PREFERENCE INFERENCE BASED ON HIERARCHICAL AND SIMPLE LEXICOGRAPHIC MODELS

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Abstract

Preference Inference involves inferring additional user preferences from elicited or observed preferences, based on assumptions regarding the form of the user's preference relation. In this paper we consider a situation in which alternatives have an associated vector of costs, each component corresponding to a different criterion, and are compared using a kind of lexicographic order, similarly to the way alternatives are compared in a Hierarchical Constraint Logic Programming model. It is assumed that the user has some (unknown) importance ordering on criteria, and that to compare two alternatives, firstly, the combined cost of each alternative with respect to the most important criteria are compared; only if these combined costs are equal, are the next most important criteria considered. The preference inference problem then consists of determining whether a preference statement can be inferred from a set of input preferences. We show that this problem is coNP-complete, even if one restricts the cardinality of the equal-importance sets to have at most two elements, and one only considers non-strict preferences. However, it is polynomial if it is assumed that the user's ordering of criteria is a total ordering (which we call a simple lexicographic model); it is also polynomial if the sets of equally important criteria are all equivalence classes of a given fixed equivalence relation. We give an efficient polynomial algorithm for these cases, which also throws light on the structure of the inference. We give a complete proof theory for the simple lexicographic model case, and analyse variations of preference inference.[1]

1 Introduction

There are increasing opportunities for decision making/support systems to take into account the preferences of individual users, with the user preferences being elicited or observed from

[1]This is an extended version of an IJCAI-2015 paper[19].

the user's behaviour. However, users tend to have limited patience for preference elicitation, so such a system will tend to have a very incomplete picture of the user preferences. *Preference Inference* involves inferring additional user preferences from elicited or observed preferences, based on assumptions regarding the form of the user's preference relation. More specifically, given a set of input preferences Γ, and a set of preference models $\mathcal{M}$ (considered as candidates for the user's preference model), we infer a preference statement φ if every model in $\mathcal{M}$ that satisfies Γ also satisfies φ. Preference Inference can take many forms, depending on the choice of $\mathcal{M}$, and on the choices of language(s) for the input and inferred statements. For instance, if we just assume that the user model is a total order (or total pre-order), we can set $\mathcal{M}$ as the set of total [pre-]orders over a set of alternatives. This leads to a relatively cautious form of inference (based on transitive closure), including, for instance, the dominance relation for CP-nets and some related systems, e.g., [3, 5, 4, 1].

Often it can be valuable to obtain a much less cautious form of inference. In recommender systems for example, we aim to present the user with a relatively small set of alternatives. We can determine this set of alternatives as the undominated alternatives of a preference inference relation based on previously expressed user preferences [7, 14], with a more adventurous form of inference generating a smaller set of alternatives. Another example arises in a multi-objective context (as in a simple form of a Multi-Attribute Utility Theory model [9]). Again, it is often better if the number of optimal (undominated) solutions is relatively small, which can be achieved with a less cautious order relation on the set of objectives. These less cautious forms of inference include assuming that the user's preference relation is a simple weighted sum as considered in [7, 13, 12], or different lexicographic forms of preference models as in [16, 14, 18]. A comparison of Pareto orders, weighted sums and lexicographic orders in an multi-objective context shows that the lexicographic case is the least cautious and results in the least undominated solutions [12]. Note that all these systems involve reasoning about what holds in a set of preference models that coincide with the user's preference statements. This contrasts with work in preference learning that typically learns a single model, with the intention that this model closely resembles the real user's preference model [11, 8, 10, 6, 2].

In this paper we consider a situation in which alternatives have an associated vector of costs, each component corresponding to a different criterion, and are compared using a kind of lexicographic order, similarly to the way alternatives (feasible solutions) are compared in a Hierarchical Constraint Logic Programming (HCLP) model [15]. It is assumed that the user has some (unknown) importance ordering on criteria, and that to compare two alternatives, firstly, the combined cost of each alternative with respect to the most important criteria are compared; only if these combined costs are equal, are the next most important criteria considered. Implicitly, we assume that the costs of the alternatives are available to the user in order to express preference statements. Also, we assume to know all criteria the user might use and their costs.

We consider the case where the input preference statements are of a simple form that one alternative is preferred to another alternative, where we allow the expression of both strict and non-strict preferences (in contrast to most related preference logics, such as [17, 3, 16, 18] where only non-strict preferences are considered). We assume that the criteria by which the alternatives are compared are unfavorable facts like costs, distances, etc. Thus the lower the values on the alternatives are the better. Accordingly, a strict preference $\alpha < \beta$ expresses that alternative α is better than β; a non-strict preference $\alpha \leq \beta$ means that α is at least as good as β. This form of preference is natural in many contexts, including for conversational recommender systems [7]. The preference inference problem then consists of determining whether a preference statement can be inferred from a set of input preferences, i.e., if every preference model (of the assumed form) satisfying the inputs also satisfies the query. We show that this problem is coNP-complete, even if one restricts the cardinality of the equal-importance sets to have at most two elements, and one only considers non-strict preferences. However, it is polynomial if it is assumed that the user's ordering of criteria is a total ordering (which we call the *simple lexicographic model* case); it is also polynomial if the sets of equally important criteria are all equivalence classes of a given fixed equivalence relation. We give an efficient polynomial algorithm for these cases, which also throws light on the structure of the inference.

Briefly, the idea behind the polynomial algorithm is as follows. Preference inference can be expressed in terms of testing consistency of a set of preference statements Γ. It turns out to be helpful to consider $\Gamma^{(\leq)}$, which is the same as Γ except that strict statements are replaced by non-strict ones on the same alternatives. We show that Γ is consistent if and only if some maximal model of $\Gamma^{(\leq)}$ satisfies Γ, which is if and only if every maximal model of $\Gamma^{(\leq)}$ satisfies Γ. Generating a maximal model of $\Gamma^{(\leq)}$ can be done in a simple and efficient way, using a greedy algorithm, thus allowing efficient testing of consistency (and thus preference inference). We also show that preference inference is compact, i.e., that if φ can be inferred from Γ then it can be inferred from a finite subset of Γ; and we analyse variations of preference inference, based on only considering maximal models, and only considering models that involve all the criteria.

We have defined our logics of preference inference in a semantic way. It is natural to consider whether we can define a complete proof theory, based on syntactic notion of consequence. We show how this can be done, if we extend the set of alternatives.

Section 2 defines our simple preference logic based on hierarchical models, along with some associated preference inference problems. Section 3 shows that in general the preference inference problem is coNP-complete. Section 4 considers the case where the importance ordering on criteria is a total order, and gives a polynomial algorithm for consistency; here we also consider variations of preference inference and relationships with a logic of disjunctive ordering constraints. In Section 5 we construct a complete proof theory, based on an extended set of alternatives. Section 6 concludes.

2 A preference logic based on hierarchical models

We consider preference models, based on an importance ordering of criteria, that is basically lexicographic, but involving a combination of criteria which are at the same level in the importance ordering. We call these "HCLP models", because models of a similar kind appear in the HCLP system [15] (though we have abstracted away some details from the latter system).

HCLP structures: Define an HCLP structure to be a tuple $\mathcal{S} = \langle \mathcal{A}, \oplus, \mathcal{C} \rangle$, where $\mathcal{A}$ (the set of *alternatives*) is a (possibly infinite) set; $\oplus$ is an associative, commutative and monotonic operation ($x \oplus y \leq z \oplus y$ if $x \leq z$) on the non-negative rational numbers $\mathbb{Q}^+$, with identity element 0; and $\mathcal{C}$ (known as the set of ($\mathcal{A}$-)*evaluations*) is a finite set[2]of functions from $\mathcal{A}$ to $\mathbb{Q}^+$. We also assume that operation $\oplus$ can be computed in linear time (which holds for natural definitions of $\oplus$, including addition and max). The evaluations in $\mathcal{C}$ may be considered as representing criteria or objectives under which the alternatives are evaluated. For $c \in \mathcal{C}$ and $\alpha \in \mathcal{A}$, if $c(\alpha) = 0$ then α fully satisfies the objective corresponding to c; more generally, the smaller the value of $c(\alpha)$, the better α satisfies the c-objective.

Example 1. *Suppose, a user wants to buy a new prepay mobile phone SIM card. She wants to make her decision between different providers based on the price per 10MB data usage* d*, the price per text message m and the price per minute for calls to the same provider* c*. These prices of* d*, m and* c *can be combined by addition. Consider four different options (providers) α, β, γ and δ with the following prices in cents.*

	α	β	γ	δ
d	18	15	13	14
m	15	17	15	13
c	10	11	14	15

In this context, the HCLP structure $\langle \mathcal{A}, \oplus, \mathcal{C} \rangle$ is given by the set of alternatives $\mathcal{A} = \{\alpha, \beta, \gamma, \delta\}$, the operator $\oplus$ being the ordinary addition on the integers and the set of evaluation functions $\mathcal{C} = \{d, m, c\}$.

[2]We could easily extend this to the case where $\mathcal{C}$ is a multi-set. (Or alternatively, we can reason about the latter case using the current formalism by adding an artificial alternative that every evaluation differs on.)

