 0 \\ 0 & 0 & b \\ 0 & w & 0 \end{bmatrix} := \begin{bmatrix} \frac{1}{2}t & [c^{1*}\, c^{2*} \, \ldots] & 0 \\ 0 & 0 & [b_1 \, b_2 \, \ldots] \\ 0 & [w_1 \, w_2 \, \ldots] & 0 \end{bmatrix}$$

where $t = \phi_0^0(1), c^i = \phi_0^i(1)$ and (b_k) is *any* sequence in $\mathcal{M} \otimes \mathsf{k}_0$ indexed by the orthonormal basis η_1 of k_1. Since each column of $\widehat{l}$ defines a bounded operator $\mathfrak{h} \to \mathfrak{h} \otimes \widehat{\mathsf{k}}$, l is regular and Proposition 3.2 guarantees the existence of a unique strong solution of the equation $dW_t = j_t(l^\mu_\nu) W_t \, d\Lambda^\nu_\mu(t)$. Note that, putting $\Delta_0 = \iota - \Delta$ and $\varphi = \widetilde{\phi}$, where $\widetilde{\phi}$ is the generator (4.1), we have

$$\widehat{\varphi}(a) = \begin{bmatrix} \widehat{\phi}(a) & 0 \\ 0 & 0 \end{bmatrix} = \Delta_0(a)\widehat{l} + \widehat{l}^{\,\dagger}\Delta_0(a) + \Big(\Delta_0(1) + \widehat{l}^{\,\dagger}\Delta(1)\Big)\, \widehat{\theta}(a)\, \Big(\Delta(1)\widehat{l} + \Delta_0(1)\Big).$$

That is, $\widehat{\varphi}(a) = \widehat{\psi}(a)$, where ψ is the mapping matrix constructed from θ and l through (3.12). Therefore, by Lemma 4.2 and Theorem 3.7, $\operatorname{Ran}(j_t(a)W_t) \subset \operatorname{Dom} W_t^*$ for all a and t, and $\widetilde{k}_t(a) = W_t^* j_t(a) W_t$ where $\widetilde{k}_t(a) = k_t(a) \odot P_t^{(1)}$. $\square$

Note that, when it is bounded, the operator l in the above proof satisfies

$$l + l^\dagger + l^\dagger \Delta(1) l = \begin{bmatrix} t & c^* & 0 \\ c & w^*w - 1 & 0 \\ 0 & 0 & b^*b - 1 \end{bmatrix}.$$

If the original process k is a contraction process then, by [LW1],[LiP], ϕ is bounded and $\phi(1) \leq 0$, in other words

$$\begin{bmatrix} t & c^* \\ c & w^*w - 1 \end{bmatrix} \leq 0.$$

Therefore if we choose b to be a contraction in $\mathcal{M} \otimes \mathcal{B}(\mathsf{k}_1; \mathsf{k}_0)$, then $l^\dagger + l + l^\dagger \Delta(1) l \leq 0$ and so W is a contraction process by Proposition 3.3. If the original process k is unital, then $\phi(1) = 0$ so $t = 0, c = 0$ and w is isometric. Thus if there is an isometry $b \in \mathcal{M} \otimes \mathcal{B}(\mathsf{k}_1; \mathsf{k}_0)$ then we can ensure that W is an isometric process by Proposition 3.4. This is certainly the case if $\dim \mathsf{k}_0 \geq \dim \mathsf{k}_1$. We have therefore established the following

Theorem 4.4. *Let $k \in \mathbb{P}_{\mathrm{b}}(\mathcal{M}, \mathsf{k}_0)$ be a normal completely positive regular Markovian contraction cocycle. If the initial space $\mathfrak{h}$ is separable then there is a separable Hilbert space k_1, a normal *-homomorphic regular Markovian cocycle $j \in \mathbb{P}_{\mathrm{b}}(\mathcal{M}, \mathsf{k}_0 \oplus \mathsf{k}_1)$ and a contraction process $W \in \mathbb{P}_{\mathrm{b}}(\mathfrak{h}, \mathsf{k}_0 \oplus \mathsf{k}_1)$ satisfying*

$$k_t(a) \otimes P_t^{(1)} = W_t^* j_t(a) W_t; \quad W_t \in \mathcal{M} \otimes \mathcal{B}_{\mathsf{k}_0 \oplus \mathsf{k}_1}.$$

If k is unital then W may be chosen to be isometric whenever $\mathcal{M} \otimes \mathcal{B}(\mathsf{k}_1; \mathsf{k}_0)$ contains an isometry.