HCLP orderings: With each subset C of $\mathcal{C}$ we define ordering $\preccurlyeq_C^{\oplus}$ on $\mathcal{A}$ by $\alpha \preccurlyeq_C^{\oplus} \beta$ if and only if $\bigoplus_{c\in C} c(\alpha) \leq \bigoplus_{c\in C} c(\beta)$. Relation $\preccurlyeq_C^{\oplus}$ represents how well the alternatives satisfy the set of evaluations C if the latter are considered equally important. $\preccurlyeq_C^{\oplus}$ is a total pre-order (a weak order, i.e., a transitive and complete binary relation). We write $\equiv_C^{\oplus}$ for the associated equivalence relation on $\mathcal{A}$, given by $\alpha \equiv_C^{\oplus} \beta \iff \alpha \preccurlyeq_C^{\oplus} \beta$ and $\beta \preccurlyeq_C^{\oplus} \alpha$. We write $\prec_C^{\oplus}$ for the associated strict weak ordering, defined by $\alpha \prec_C^{\oplus} \beta \iff \alpha \preccurlyeq_C^{\oplus} \beta$ and $\beta \not\preccurlyeq_C^{\oplus} \alpha$. Thus, $\alpha \equiv_C^{\oplus} \beta$ if and only if $\bigoplus_{c\in C} c(\alpha) = \bigoplus_{c\in C} c(\beta)$; and $\alpha \prec_C^{\oplus} \beta$ if and only if $\bigoplus_{c\in C} c(\alpha) < \bigoplus_{c\in C} c(\beta)$.

HCLP models: An HCLP model H based on $\langle \mathcal{A}, \oplus, \mathcal{C} \rangle$ is defined to be an ordered partition $(C_1, \ldots, C_k)$ of a (possibly empty) subset of $\mathcal{C}$; we label this subset as $\sigma(H)$, so that $\sigma(H) = C_1 \cup \cdots \cup C_k$. The sets C_i are called the *levels of* H, which are thus non-empty, disjoint and have union $\sigma(H)$. If $c \in C_i$ and $c' \in C_j$, and $i < j$, then we say that c *appears before* c' (and c' *appears after* c) in H. Associated with H is an ordering relation $\preccurlyeq_H^{\oplus}$ on $\mathcal{A}$ given by:

$\alpha \preccurlyeq_H^{\oplus} \beta$ if and only if either:

(I) for all $i = 1, \ldots, k$, $\alpha \equiv_{C_i}^{\oplus} \beta$; or

(II) there exists some $i \in \{1, \ldots, k\}$ such that (i) $\alpha \prec_{C_i}^{\oplus} \beta$ and (ii) for all j with $1 \leq j < i$, $\alpha \equiv_{C_j}^{\oplus} \beta$.

Relation $\preccurlyeq_H^{\oplus}$ is a kind of lexicographic order on $\mathcal{A}$, where the set C_i of evaluations at the same level are first combined into a single evaluation. $\preccurlyeq_H^{\oplus}$ is a weak order on $\mathcal{A}$. We write $\equiv_H^{\oplus}$ for the associated equivalence relation (corresponding with condition (I)), and $\prec_H^{\oplus}$ for the associated strict weak order (corresponding with condition (II)), so that $\preccurlyeq_H^{\oplus}$ is the disjoint union of $\prec_H^{\oplus}$ and $\equiv_H^{\oplus}$. If $\sigma(H) = \emptyset$ then the first condition for $\alpha \preccurlyeq_H^{\oplus} \beta$ holds vacuously (since $k = 0$), so we have $\alpha \preccurlyeq_H^{\oplus} \beta$ for all $\alpha, \beta \in \mathcal{A}$, and $\prec_H^{\oplus}$ is the empty relation.

Preference language inputs: Let $\mathcal{A}$ be a set of alternatives. We define $\mathcal{L}_{\leq}^{\mathcal{A}}$ to be the set of statements of the form $\alpha \leq \beta$ ("α is preferred to β"), for $\alpha, \beta \in \mathcal{A}$ (the *non-strict* statements); we write $\mathcal{L}_{<}^{\mathcal{A}}$ for the set of statements of the form $\alpha < \beta$ ("α is strictly preferred to β"), for $\alpha, \beta \in \mathcal{A}$ (the *strict* statements); and we let $\mathcal{L}^{\mathcal{A}} = \mathcal{L}_{\leq}^{\mathcal{A}} \cup \mathcal{L}_{<}^{\mathcal{A}}$. If φ is the preference statement $\alpha \leq \beta$ then $\neg\varphi$ is defined to be the preference statement $\beta < \alpha$. If φ is the preference statement $\alpha < \beta$ then $\neg\varphi$ is defined to be the preference statement $\beta \leq \alpha$.

Satisfaction of preference statements: For an HCLP model H over the HCLP structure $\langle \mathcal{A}, \oplus, \mathcal{C} \rangle$, we say that H *satisfies* $\alpha \leq \beta$ (written $H \models^{\oplus} \alpha \leq \beta$) if $\alpha \preccurlyeq_H^{\oplus} \beta$ holds.

Similarly, we say that H *satisfies* $\alpha < \beta$ (written $H \models^{\oplus} \alpha < \beta$) if $\alpha \prec_H^{\oplus} \beta$. For $\Gamma \subseteq \mathcal{L}^{\mathcal{A}}$, we say that H *satisfies* Γ (written $H \models^{\oplus} \Gamma$) if H satisfies φ for all $\varphi \in \Gamma$. If $H \models^{\oplus} \varphi$ then we sometimes say that H *is a model of* φ (and similarly, if $H \models^{\oplus} \Gamma$).

Satisfaction of negated preference statements behaves as one would expect:

Lemma 1. *Let H be a HCLP model over HCLP structure $\mathcal{S}$. Then, H satisfies φ if and only if H does not satisfy $\neg\varphi$.*

Proof: Write $\mathcal{S}$ as $\langle \mathcal{A}, \oplus, \mathcal{C} \rangle$. It is sufficient to show that, for any $\alpha, \beta \in \mathcal{A}$, H satisfies $\alpha \leq \beta$ if and only if H does not satisfy $\beta < \alpha$. We have that H satisfies $\alpha \leq \beta$ if and only if $\alpha \preccurlyeq_H^{\oplus} \beta$, which, since $\preccurlyeq_H^{\oplus}$ is a weak order, is if and only if $\beta \not\prec_H^{\oplus} \alpha$, i.e., H does not satisfy $\beta < \alpha$. $\square$

Example 2. *Consider Example 1 of a user choosing between different providers to buy a prepay SIM card. Suppose that the user is not interested in using data, and regards m and c as equally important. She can express her preferences by the corresponding HCLP model $H = (\{m, c\})$. Since $m(\alpha) + c(\alpha) = 25 < m(\beta) + c(\beta) = 28 = m(\delta) + c(\delta) = 28 < m(\gamma) + c(\gamma) = 29$, H satisfies $\alpha \prec_H^{\oplus} \beta \equiv_H^{\oplus} \delta \prec_H^{\oplus} \gamma$. The evaluations involved in H are $\sigma(H) = \{m, c\}$. If the user is most interested in the text message prices, and only if these are equal in the call prices, and only if these are also equal in the data prices, then the corresponding HCLP model is $H' = (\{m\}, \{c\}, \{d\})$. The induced order relation for this model satisfies $\delta \prec_{H'}^{\oplus} \alpha \prec_{H'}^{\oplus} \gamma \prec_{H'}^{\oplus} \beta$, since $m(\delta) < m(\alpha) = m(\gamma) < m(\beta)$ and $c(\alpha) < c(\gamma)$. The evaluations involved in H' are $\sigma(H') = \{d, m, c\}$.*

Preference inference/deduction relation: We are interested in different restrictions on the set of models, and the corresponding inference relations. Let $\mathcal{M}$ be a set of HCLP models over HCLP structure $\langle \mathcal{A}, \oplus, \mathcal{C} \rangle$. For $\Gamma \subseteq \mathcal{L}^{\mathcal{A}}$, and $\varphi \in \mathcal{L}^{\mathcal{A}}$, we say that $\Gamma \models_{\mathcal{M}}^{\oplus} \varphi$, if H satisfies φ for every $H \in \mathcal{M}$ satisfying Γ. Thus, if we elicit some preference statements Γ of a user, and we assume that their preference relation is an HCLP model in $\mathcal{M}$ (based on the HCLP structure), then $\Gamma \models_{\mathcal{M}}^{\oplus} \varphi$ holds if and only if we can deduce (with certainty) that the user's HCLP model H satisfies φ.

Consistency: For set of HCLP models $\mathcal{M}$ over HCLP structure $\langle \mathcal{A}, \oplus, \mathcal{C} \rangle$, and set of preference statements $\Gamma \subseteq \mathcal{L}^{\mathcal{A}}$, we say that Γ is $(\mathcal{M}, \oplus)$-consistent if there exists $H \in \mathcal{M}$ such that $H \models^{\oplus} \Gamma$; otherwise, we say that Γ is $(\mathcal{M}, \oplus)$-inconsistent. In the usual way, because of the existence of a negation operator, deduction can be reduced to checking (in)consistency.

Proposition 1. $\Gamma \models_{\mathcal{M}}^{\oplus} \varphi$ *if and only if* $\Gamma \cup \{\neg\varphi\}$ *is* $(\mathcal{M}, \oplus)$*-inconsistent.*

Proof: Suppose that $\Gamma \models^{\oplus}_{\mathcal{M}} \varphi$. By definition, H satisfies φ for every $H \in \mathcal{M}$ satisfying (every element of) Γ. Thus, using Lemma 1, there exists no $H \in \mathcal{M}$ that satisfies Γ and $\neg\varphi$, which implies that $\Gamma \cup \{\neg\varphi\}$ is $(\mathcal{M}, \oplus)$-inconsistent.