To realise k_t in the Stinespring form (0.1), let $E^{(0)} : \mathfrak{h} \otimes \mathcal{F}_{\mathsf{k}_0} \to \mathfrak{h} \otimes \mathcal{F}_{\mathsf{k}_0 \oplus \mathsf{k}_1}$ be the isometric embedding $\xi \mapsto \xi \otimes \Omega_{\mathsf{k}_1}$, then $\{W_t^{(0)} := W_t E^{(0)}\}_{t \geq 0} \subset \mathcal{L}(\mathfrak{h} \odot \mathcal{E}[\eta_0]; \mathfrak{h} \otimes \mathcal{F}_{\mathsf{k}_0 \oplus \mathsf{k}_1})$ is a family of linear maps such that

$$k_t(a) = W_t^{(0)*} j_t(a) W_t^{(0)}.$$

Alternatively we may modify the construction in Theorem 4.3 as follows. Let $(\mathsf{k}_1, \pi, h, d, \{w_j\})$ be a quintuple associated to k through Theorem 4.1, and ensure that $\dim \mathsf{k}_0 \geq \dim \mathsf{k}_1$ by replacing $k_t(a)$ by $k_t(a) \odot 1$ if necessary, and putting zeros in the new places of the expanded mapping matrix. Choose an isometry $V : \mathsf{k}_1 \to \mathsf{k}_0$ and define a quintuple $(\mathsf{k}_0, {}^0\pi, h, {}^0d, \{{}^0w_j\})$ by

$${}^0\pi(a) = (1 \otimes V)\pi(a)(1 \otimes V^*), \quad {}^0d = (1 \otimes V)d, \quad {}^0w_j = (1 \otimes V)w_j.$$

Then ${}^0\pi : \mathcal{M} \to \mathcal{M} \otimes \mathcal{B}(\mathsf{k}_0)$ is a *-homomorphism and ${}^0d, {}^0w_j \in \mathcal{M} \otimes \mathsf{k}_0$. Now define a bounded mapping matrix ${}^0\theta \in \mathcal{B}(\mathcal{M}; \mathcal{M} \otimes \widehat{\mathsf{k}_0})$ and a *regular* algebra matrix ${}^0\ell$ by

$$\widehat{{}^0\theta} = \begin{bmatrix} \mathcal{L}_{{}^0d, {}^0\pi, h} & \delta^\dagger_{{}^0d, {}^0\pi} \\ \delta_{{}^0d, {}^0\pi} & {}^0\pi \end{bmatrix}, \quad \widehat{{}^0\ell} = \begin{bmatrix} \frac{1}{2}t & c^* \\ 0 & {}^0w \end{bmatrix},$$

let ${}^0j \in \mathbb{P}_{\mathrm{cb}}(\mathcal{M}, \mathsf{k}_0)$ be the normal *-homomorphic flow generated by ${}^0\theta$, and let ${}^0W \in \mathbb{P}(\mathfrak{h}, \mathcal{E}[\eta_0])$ be the strong solution of $d{}^0W_t = {}^0j_t({}^0\ell^\alpha_\beta){}^0W_t \, d\Lambda^\beta_\alpha(t)$. It is straightforward to check that $\phi = \psi$, where ψ is the mapping matrix constructed from ${}^0\theta$ and ${}^0\ell$ through the identity (3.12). Thus, by Theorem 3.7,

$$k_t(a) = {}^0W_t^* \, {}^0j_t(a){}^0W_t$$

(or $k_t(a) \odot 1 = {}^0W_t^* \, {}^0j_t(a){}^0W_t$ if $\dim \mathsf{k}_1 > \dim \mathsf{k}_0$).

Our final result exploits the fact that we are working with ultraweakly continuous maps on a von Neumann algebra to obtain a form of our stochastic Stinespring decomposition in which j is trivial. Similar results have been obtained by Belavkin in the case when $\mathcal{M} = \mathcal{B}(\mathfrak{h})$ ([Be2]).

Theorem 4.5. *Let $\phi \in \mathrm{M}_D(\mathcal{B}(\mathcal{M}))$. Suppose that $k \in \mathbb{P}(\mathcal{M}, \mathcal{E}[\eta_0])$ is a completely positive weakly regular process weakly satisfying the equation $dk_t = k_t \circ \phi^\alpha_\beta \, d\Lambda^\beta_\alpha(t)$, each ϕ^α_β ultraweakly continuous, and the initial space $\mathfrak{h}$ is separable. Then there is a separable Hilbert space k containing k_0 (equal to k_0 if k_0 is infinite dimensional) and a process $W \in \mathbb{P}(\mathfrak{h}, \mathcal{E}[\eta])$ such that*

$$k_t(a) \odot 1_{\mathsf{k} \ominus \mathsf{k}_0} = W_t^*(a \otimes 1_{\mathsf{k}})W_t,$$

where η is a basis for k containing η_0.

Proof. Let $(\mathsf{k}_1, \pi, h, d, \{w_j\})$ be a quintuple associated to k through Theorem 4.1. Since π is a normal representation taking values in $\mathcal{B}(\mathfrak{h} \otimes \mathsf{k}_1)$, with $\mathfrak{h} \otimes \mathsf{k}_1$ separable, there is a further separable Hilbert space k_2 and a partial isometry $S : \mathfrak{h} \otimes \mathsf{k}_1 \to \mathfrak{h} \otimes \mathsf{k}_2$ satisfying $\pi(a) = S^*(a \otimes 1)S$ and $SS^* \in \mathcal{M}' \otimes \mathcal{B}(\mathsf{k}_2)$ (see [Dix], p.61). By enlarging k_0 if necessary we may assume that $\dim \mathsf{k}_0 \geq \dim \mathsf{k}_2$, and thus can choose an isometry $U : \mathsf{k}_2 \to \mathsf{k}_0$. The map ${}^1S = (1 \otimes U)S : \mathfrak{h} \otimes \mathsf{k}_1 \to \mathfrak{h} \otimes \mathsf{k}_0$ is then a partial isometry such that $\pi(a) = {}^1S^*(a \otimes 1){}^1S$. So define ${}^1\pi : \mathcal{M} \to \mathcal{M} \otimes \mathcal{B}(\mathsf{k}_0)$ and ${}^1d, {}^1w_j \in \mathcal{M} \otimes \mathsf{k}_0$ by