Conversely, suppose $\Gamma \cup \{\neg\varphi\}$ is $(\mathcal{M}, \oplus)$-inconsistent. By definition, there exists no $H \in \mathcal{M}$ that satisfies $\Gamma \cup \neg\varphi$. Thus, every $H \in \mathcal{M}$ that satisfies Γ does not satisfy $\neg\varphi$, and therefore satisfies φ, by Lemma 1. Hence, $\Gamma \models^{\oplus}_{\mathcal{M}} \varphi$. □

Let t be some number in $\{1, 2, \dots, |\mathcal{C}|\}$. We define $\mathcal{C}(t)$ to be the set of all HCLP models $(C_1, \dots, C_k)$ based on HCLP structure $\langle \mathcal{A}, \oplus, \mathcal{C} \rangle$ such that $|C_i| \leq t$, for all $i = 1, \dots, k$. An element of $\mathcal{C}(1)$ thus corresponds to a sequence of singleton sets of evaluations; we identify it with a sequence of evaluations $(c_1, \dots, c_k)$ in $\mathcal{C}$. Thus, $\Gamma \models^{\oplus}_{\mathcal{C}(t)} \varphi$ if and only if $H \models^{\oplus} \varphi$ for all $H \in \mathcal{C}(t)$ such that $H \models^{\oplus} \Gamma$. Note that for $t = 1$, these definitions do not depend on $\oplus$ (since there is no combination of evaluations involved), so we may drop any mention of $\oplus$.

Let $\equiv$ be an equivalence relation on $\mathcal{C}$, and let $\mathcal{E}$ be the set of equivalence classes of $\equiv$. Thus, for each $c \in \mathcal{C}$ there exists a unique element $E \in \mathcal{E}$ such that $E \ni c$, and $E = \{c' \in \mathcal{C} : c' \equiv c\}$. We define $\mathcal{C}(\equiv)$ to be the set of all HCLP models $(C_1, \dots, C_k)$ such that each C_i is an equivalence class with respect to $\equiv$, i.e., $C_i \in \mathcal{E}$. It is easy to see that the relation $\models^{\oplus}_{\mathcal{C}(\equiv)}$ is the same as the relation $\models_{C'(1)}$ where C' is defined as follows. C' is in 1-1 correspondence with $\mathcal{E}$. If E is the $\equiv$-equivalence class of $\mathcal{C}$ corresponding with $c' \in C'$ then, for $\alpha \in \mathcal{A}$, $c'(\alpha)$ is defined to be $\bigoplus_{c \in E} c(\alpha)$, so that each C_i in an HCLP model is replaced by a single evaluation equivalent to the combination of all the elements of C_i.

For $\models$ either being $\models^{\oplus}_{\mathcal{C}(t)}$ for some $t \in \{1, 2, \dots, |\mathcal{C}|\}$, or being $\models^{\oplus}_{\mathcal{C}(\equiv)}$ for some equivalence relation $\equiv$ on $\mathcal{C}$, we consider the following decision problem.

HCLP-DEDUCTION FOR $\models$: Given $\mathcal{C}$, Γ and φ is it the case that $\Gamma \models \varphi$?

In Section 4, we will show that this problem is polynomial for $\models$ being $\models^{\oplus}_{\mathcal{C}(t)}$ when $t = 1$. Thus it is polynomial also for $\models^{\oplus}_{\mathcal{C}(\equiv)}$, for any equivalence relation $\equiv$. It is coNP-complete for $\models$ being $\models^{\oplus}_{\mathcal{C}(t)}$ when $t > 1$, as shown below in Section 3.

Theorem 1. HCLP-DEDUCTION FOR $\models^{\oplus}_{\mathcal{C}(t)}$ *is polynomial when* $t = 1$*, and is* coNP-*complete for any* $t > 1$*, even if we restrict the language to non-strict preference statements.* HCLP-DEDUCTION FOR $\models^{\oplus}_{\mathcal{C}(\equiv)}$ *is polynomial for any equivalence relation* $\equiv$.

Example 3. *Consider the HCLP structure of Example 1. Suppose, the user states that she prefers* α *to* β*, i.e.* $\alpha \leq \beta$*, and strictly prefers* β *to* γ*, i.e.* $\beta < \gamma$*. Only the HCLP models of the forms* $(\{c\}, \dots)$*,* $(\{m\}, \dots)$*,* $(\{c, m\}, \dots)$ *or* $(\{d, m, c\})$ *satisfy* $\alpha \leq \beta$*. Only the HCLP models* $(\{c\}, \dots)$*,* $(\{c, d\}, \dots)$ *or* $(\{c, m\}, \dots)$ *satisfy* $\beta < \gamma$*. Thus, the models* $(\{c\}, \dots)$ *and* $(\{c, m\}, \dots)$ *are the only ones that satisfy the set* $\Gamma = \{\alpha \leq \beta, \beta < \gamma\}$

of the user's input preferences. Let $t \in \{1,2,3\}$. Then $\Gamma \not\models^{\oplus}_{\mathcal{C}(t)} \delta \leq \beta$ since the model $H = (\{c\}) \in \mathcal{C}(1) \subseteq \mathcal{C}(t)$ satisfies Γ and $\beta \prec^{\oplus}_{H} \delta$, i.e., $H \not\models^{\oplus} \delta \leq \beta$. Furthermore, $\Gamma \not\models^{\oplus}_{\mathcal{C}(2)} \beta \leq \delta$ since the model $H' = (\{c, m\}, \{d\}) \in \mathcal{C}(2)$ satisfies Γ and $\delta \prec^{\oplus}_{H'} \beta$, i.e., $H' \not\models^{\oplus} \beta \leq \delta$. However, we can infer $\Gamma \models^{\oplus}_{\mathcal{C}(1)} \beta \leq \delta$, and even $\Gamma \models^{\oplus}_{\mathcal{C}(1)} \beta < \delta$, since all Γ-satisfying HCLP models in $\mathcal{C}(1)$, i.e., $(\{c\})$, $(\{c\}, \{m\})$, $(\{c\}, \{d\})$, $(\{c\}, \{m\}, \{d\})$, and $(\{c\}, \{d\}, \{m\})$, satisfy the relation $\beta < \delta$.

3 Proving coNP-completeness of HCLP-deduction for $\models^{\oplus}_{\mathcal{C}(t)}$ for $t > 1$

Given an arbitrary 3-SAT instance we will show that we can construct a set Γ and a statement $\alpha \leq \beta$ such that the 3-SAT instance has a satisfying truth assignment if and only if $\Gamma \not\models^{\oplus}_{\mathcal{C}(t)} \alpha \leq \beta$ (see Proposition 2 below). This then implies that determining if $\Gamma \not\models^{\oplus}_{\mathcal{C}(t)} \alpha \leq \beta$ holds is NP-hard.

We have that $\Gamma \not\models^{\oplus}_{\mathcal{C}(t)} \alpha \leq \beta$ if and only if there exists an HCLP-model $H \in \mathcal{C}(t)$ such that $H \models^{\oplus} \Gamma$ and $H \not\models^{\oplus} \alpha \leq \beta$. For any given H, checking that $H \models^{\oplus} \Gamma$ and $H \not\models^{\oplus} \alpha \leq \beta$ can be performed in polynomial time. This implies that determining if $\Gamma \not\models^{\oplus}_{\mathcal{C}(t)} \alpha \leq \beta$ holds is in NP, and therefore is NP-complete, and thus determining if $\Gamma \models^{\oplus}_{\mathcal{C}(t)} \alpha \leq \beta$ holds is coNP-complete.

Consider an arbitrary 3-SAT instance based on propositional variables $p_1, \ldots, p_r$, consisting of clauses Λ_j, for $j = 1, \ldots, s$. For each propositional variable p_i we associate two evaluations q_i^+ and q_i^-, where q_i^- corresponds with literal $\neg p_i$, and q_i^+ corresponds with literal p_i.

The idea behind the construction is as follows: we generate a (polynomial size) set $\Gamma \subseteq \mathcal{L}^{\mathcal{A}}_{\leq}$ as the disjoint union of sets Γ_1, Γ_2 and Γ_3, and we choose a non-strict statement $\alpha \leq \beta$. For the remainder of this section, let H be an arbitrary HCLP-model in $\mathcal{C}(t)$. Γ_1 is chosen so that if $H \models^{\oplus} \Gamma_1$ then, for each $i = 1, \ldots, r$, $\sigma(H)$ cannot contain both q_i^+ and q_i^-, i.e., q_i^+ and q_i^- do not both appear in H. (Recall H is an ordered partition of $\sigma(H)$, so that $\sigma(H)$ is the subset of $\mathcal{C}$ that appears in H.) If $H \models^{\oplus} \Gamma_2$ and $H \models^{\oplus} \beta < \alpha$ then $\sigma(H)$ contains either q_i^+ or q_i^-. Together, this implies that if $H \models^{\oplus} \Gamma$ and $H \not\models^{\oplus} \alpha \leq \beta$ then for each propositional variable p_i, model H involves either q_i^+ or q_i^-, but not both. Γ_3 is used to make the correspondence with the clauses. For instance, if one of the clauses is $p_2 \vee \neg p_5 \vee p_6$ then any HCLP model $H \in \mathcal{C}(t)$ of $\Gamma \cup \{\beta < \alpha\}$ will involve either q_2^+, q_5^-, or q_6^+.

Suppose that H satisfies Γ but not $\alpha \leq \beta$. We can generate a satisfying assignment of the 3-SAT instance, by assigning p_i to 1 (TRUE) if and only if q_i^+ appears in H.

The monotonicity assumption for operation $\oplus$ implies that $1 \oplus 1 > 0$, since we have $1 \oplus 1 \geq 1 \oplus 0 = 1 > 0$. In fact, in the proof below we do not need to assume monotonicity of $\oplus$; it is sufficient to just assume that $1 \oplus 1 > 0$.

We describe the construction more formally below.

Defining $\mathcal{A}$ and $\mathcal{C}$: The set of alternatives $\mathcal{A}$ is defined to be the union of the following sets

- $\{\alpha, \beta\} \cup \{\alpha_i, \beta_i, \delta_i \,:\, i = 1, \ldots, r\}$
- $\{\gamma_i^k \,:\, i = 1, \ldots, r, k = 1, \ldots, t-1\}$
- $\{\theta_j, \tau_j \,:\, j = 1, \ldots, s\}$.

We define the set of evaluations $\mathcal{C}$ to be $\{c^*\} \cup \{q_i^+, q_i^- \,:\, i = 1, \ldots, r\} \cup A_1 \cup \cdots \cup A_r$, where $A_i = \{a_i^k \,:\, k = 1, \ldots, t-1\}$. Both $\mathcal{A}$ and $\mathcal{C}$ are of polynomial size.

Satisfying $\beta < \alpha$: The evaluations on α and β are defined as follows:

- $c^*(\alpha) = 1$, and for all $c \in \mathcal{C} - \{c^*\}$, $c(\alpha) = 0$.
- For all $c \in \mathcal{C}$, $c(\beta) = 0$.

It immediately follows that: $H \models^{\oplus} \beta < \alpha \iff \sigma(H) \ni c^*$.

The construction of Γ_1: We define $\Gamma_1 = \bigcup_{i=1}^r \Gamma_1^i$ where, for each $i = 1, \ldots, r$, we define $\Gamma_1^i = \{\delta_i \leq \gamma_i^k,\ \gamma_i^k \leq \delta_i \,:\, k = 1, \ldots, t-1\}$. We make use of auxiliary evaluations $A_i = \{a_i^1, \ldots, a_i^{t-1}\}$. The values of the evaluations on γ_i^k and δ_i are defined as follows:

- $a_i^k(\gamma_i^k) = 1$, and for all $c \in \mathcal{C} - \{a_i^k\}$ we set $c(\gamma_i^k) = 0$.
- $q_i^+(\delta_i) = q_i^-(\delta_i) = 1$, and for other $c \in \mathcal{C}$, $c(\delta_i) = 0$.

Thus, for any $B \subseteq A_i$, we have $(\bigoplus_{a \in B} a \oplus q_i^+)(\delta_i) = \bigoplus_{a \in B} a(\delta_i) \oplus q_i^+(\delta_i) = 0 \oplus \cdots \oplus 0 \oplus 1 = 1$. Similarly, $(\bigoplus_{a \in B} a \oplus q_i^-)(\delta_i) = 1$. Furthermore, $(\bigoplus_{a \in B} a \oplus q_i^+)(\gamma_i^k) = 1 \Leftrightarrow a_i^k \in B$ and $(\bigoplus_{a \in B} a \oplus q_i^-)(\gamma_i^k) = 1 \Leftrightarrow a_i^k \in B$.

Lemma 2. *$H \models^{\oplus} \Gamma_1^i$ if and only if either (i) $\sigma(H)$ does not contain any element in A_i or q_i^+ or q_i^-, i.e., $\sigma(H) \cap (A_i \cup \{q_i^+, q_i^-\}) = \emptyset$; or (ii) $A_i \cup \{q_i^+\}$ is a level of H, and $\sigma(H) \not\ni q_i^-$; or (iii) $A_i \cup \{q_i^-\}$ is a level of H, and $\sigma(H) \not\ni q_i^+$. In particular, if $H \models^{\oplus} \Gamma_1^i$ then $\sigma(H)$ does not contain both q_i^+ and q_i^-.*

Proof: Consider any $H \in \mathcal{C}(t)$, so that for each level E of H we have $|E| \leq t$. We have that $H \models^{\oplus} \Gamma_1^i$ if and only if for each level E of H and for all $k = 1, \ldots, t-1$, $\delta_i \equiv_E^{\oplus} \gamma_i^k$. Now, $\delta_i \equiv_E^{\oplus} \gamma_i^k$ if and only if $\bigoplus_{c \in E} c(\delta_i) = \bigoplus_{c \in E} c(\gamma_i^k)$. Also, $\bigoplus_{c \in E} c(\delta_i) = 0$ unless E contains either q_i^+ or q_i^-; and $\bigoplus_{c \in E} c(\delta_i) = 1 \oplus 1 > 0$ if E contains both q_i^+ and q_i^-; and equals 1 if E contains either q_i^+ or q_i^-, but not both. $\bigoplus_{c \in E} c(\gamma_i^k)$ equals 1 if and only if E contains a_i^k, and equals 0 otherwise.

This implies that if for all $k = 1, \ldots, t-1$, $\delta_i \equiv_E^{\oplus} \gamma_i^k$ and E contains q_i^+ or q_i^- then for all $k = 1, \ldots, t-1$, E contains a_i^k, and so $E \supseteq A_i$. Because of the condition that $|E| \leq t$ (since $H \in \mathcal{C}(t)$), and $|A_i| = t-1$, we then have that E equals either $A_i \cup \{q_i^+\}$ or $A_i \cup \{q_i^-\}$.

Similarly, if for all $k = 1, \ldots, t-1$, $\delta_i \equiv_E^{\oplus} \gamma_i^k$ and E contains a_i^k for some $k \in \{1, \ldots, t-1\}$, then E contains q_i^+ or q_i^-, and so, by the previous paragraph, E equals either $A_i \cup \{q_i^+\}$ or $A_i \cup \{q_i^-\}$.

Thus, if $H \models^{\oplus} \Gamma_1^i$, then for at most one level E of H do we have $E \cap (A_i \cup \{q_i^+, q_i^-\})$ non-empty (else we would have two levels both containing A_i, contradicting disjointness of levels); also if $E \cap (A_i \cup \{q_i^+, q_i^-\})$ is non-empty then E equals either $A_i \cup \{q_i^+\}$ or $A_i \cup \{q_i^-\}$. In particular, if $H \models^{\oplus} \Gamma_1^i$ then $\sigma(H)$ does not contain both q_i^+ and q_i^-.

Regarding the converse, let us suppose first that (i) $\sigma(H)$ does not intersect with $A_i \cup \{q_i^+, q_i^-\}$. Then for all levels E of H, and for all $k = 1, \ldots, t-1$, we have $\bigoplus_{c \in E} c(\delta_i) = \bigoplus_{c \in E} c(\gamma_i^k) = 0$, and thus $\delta_i \equiv_E^{\oplus} \gamma_i^k$, which implies $H \models^{\oplus} \Gamma_1^i$.

Now suppose (ii) that $A_i \cup \{q_i^+\}$ is a level E' of H and $\sigma(H) \not\ni q_i^-$. Then every other level E is disjoint from $A_i \cup \{q_i^+, q_i^-\}$, so for all $k = 1, \ldots, t-1$, $\bigoplus_{c \in E} c(\delta_i) = \bigoplus_{c \in E} c(\gamma_i^k) = 0$, and thus $\delta_i \equiv_E^{\oplus} \gamma_i^k$. Also, $\bigoplus_{c \in E'} c(\delta_i) = \bigoplus_{c \in E'} c(\gamma_i^k) = 1$, and thus $H \models^{\oplus} \Gamma_1^i$. Case (iii), when $A_i \cup \{q_i^-\}$ is a level E' of H and $\sigma(H) \not\ni q_i^+$, is essentially identical to Case (ii), just switching the roles of q_i^+ and q_i^+. □

The construction of Γ_2: For each $i = 1, \ldots, r$, define φ_i to be $\alpha_i \leq \beta_i$. We let $\Gamma_2 = \{\varphi_i : i = 1, \ldots, r\}$. The values of the evaluations on α_i and β_i are defined as follows. We define $c^*(\alpha_i) = 1$, and for all $c \in \mathcal{C} - \{c^*\}$, $c(\alpha_i) = 0$. Define $q_i^+(\beta_i) = q_i^-(\beta_i) = 1$, and for all $c \in \mathcal{C} - \{q_i^+, q_i^-\}$, $c(\beta_i) = 0$. Thus, similarly to the previous observations for Γ_1, $(c^* \oplus q_i^+)(\beta_i) = (c^* \oplus q_i^-)(\beta_i) = 1$ and $(c^* \oplus q_i^+)(\alpha_i) = (c^* \oplus q_i^-)(\alpha_i) = 1$. Also, $(q_i^+ \oplus q_i^-)(\alpha_i) = 0$ and $(q_i^+ \oplus q_i^-)(\beta_i) \geq 1$, because of the monotonicity of $\oplus$, and $(c^* \oplus q_i^+ \oplus q_i^-)(\alpha_i) = 1$ and $(c^* \oplus q_i^+ \oplus q_i^-)(\beta_i) \geq 1$.

The following result easily follows.

Lemma 3. *If q_i^+ or q_i^- appears before c^* in H then $H \models^{\oplus} \varphi_i$. If $\sigma(H) \ni c^*$ and $H \models^{\oplus} \varphi_i$ then $\sigma(H) \ni q_i^+$ or $\sigma(H) \ni q_i^-$.*

Proof: Consider any $H \in \mathcal{C}(t)$, and consider any $i \in \{1, \ldots, r\}$. Then the following hold for any level E of H.

(I) If E does not contain any of $\{c^*, q_i^+, q_i^-\}$ then $\bigoplus_{c \in E} c(\alpha_i) = \bigoplus_{c \in E} c(\beta_i) = 0$ so $\alpha_i \equiv_E^\oplus \beta_i$.

(II) If E contains c^* but neither of q_i^+ or q_i^-, then $\bigoplus_{c \in E} c(\alpha_i) = 1$ and $\bigoplus_{c \in E} c(\beta_i) = 0$, so $\alpha_i \not\preccurlyeq_E^\oplus \beta_i$.

(III) If E contains q_i^+ or q_i^- but not c^* then $\bigoplus_{c \in E} c(\alpha_i) = 0$ and $\bigoplus_{c \in E} c(\beta_i) > 0$ using the fact that $1 \oplus 1 > 0$, so $\alpha_i \prec_E^\oplus \beta_i$.

Assume that $\sigma(H) \ni c^*$. If $\sigma(H) \cap \{q_i^+, q_i^-\} = \emptyset$ then by considering the level containing c^* we can see, using (I) and (II), that $\alpha_i \not\preccurlyeq_H^\oplus \beta_i$, so $H \not\models^\oplus \varphi_i$. This proves the second half of the lemma.