$${}^1\pi(a) = a \otimes 1, \quad {}^1d = {}^1Sd, \quad {}^1w_j = {}^1Sw_j,$$

and since $\pi(1)d = d$ and $\pi(1)w_j = w_j$, it follows that

$$\widehat{\phi}(a) = \begin{bmatrix} \mathcal{L}_{{}^1d, {}^1\pi, h}(a) + \frac{1}{2}\{t, a\} & \delta^\dagger_{{}^1d, {}^1\pi}(a){}^1w + ac^* \\ {}^1w^* \delta_{{}^1d, {}^1\pi}(a) + ca & {}^1w^* \, {}^1\pi(a){}^1w \end{bmatrix}$$

Let $L \in \mathrm{M}_D(\mathcal{B}(\mathfrak{h}))$ be the regular *operator matrix* defined by

$$\widehat{L} = \begin{bmatrix} \frac{1}{2}(t - {}^1d^* \, {}^1d) + ih & c^* - {}^1d^* \, {}^1w \\ {}^1d & {}^1w \end{bmatrix},$$

and let ${}^1W$ be the strongly regular strong solution to $d{}^1W_t = L^\alpha_\beta \, {}^1W_t \, d\Lambda^\beta_\alpha(t)$ ([LW1], Theorem 7.1, or Proposition 3.2 of this paper with $\mathcal{A} = \mathcal{B}(\mathfrak{h})$ and $j_t(a) = a \otimes 1$). Define ${}^1\phi \in \mathrm{M}_D(\mathcal{B}(\mathcal{N}))$, where $\mathcal{N} = \mathcal{B}(\mathfrak{h})$, by

$${}^1\phi(b) = \iota(b)L + L^\dagger \iota(b) + L^\dagger \Delta(b) L, \quad b \in \mathcal{B}(\mathfrak{h}).$$

Now note that although we do not know if $L^\alpha_\beta \in \mathcal{M}$ for all α, β, we do have

$$^1\phi^\alpha_\beta(a) = \phi^\alpha_\beta(a), \quad \forall a \in \mathcal{M}, \alpha, \beta \geq 0.$$

by construction. Hence, by (2.7),

$$\begin{aligned}
&\langle {}^1W_t u\varepsilon(f), (a \otimes 1_{\mathsf{k}}) {}^1W_t v\varepsilon(g)\rangle - \langle u\varepsilon(f), k_t(a) v\varepsilon(g)\rangle \\
&= \int_0^t f_\alpha(s) g^\beta(s) \Big\{ \langle {}^1W_s u\varepsilon(f), \phi^\alpha_\beta(a)\, {}^1W_s v\varepsilon(g)\rangle - \langle u\varepsilon(f), k_s(\phi^\alpha_\beta(a)) v\varepsilon(g)\rangle \Big\}\, ds.
\end{aligned}$$

An iteration argument as in the proof of Theorem 3.7 shows that Dom ${}^1W_t^* \supset$ Ran$((a \otimes 1_{\mathsf{k}}){}^1W_t)$ for all $a \in \mathcal{M}$, and the result follows. □

Remarks. (i) In contrast to the earlier theorems we are not able to show that $L^\alpha_\beta \in \mathcal{M}$, and so the process ${}^1W$ may not be affiliated to the algebra $\mathcal{M} \otimes \mathcal{B}$.

(ii) It follows from standard facts about positivity preserving maps between C^*-algebras that k is a contraction process if and only if W is a contraction process. Similarly k is unital if and only if W is an isometry process.

Corollary 4.6 ([GoS]). *If $j \in \mathbb{P}_{\mathrm{b}}(\mathcal{M}, \mathsf{k}_0)$ is a normal *-homomorphic regular Markovian cocycle, and the initial Hilbert space is separable, then there is a separable Hilbert space k containing k_0 (equal to k_0 if k_0 is infinite dimensional) and a partial isometry valued process $W \in \mathbb{P}_{\mathrm{b}}(\mathfrak{h}, \mathsf{k})$ such that*

$$j_t(a) \otimes 1_{\mathsf{k} \ominus \mathsf{k}_0} = W_t^*(a \otimes 1_{\mathsf{k}})\, W_t.$$

Proof. To see that in this case W is partial isometry valued note that

$$W_t^* W_t = j_t(1) \otimes 1 = (j_t(1) \otimes 1)^2 = (W_t^* W_t)^2. \quad \square$$

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