If q_i^+ or q_i^- (or both) appear before c^* in H then (I) and (III) imply that $\alpha_i \prec_H^\oplus \beta_i$ and thus $H \models^\oplus \varphi_i$. □

The construction of Γ_3: For each $i = 1, \ldots, r$, define $Q(p_i) = q_i^+$ and $Q(\neg p_i) = q_i^-$. This defines the function Q over all literals. Let us write the jth clause as $l_1 \vee l_2 \vee l_3$ for literals l_1, l_2 and l_3. Define $Q_j = \{Q(l_1), Q(l_2), Q(l_3)\}$. For example, if the jth clause were $p_2 \vee \neg p_5 \vee p_6$ then $Q_j = \{q_2^+, q_5^-, q_6^+\}$. We define ψ_j to be $\theta_j \leq \tau_j$, and $\Gamma_3 = \{\psi_j : j = 1, \ldots s\}$. Define $c^*(\theta_j) = 1$ and $c(\theta_j) = 0$ for all $c \in \mathcal{C} - \{c^*\}$. Define $q(\tau_j) = 1$ for $q \in Q_j$, and for all other c (i.e., $c \in \mathcal{C} - Q_j$), define $c(\tau_i) = 0$.

Lemma 4. *If some element of Q_j appears in H before c^*, and no level of H contains more than one element of Q_j, then $H \models^\oplus \psi_j$. If $\sigma(H) \ni c^*$ and $H \models^\oplus \psi_j$ then $\sigma(H)$ contains some element of Q_j.*

Proof: The proof of this result is similar to that of Lemma 3. Consider any $H \in \mathcal{C}(t)$ any clause j. Then the following hold for any level E of H.

(I) If E does not contain any element of $Q_j \cup \{c^*\}$ then $\bigoplus_{c \in E} c(\theta_j) = \bigoplus_{c \in E} c(\tau_j) = 0$ so $\theta_j \equiv_E^\oplus \tau_j$.

(II) If E contains c^* but no element of Q_j neither of q_i^+ or q_i^-, then $\bigoplus_{c \in E} c(\theta_j) = 1$ and $\bigoplus_{c \in E} c(\tau_j) = 0$, so $\theta_j \not\preccurlyeq_E^\oplus \tau_j$.

(III) If E contains exactly one element of Q_j but not c^* then $\bigoplus_{c \in E} c(\theta_j) = 0$ and $\bigoplus_{c \in E} c(\tau_j) = 1$, so $\theta_j \prec_E^\oplus \tau_j$.

Assume that $\sigma(H) \ni c^*$. If $\sigma(H) \cap Q_j = \emptyset$ then by considering the level containing c^* we can see, using (I) and (II), that $\theta_j \not\preccurlyeq^{\oplus}_{H} \tau_j$, so $H \not\models^{\oplus} \varphi_i$. This argument proves that if $\sigma(H) \ni c^*$ and $H \models^{\oplus} \psi_j$ then $\sigma(H)$ contains some element of Q_j.

If some element of Q_j appears in H before c^*, and no level of H contains more than one element of Q_j, then (I) and (III) imply that $\theta_j \prec^{\oplus}_{H} \tau_j$ and thus $H \models^{\oplus} \varphi_i$. □

We set $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. The following result implies that the HCLP deduction problem is coNP-hard (even if we restrict to the case when $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathcal{A}}_{\leq}$).

Proposition 2. *Using the notation defined above, the 3-SAT instance is satisfiable if and only if* $\Gamma \not\models^{\oplus}_{\mathcal{C}(t)} \alpha \leq \beta$.

Proof: First let us assume that $\Gamma \not\models^{\oplus}_{\mathcal{C}(t)} \alpha \leq \beta$. Then by definition, there exists an HCLP model $H \in \mathcal{C}(t)$ with $H \models^{\oplus} \Gamma$ and $H \not\models^{\oplus} \alpha \leq \beta$. Since $H \not\models^{\oplus} \alpha \leq \beta \iff H \models^{\oplus} \beta < \alpha$, we have $H \models^{\oplus} \Gamma \cup \{\beta < \alpha\}$. Because $H \models^{\oplus} \beta < \alpha$, we have $\sigma(H) \ni c^*$.

Because also $H \models^{\oplus} \Gamma_2^i$, either $\sigma(H) \ni q_i^+$ or $\sigma(H) \ni q_i^-$, by Lemma 3. Since $H \models^{\oplus} \Gamma_1^i$, the set $\sigma(H)$ does not contain both q_i^+ and q_i^-, by Lemma 2.

Let us define a truth function $f : \mathcal{P} \to \{0, 1\}$ as follows: $f(p_i) = 1 \iff \sigma(H) \ni q_i^+$. Since $\sigma(H)$ contains exactly one of q_i^+ and q_i^-, we have $f(p_i) = 0 \iff \sigma(H) \ni q_i^-$. We extend f to negative literals in the obvious way: $f(\neg p_i) = 1 - f(p_i)$, and thus, $f(\neg p_i) = 1 \iff \sigma(H) \ni q_i^-$.

Since $H \models^{\oplus} \Gamma_3$ and $\sigma(H) \ni c^*$, then $\sigma(H)$ contains at least one element of each Q_j, by Lemma 4. Thus for each j, $f(l) = 1$ for at least one literal l in the jth clause, and hence f satisfies clause Λ_j. We have shown that f satisfies each clause of the 3-SAT instance, proving that the instance is satisfiable.

Conversely, suppose that the 3-SAT instance is satisfiable, so there exists a truth function f satisfying it. We will construct an HCLP model $H \in \mathcal{C}(t)$ such that $H \models^{\oplus} \Gamma \cup \{\beta < \alpha\}$, and thus $H \not\models^{\oplus} \alpha \leq \beta$, proving that $\Gamma \not\models^{\oplus}_{\mathcal{C}(t)} \alpha \leq \beta$.

For $i = 1, \dots, r$, let $S_i = A_i \cup \{q_i^+\}$ if $f(p_i) = 1$, and otherwise, let $S_i = A_i \cup \{q_i^-\}$. Thus, if $f(p_i) = 1$ then $Q(p_i) \in S_i$; and if $f(\neg p_i) = 1$ then $Q(\neg p_i) \in S_i$. We then define H to be the sequence $S_1, S_2, \dots, S_r, \{c^*\}$. Since $\sigma(H) \ni c^*$, we have that $H \models^{\oplus} \beta < \alpha$. By Lemma 2, for all $i = 1, \dots, r$, $H \models^{\oplus} \Gamma_1^i$ and so $H \models^{\oplus} \Gamma_1$. By Lemma 3, for all $i = 1, \dots, r$, $H \models^{\oplus} \varphi_i$, so $H \models^{\oplus} \Gamma_2$.

Consider any $j \in \{1, \dots, s\}$, and, as above, write the jth clause as $l_1 \vee l_2 \vee l_3$. Truth assignment f satisfies this clause, so there exists $k \in \{1, 2, 3\}$ such that $f(l_k) = 1$. Then $Q(l_k)$ appears in H before c^*, so, by Lemma 4, $H \models^{\oplus} \psi_j$. Thus $H \models^{\oplus} \Gamma_3$. Since $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, we have shown that $H \models^{\oplus} \Gamma \cup \{\beta < \alpha\}$, proving that $\Gamma \not\models^{\oplus}_{\mathcal{C}(t)} \alpha \leq \beta$.□

Example 4. *Let $(p_1 \vee p_2 \vee \neg p_3) \wedge (\neg p_1 \vee p_2 \vee p_3)$ be an instance of 3-SAT with the three propositional variables p_1, p_2, p_3 and clauses Λ_1, Λ_2. From this we construct a $\mathcal{C}(2)$ HCLP-Deduction instance as in the previous paragraphs (so with $t = 2$). Corresponding to the two possible assignments of each of the propositional variables p_1, p_2, p_3, we construct evaluation functions q_1^+, q_2^+, q_3^+ and q_1^-, q_2^-, q_3^-. We also introduce the additional evaluation functions c^* and $A_1 = \{a_1^1, a_2^1, a_3^1\}$. Furthermore, we construct alternatives α, β, $\alpha_1, \alpha_2, \alpha_3$, $\beta_1, \beta_2, \beta_3$, $\delta_1, \delta_2, \delta_3$, $\gamma_1^1, \gamma_2^1, \gamma_3^1$, θ_1, θ_2, τ_1, τ_2 for the preference statements $\alpha > \beta$, Γ_1, Γ_2 and Γ_3, with the values of the evaluation functions given as follows:*

				Γ_2									Γ_1									Γ_3					
	α	$>$	β	α_1	$\leq$	β_1	α_2	$\leq$	β_2	α_3	$\leq$	β_3	δ_1	$\leq, \geq$	γ_1^1	δ_2	$\leq, \geq$	γ_2^1	δ_3	$\leq, \geq$	γ_3^1	θ_1	$\leq$	τ_1	θ_2	$\leq$	τ_2
q_1^+	0		0	0		1	0		0	0		0	1		0	0		0	0		0	0		1	0		0
q_2^+	0		0	0		0	0		1	0		0	0		0	1		0	0		0	0		1	0		1
q_3^+	0		0	0		0	0		0	0		1	0		0	0		0	1		0	0		0	0		1
q_1^-	0		0	0		1	0		0	0		0	1		0	0		0	0		0	0		0	0		1
q_2^-	0		0	0		0	0		1	0		0	0		0	1		0	0		0	0		0	0		0
q_3^-	0		0	0		0	0		0	0		1	0		0	0		0	1		0	0		1	0		0
c^*	1		0	1		0	1		0	1		0	0		0	0		0	0		0	1		0	1		0
a_1^1	0		0	0		0	0		0	0		0	0		1	0		0	0		0	0		0	0		0
a_2^1	0		0	0		0	0		0	0		0	0		0	0		1	0		0	0		0	0		0
a_3^1	0		0	0		0	0		0	0		0	0		0	0		0	0		1	0		0	0		0

Here, the values of τ_1 and τ_2 correspond to the occurrences of the literals p_i or $\neg p_i$ in the clauses Λ_1 and Λ_2, respectively. Since the statement $\alpha > \beta$ is strict, the evaluation c^* has to be included in any satisfying HCLP model H of $\Gamma \cup \{\alpha > \beta\}$, where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. To satisfy a non-strict preference statement $\nu \leq \rho$, if a level contains an evaluation function with value 1 for ν then the same or an earlier level must contain an evaluation function with value 1 for ρ. The preference statement e.g., $\alpha_1 \leq \beta_1$ in Γ_2 then enforces that either q_1^+ or q_1^- appears in some level of H (and no later than c^*) because $c^*(\alpha_1) = 1$ and $q_1^+(\beta_1) = q_1^-(\beta_1) = 1$. Since Γ_1 contains $\delta_1 \leq \gamma_1^1$ and $\gamma_1^1 \leq \delta_1$, a $\mathcal{C}(2)$-HCLP model H satisfying $\Gamma \cup \{\alpha > \beta\}$ must have a_1^1 appearing in the same level as q_1^+ or q_1^-, and both q_1^+ and q_1^- cannot then appear in H. Thus H involves either q_1^+ or q_1^- but not both. Γ_3 contains ψ_1, i.e., $\theta_1 \leq \tau_1$, which ensures that at least one element in $Q_1 = \{q_1^+, q_2^+, q_3^-\}$ appears in some level of a satisfying HCLP model, which corresponds to satisfying the first clause. The assignment $p_1 =$ true, $p_2 =$ true, $p_3 =$ false satisfies the instance $(p_1 \vee p_2 \vee \neg p_3) \wedge (\neg p_1 \vee p_2 \vee p_3)$. A corresponding $\Gamma \cup \{\alpha > \beta\}$-satisfying HCLP model in $\mathcal{C}(2)$ is $(\{q_1^+, a_1^1\}, \{q_2^+, a_2^1\}, \{q_3^-, a_3^1\}, \{c^*\})$.

4 Simple lexicographic models

In this section, we consider the case where we restrict to HCLP models which consist of a sequence of singletons; thus each model corresponds to a sequence of evaluations, and generates a lexicographic order based on these. We call such models: *simple lexicographic models*.

Let $\mathcal{C}$ be a set of evaluations on $\mathcal{A}$. To simplify notation, we redefine a $\mathcal{C}(1)$-model to be a sequence of different elements of $\mathcal{C}$ (rather than a sequence of singleton sets). As mentioned earlier, the operation $\oplus$ plays no part, so we can harmlessly abbreviate ordering $\preccurlyeq_H^{\oplus}$ to just $\preccurlyeq_H$, for any $\mathcal{C}(1)$-model H, and similarly for $\prec_H$ and $\equiv_H$. The deduction problem for the sequence of singletons case is thus as follows. Given $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathcal{A}}$, is it the case that $\Gamma \models_{\mathcal{C}(1)} \varphi$? That is, is it the case that for all $\mathcal{C}(1)$-models H (over $\mathcal{A}$), if H satisfies Γ then H satisfies φ?

Given set of evaluations $\mathcal{C}$ and set of preference statements Γ, we introduce in Section 4.1 the important concept of *maximal inconsistency base* $(\Gamma^{\perp}, C^{\perp})$, where $\Gamma^{\perp} \subseteq \Gamma$ and $C^{\perp} \subseteq \mathcal{C}$. No model of Γ involves any element of $C^{\perp}$, and it turns out (Corollary 1) that Γ is $\mathcal{C}(1)$-inconsistent if and only if $\Gamma^{\perp}$ contains a strict element. It is helpful (see Section 4.2) to consider $\Gamma^{(\leq)}$, a version of Γ where each strict element is replaced by the corresponding non-strict one. Models of $\Gamma^{(\leq)}$ can be generated in a simple iterative way. If one model of $\Gamma^{(\leq)}$ extends another, then the former satisfies at least as many elements of Γ as the latter does. It is natural to then consider maximal models of $\Gamma^{(\leq)}$. We show (Proposition 8) that maximal models of $\Gamma^{(\leq)}$ involve every evaluation except the ones in $C^{\perp}$, and satisfy every element of Γ except the strict statements in $\Gamma^{\perp}$. This implies that all maximal models of $\Gamma^{(\leq)}$ involve the same evaluations and satisfy the same subset of Γ. Thus to determine if Γ is $\mathcal{C}(1)$-consistent, we just have to generate any maximal model of $\Gamma^{(\leq)}$ (see Theorems 2 and 3), which can be done with a simple greedy algorithm, and test if this model satisfies Γ.

A nice mathematical property of this form of preference inference is compactness (see Corollary 2): any inference from an infinite set Γ also follows from some finite subset of it.

Our notion of preference inference is an intuitive one; however, there are also natural variations based on only considering models that involve all the evaluations; or alternatively, only considering maximal models. We explore such variations of preference inference in Section 4.3, and show strong connections with the main notion of preference inference. In Section 4.4 we show how the preference inference based on simple lexicographic models is very closely related to a logic based on disjunctive ordering statements.

4.1 Some basic definitions and results

We write $\varphi \in \mathcal{L}^{\mathcal{A}}$ as $\alpha_\varphi < \beta_\varphi$, if φ is strict, or as $\alpha_\varphi \leq \beta_\varphi$, if φ is non-strict. We consider a set $\Gamma \subseteq \mathcal{L}^{\mathcal{A}}$, and a set $\mathcal{C}$ of evaluations on $\mathcal{A}$.

***Supp*$^\varphi_\mathcal{C}$, *Opp*$^\varphi_\mathcal{C}$ and *Ind*$^\varphi_\mathcal{C}$:** For $\varphi \in \Gamma$, define *Supp*$^\varphi_\mathcal{C}$ to be $\{c \in \mathcal{C} : c(\alpha_\varphi) < c(\beta_\varphi)\}$; define *Opp*$^\varphi_\mathcal{C}$ to be $\{c \in \mathcal{C} : c(\alpha_\varphi) > c(\beta_\varphi)\}$; and define *Ind*$^\varphi_\mathcal{C}$ to be $\{c \in \mathcal{C} : c(\alpha_\varphi) = c(\beta_\varphi)\}$. Thus, *Supp*$^\varphi_\mathcal{C}$, *Opp*$^\varphi_\mathcal{C}$ and *Ind*$^\varphi_\mathcal{C}$ form a partition of $\mathcal{C}$, for any $\varphi \in \mathcal{L}^{\mathcal{A}}$. Note that these three sets do not depend on whether φ is strict or not. We may abbreviate *Supp*$^\varphi_\mathcal{C}$ to *Supp*$^\varphi$, and similarly for *Opp*$^\varphi_\mathcal{C}$ and *Ind*$^\varphi_\mathcal{C}$. *Supp*$^\varphi$ are the evaluations that support φ; *Opp*$^\varphi$ are the evaluations that oppose φ. *Ind*$^\varphi$ are the other evaluations, that are indifferent regarding φ. For a model H to satisfy φ it is necessary that no evaluation that opposes φ appears before all evaluations that support φ. More precisely, we have the following:

Lemma 5. *Let H be an element of $\mathcal{C}(1)$, i.e., a sequence of different elements of $\mathcal{C}$. For strict φ, $H \models \varphi$ if and only if an element of Supp$^\varphi_\mathcal{C}$ appears in H which appears before any (if there are any) element in Opp$^\varphi_\mathcal{C}$ that appears. For non-strict φ, $H \models \varphi$ if and only if an element of Supp$^\varphi_\mathcal{C}$ appears in H before any element in Opp$^\varphi_\mathcal{C}$ appears, or no element of Opp$^\varphi_\mathcal{C}$ appears in H (i.e., $\sigma(H) \cap$ Opp$^\varphi_\mathcal{C} = \emptyset$).*

Proof: Let $H = (c_1, \dots c_k)$ be a $\mathcal{C}(1)$-model. Suppose that φ is a strict statement. Then $H \models \varphi$, i.e., $\alpha_\varphi \prec_H \beta_\varphi$, if and only if there exists some $i \in \{1, \dots, k\}$ such that $\{c_1, \dots c_{i-1}\} \subseteq$ *Ind*$^\varphi$ and $c_i \in$ *Supp*$^\varphi_\mathcal{C}$, which is if and only if an element of *Supp*$^\varphi_\mathcal{C}$ appears in H before any element in *Opp*$^\varphi_\mathcal{C}$ appears.

Now suppose that φ is a non-strict statement. Then $H \models \varphi$, i.e., $\alpha_\varphi \preccurlyeq_H \beta_\varphi$, if and only if either (i) for all $i = 1, \dots, k$, $\alpha \equiv_{c_i} \beta$; or (ii) there exists some $i \in \{1, \dots, k\}$ such that $\alpha \prec_{c_i} \beta$ and for all j such that $1 \leq j < i$, $\alpha \equiv_{c_j} \beta$. (i) holds if and only if $\sigma(H) \subseteq$ *Ind*$^\varphi$, i.e., no element of *Supp*$^\varphi_\mathcal{C}$ or *Opp*$^\varphi_\mathcal{C}$ appears in H. (ii) holds if and only if an element of *Supp*$^\varphi_\mathcal{C}$ appears in H before any element in *Opp*$^\varphi_\mathcal{C}$ appears, and some element of *Supp*$^\varphi_\mathcal{C}$ appears in H. Thus, $H \models \varphi$ holds if and only if either no element in *Opp*$^\varphi_\mathcal{C}$ appears in H or some element of *Supp*$^\varphi_\mathcal{C}$ appears in H and the first such element appears before any element in *Opp*$^\varphi_\mathcal{C}$ appears. $\square$

The following defines *inconsistency bases*, which are concerned with evaluations that cannot appear in any model satisfying the set of preference statements Γ (see Proposition 3 below). They are a valuable tool in understanding the structure of the set of satisfying models (see e.g., Proposition 8 below).

Definition 1. *Let $\Gamma \subseteq \mathcal{L}^{\mathcal{A}}$, and let $\mathcal{C}$ be a set of $\mathcal{A}$-evaluations. We say that (Γ', C') is an inconsistency base for $(\Gamma, \mathcal{C})$ if $\Gamma' \subseteq \Gamma$, and $C' \subseteq \mathcal{C}$, and* *(i) for all $\varphi \in \Gamma'$, $\mathit{Supp}^{\varphi}_{\mathcal{C}} \cup \mathit{Opp}^{\varphi}_{\mathcal{C}} \subseteq C'$ (and thus $\mathcal{C} - C' \subseteq \mathit{Ind}^{\varphi}_{\mathcal{C}}$); and*

(ii) for all $c \in C'$, there exists $\varphi \in \Gamma'$ such that $\mathit{Opp}^{\varphi}_{\mathcal{C}} \ni c$.

Thus, for all $\varphi \in \Gamma'$, the set C' contains all evaluations that are not indifferent regarding φ, and for all $c \in C'$ there is some element of Γ' that is opposed by c.

Example 5. *Consider evaluations $\mathcal{C} = \{e, f, g, h\}$ with values for alternatives α, β, γ and δ as in the following table.*

	α	β	γ	δ
e	2	2	3	3
f	0	3	1	1
g	0	2	2	0
h	1	1	3	2

Consider the strict preference statement $\varphi_1 : \alpha < \beta$, and the non-strict preference statements $\varphi_2 : \beta \leq \gamma$, $\varphi_3 : \gamma \leq \delta$. Let $\Gamma = \{\varphi_1, \varphi_2, \varphi_3\}$. Then, $\mathit{Opp}^{\varphi_1}_{\mathcal{C}} = \emptyset$, $\mathit{Supp}^{\varphi_1}_{\mathcal{C}} = \{f, g\}$ and $\mathit{Ind}^{\varphi_1}_{\mathcal{C}} = \{e, h\}$. Similarly, $\mathit{Opp}^{\varphi_2}_{\mathcal{C}} = \{f\}$, $\mathit{Supp}^{\varphi_2}_{\mathcal{C}} = \{e, h\}$ and $\mathit{Ind}^{\varphi_2}_{\mathcal{C}} = \{g\}$. For φ_3, $\mathit{Opp}^{\varphi_3}_{\mathcal{C}} = \{g, h\}$, $\mathit{Supp}^{\varphi_3}_{\mathcal{C}} = \emptyset$ and $\mathit{Ind}^{\varphi_3}_{\mathcal{C}} = \{e, f\}$.

The HCLP model (e, f) satisfies Γ. As stated in Lemma 5, the evaluation $e \in \mathit{Supp}^{\varphi_2}_{\mathcal{C}}$ precedes the only element f in $\mathit{Opp}^{\varphi_2}_{\mathcal{C}}$. The tuple $(\Gamma', \mathcal{C}') = (\{\varphi_3\}, \{g, h\})$ is an inconsistency base of $(\Gamma, \mathcal{C})$. Condition (i) of Definition 1 is satisfied by $\mathit{Supp}^{\varphi_3}_{\mathcal{C}} \cup \mathit{Opp}^{\varphi_3}_{\mathcal{C}} = \{g, h\} \subseteq C'$. Since for $g, h \in C'$, $g \in \mathit{Opp}^{\varphi_3}_{\mathcal{C}}$ and $h \in \mathit{Opp}^{\varphi_3}_{\mathcal{C}}$, condition (ii) is satisfied as well.

The following result motivates the definition of inconsistency bases, showing that no model of Γ can involve any element of C', and that if Γ' contains a strict element then Γ is $\mathcal{C}(1)$-inconsistent.

Proposition 3. *Let (Γ', C') be an inconsistency base for $(\Gamma, \mathcal{C})$. Let H be an element of $\mathcal{C}(1)$. If $H \models \Gamma'$ then $C' \cap \sigma(H) = \emptyset$ and for any $\varphi \in \Gamma'$, $\alpha_\varphi \equiv_H \beta_\varphi$, so $H \not\models \alpha_\varphi < \beta_\varphi$. In particular, no $\mathcal{C}(1)$ model of Γ can involve any element of C'. Also, if Γ is $\mathcal{C}(1)$-consistent then Γ' contains no strict preference statements.*

Proof: Let (Γ', C') be an inconsistency base for $(\Gamma, \mathcal{C})$. Let $H = (c_1, \ldots c_k)$ be an element of $\mathcal{C}(1)$ with $H \models \Gamma'$. Suppose H contains some element in C' and let c_i be the element in $C' \cap \sigma(H)$ with the smallest index. By Definition 1(ii), there exists $\varphi \in \Gamma'$ such that $\mathit{Opp}^{\varphi}_{\mathcal{C}} \ni c_i$. Furthermore, since $c_j \notin C'$ for all $1 \leq j < i$, Definition 1(i) implies $c_j \in \mathit{Ind}^{\varphi}_{\mathcal{C}}$.

But then, an evaluation that opposes φ appears before all evaluations that support φ. By Lemma 5, this is a contradiction to $H \models \Gamma'$; hence we must have $C' \cap \sigma(H) = \emptyset$. Also, for all $\varphi \in \Gamma'$, $\sigma(H) \subseteq \mathcal{C} - C' \subseteq \mathit{Ind}_{\mathcal{C}}^{\varphi}$ by Definition 1(i). Therefore, for any $\varphi \in \Gamma'$, $\alpha_\varphi \equiv_H \beta_\varphi$, and thus $H \not\models \alpha_\varphi < \beta_\varphi$. Since $H \models \Gamma'$, this implies that Γ' contains no strict elements. The last parts follow from the fact that Γ' is a subset of Γ, so if $H \models \Gamma$ then $H \models \Gamma'$. □

We next give a small technical lemma that will be useful later. In particular, part (i) will be used in proving compactness of preference inference.

Lemma 6. *Assume that* (Γ', C') *is an inconsistency base for* $(\Gamma, \mathcal{C})$. *Then the following hold.*

(i) There exists a finite set $\Gamma'' \subseteq \Gamma$ *such that* (Γ'', C') *is an inconsistency base for* $(\Gamma, \mathcal{C})$, *and if* Γ' *contains a strict statement then* Γ'' *does also.*

(ii) For any Δ *such that* $\Gamma' \subseteq \Delta \subseteq \Gamma$, (Γ', C') *is an inconsistency base for* $(\Delta, \mathcal{C})$.

Proof: (i): By condition (ii) of the definition of an inconsistency base, for each $c \in C'$, there exists $\varphi_c \in \Gamma'$ such that $\mathit{Opp}_{\mathcal{C}}^{\varphi_c} \ni c$. If Γ' contains a strict statement ψ then let $\Gamma'' = \{\psi\} \cup \{\varphi_c : c \in C'\}$; else let $\Gamma'' = \{\varphi_c : c \in C'\}$. Because $\mathcal{C}$ is finite, Γ'' is finite. The definition implies that (Γ'', C') is an inconsistency base for $(\Gamma, \mathcal{C})$.

Part (ii) follows immediately from Definition 1, since conditions (i) and (ii) of the definition do not directly refer to Γ, but just to Γ', which is a subset of Γ. □

We will show there is, in a natural sense, a unique maximal inconsistency base for $(\Gamma, \mathcal{C})$.

For inconsistency bases (Γ_1, C_1) and (Γ_2, C_2) for $(\Gamma, \mathcal{C})$, define $(\Gamma_1, C_1) \cup (\Gamma_2, C_2)$ to be $(\Gamma_1 \cup \Gamma_2, C_1 \cup C_2)$. More generally, for inconsistency bases (Γ_i, C_i), $i \in I$, we define $\cup_{i \in I}(\Gamma_i, C_i)$ to be $(\cup_{i \in I}\Gamma_i, \cup_{i \in I} C_i)$, which can be easily shown to be an inconsistency base.

Lemma 7. *Suppose, for some (finite or infinite) non-empty index set* I, *and for all* $i \in I$, *that* (Γ_i, C_i) *is an inconsistency base. Then* $\cup_{i \in I}(\Gamma_i, C_i)$ *is an inconsistency base.*

Proof: For all $i \in I$, by Definition 1(i), for all $\varphi \in \Gamma_i$, $\mathit{Supp}_{\mathcal{C}}^{\varphi} \cup \mathit{Opp}_{\mathcal{C}}^{\varphi} \subseteq C_i$; thus, for all $\varphi \in \cup_{i \in I}\Gamma_i$, $\mathit{Supp}_{\mathcal{C}}^{\varphi} \cup \mathit{Opp}_{\mathcal{C}}^{\varphi} \subseteq \cup_{i \in I} C_i$. This proves condition (i). To prove condition (ii): for all $i \in I$, by Definition 1(ii), for all $c \in C_i$, there exists $\varphi \in \Gamma_i$ such that $\mathit{Opp}_{\mathcal{C}}^{\varphi} \ni c$. Thus, for all $c \in \cup_{i \in I} C_i$, there exists $\varphi \in \cup_{i \in I}\Gamma_i$ such that $\mathit{Opp}_{\mathcal{C}}^{\varphi} \ni c$. □

Define $\mathit{MIB}(\Gamma, \mathcal{C})$, the *maximal inconsistency base for* $(\Gamma, \mathcal{C})$, to be the union of all inconsistency bases for $(\Gamma, \mathcal{C})$, i.e., $\bigcup\{(\Gamma', C') \in \mathcal{I}\}$, where $\mathcal{I}$ is the set of inconsistency bases for $(\Gamma, \mathcal{C})$. This is well-defined, because $\mathcal{I}$ is non-empty, since it always contains the tuple $(\emptyset, \emptyset)$.

The next result states that *MIB*$(\Gamma, \mathcal{C})$ is an inconsistency base for $(\Gamma, \mathcal{C})$.

Proposition 4. *MIB$(\Gamma, \mathcal{C})$ is an inconsistency base for $(\Gamma, \mathcal{C})$, which is maximal in the following sense: if (Γ_1, C_1) is an inconsistency base for $(\Gamma, \mathcal{C})$ then $\Gamma_1 \subseteq \Gamma^\perp$ and $C_1 \subseteq C^\perp$, where MIB$(\Gamma, \mathcal{C}) = (\Gamma^\perp, C^\perp)$.*

Proof: By Lemma 7, the union of an arbitrary set of inconsistency bases is an inconsistency base. Consequently, *MIB*$(\Gamma, \mathcal{C})$ is an inconsistency base. Let *MIB*$(\Gamma, \mathcal{C}) = (\Gamma^\perp, C^\perp)$. The definition immediately implies that if (Γ_1, C_1) is an inconsistency base for $(\Gamma, \mathcal{C})$, then $\Gamma_1 \subseteq \Gamma^\perp$ and $C_1 \subseteq C^\perp$. □

By Proposition 3, if Γ is $\mathcal{C}(1)$-consistent then $\Gamma^\perp$ contains no strict elements, proving the next result. The converse also holds—see Corollary 1.

Proposition 5. *Suppose that Γ is $\mathcal{C}(1)$-consistent, i.e., there exists a $\mathcal{C}(1)$ model of Γ. Then for any inconsistency base (Γ', C') of $(\Gamma, \mathcal{C})$, $\Gamma' \cap \mathcal{L}^{\mathcal{A}}_{<} = \emptyset$. In particular, if MIB$(\Gamma, \mathcal{C}) = (\Gamma^\perp, C^\perp)$ then $\Gamma^\perp \cap \mathcal{L}^{\mathcal{A}}_{<} = \emptyset$.*

Example 6. *Consider the HCLP structure and preference statements as in Example 5. The only inconsistency bases of $(\Gamma, \mathcal{C})$ are $(\emptyset, \emptyset)$ and $(\{\varphi_3\}, \{g, h\})$. Thus, $(\{\varphi_3\}, \{g, h\})$ is the maximal inconsistency base MIB$(\Gamma, \mathcal{C})$ and does not contain any strict statements of Γ.*

In the following sections, it will be important to consider models extending other models.

Definition 2. *For $H, H' \in \mathcal{C}(1)$, write H as $(c_1, \ldots, c_k)$ and $H' = (c'_1, \ldots, c'_l)$. we say that H' extends H if $l > k$ and for all $j = 1, \ldots, k$, $c'_j = c_j$.*

Lemma 8. *Suppose that $H, H' \in \mathcal{C}(1)$ and that H' extends H. Then,*

(i) If $H \models \alpha < \beta$ then $H' \models \alpha < \beta$.

(ii) If $H' \models \alpha \leq \beta$ then $H \models \alpha \leq \beta$.

Proof: (i) Suppose that $H \models \alpha < \beta$, so that $\alpha \prec_H \beta$. Write H as $(c_1, \ldots, c_k)$. For some i, $c_i(\alpha) \neq c_i(\beta)$; and let i be minimal such that $c_i(\alpha) \neq c_i(\beta)$. Since $\alpha \prec_H \beta$, we have $c_i(\alpha) < c_i(\beta)$. Because, H' extends H, this implies that $\alpha \prec_{H'} \beta$, i.e., $H' \models \alpha < \beta$.

(ii) Suppose that $H' \models \alpha \leq \beta$. Then $H' \not\models \beta < \alpha$, by Lemma 1. Part (i) implies that $H \not\models \beta < \alpha$, and thus $H \models \alpha \leq \beta$, using Lemma 1 again. □

4.2 Towards a polynomial algorithm for consistency and deduction

Throughout this section we consider a set $\Gamma \subseteq \mathcal{L}^{\mathcal{A}}$ of input preference statements, and a set $\mathcal{C}$ of $\mathcal{A}$-evaluations.

Define $Opp_{\Gamma}(c)$ (usually abbreviated to $Opp(c)$) to be the set of elements opposed by c, i.e., $\varphi \in \Gamma$ such that $c(\alpha_\varphi) > c(\beta_\varphi)$, and define $Supp_{\Gamma}(c)$ (abbreviated to $Supp(c)$) to be the set of elements φ of Γ supported by c, (i.e., $c(\alpha_\varphi) < c(\beta_\varphi)$). For for $C' \subseteq \mathcal{C}$, we define $Supp_{\Gamma}(C')$ to be the elements of Γ that are supported by some element of C', i.e., $Supp(C') = \bigcup_{c \in C'} Supp(c)$. Also, for sequence of evaluations $(c_1, \ldots, c_k)$, we define $Supp(c_1, \ldots, c_k)$ to be $\bigcup_{i=1}^{k} Supp(c_i)$, which equals $Supp(\{c_1, \ldots, c_k\})$.

We thus have $\varphi \in Supp(c) \iff c(\alpha_\varphi) < c(\beta_\varphi) \iff c \in Supp^{\varphi}$; and $\varphi \in Opp(c) \iff c(\alpha_\varphi) > c(\beta_\varphi) \iff c \in Opp^{\varphi}$.

$\Gamma^{(\leq)}$, the non-strict version of Γ: It turns out to be helpful to consider a non-strict version of Γ; we define $\Gamma^{(\leq)}$ to be $\{\alpha_\varphi \leq \beta_\varphi : \varphi \in \Gamma\}$, i.e., Γ where the strict statements are replaced by corresponding non-strict statements. Clearly, if $H \models \Gamma$ then $H \models \Gamma^{(\leq)}$ (since $H \models \alpha < \beta$ implies $H \models \alpha \leq \beta$).

The next lemma follows immediately, since the definition of maximal inconsistency base does not depend on whether elements of Γ are strict or not.

Lemma 9. *For any Γ and $\mathcal{C}$, $MIB(\Gamma^{(\leq)}, \mathcal{C}) = MIB(\Gamma, \mathcal{C})$.*

In order to determine the consistency of set of preference statements Γ, we want a method for generating a model $H \in \mathcal{C}(1)$ satisfying Γ. (Determining (non-)inference can be similarly performed by generating a model satisfying $\Gamma \cup \{\neg\varphi\}$, using Proposition 1.) A necessary condition for $H \models \Gamma$ is $H \models \Gamma^{(\leq)}$. There is a simple necessary and sufficient condition for $H \models \Gamma^{(\leq)}$, where $H = (c_1, \ldots, c_k)$, which is that every $\varphi \in \Gamma$ that is opposed by c_j is supported by some earlier element in the sequence (see Proposition 6). This condition allows one to easily incrementally grow models of $\Gamma^{(\leq)}$, until one has a maximal model of $\Gamma^{(\leq)}$. We only need to consider maximal models because if a model H of $\Gamma^{(\leq)}$ satisfies Γ then any maximal model of $\Gamma^{(\leq)}$ extending H satisfies Γ (see Lemma 11). The results about maximal inconsistency bases allow us to show (Theorem 2) that if Γ is consistent then any maximal model of $\Gamma^{(\leq)}$ satisfies Γ, so to determine consistency of Γ we just need to generate any maximal model of $\Gamma^{(\leq)}$, which can be done in a straight-forward iterative way. This is the basis of the algorithm.

4.2.1 Γ-allowed sequences, i.e., models of $\Gamma^{(\leq)}$

We define the notion of Γ-allowed sequence, which turns out to be the same as a model of $\Gamma^{(\leq)}$ (see Proposition 6), and derive important properties (Proposition 7), which are useful for deriving the main results about maximal Γ-allowed sequences in Section 4.2.2.

Define $\mathit{Next}_\Gamma(C')$ to be the set of all $c \in \mathcal{C} - C'$ such that $\mathit{Opp}(c) \subseteq \mathit{Supp}(C')$, i.e., the set of $c \in \mathcal{C} - C'$ that only oppose elements in Γ that are supported by elements of C'. The following result gives an equivalent condition for $c \in \mathit{Next}_\Gamma(C')$.

Lemma 10. *Consider any $c \in \mathcal{C}$. Then, $c \in \mathit{Next}_\Gamma(C')$, i.e., $\mathit{Opp}(c) \subseteq \mathit{Supp}(C')$, if and only if for all $\varphi \in \Gamma - \mathit{Supp}(C')$, $c \in \mathit{Supp}^\varphi \cup \mathit{Ind}^\varphi$.*

Proof: Suppose first that $\mathit{Opp}(c) \subseteq \mathit{Supp}(C')$, and consider any $\varphi \in \Gamma - \mathit{Supp}(C')$. Since $\varphi \notin \mathit{Supp}(C')$, then $\varphi \notin \mathit{Opp}(c)$, and thus, $c \notin \mathit{Opp}^\varphi$. This implies that $c \in \mathit{Supp}^\varphi \cup \mathit{Ind}^\varphi$.

Conversely, suppose that for all $\varphi \in \Gamma - \mathit{Supp}(C')$, $c \in \mathit{Supp}^\varphi \cup \mathit{Ind}^\varphi$. Consider any $\varphi \in \mathit{Opp}(c)$. Then $c \in \mathit{Opp}^\varphi$ and so $c \notin \mathit{Supp}^\varphi \cup \mathit{Ind}^\varphi$, and therefore, $\varphi \in \mathit{Supp}(C')$. □

Consider an arbitrary sequence $H = (c_1, \ldots, c_k)$ of elements of $\mathcal{C}$. Let us say that H is a *Γ-allowed sequence (of $\mathcal{C}$)* if for all $j = 1, \ldots, k$, $c_j \in \mathit{Next}(\{c_1, \ldots, c_{j-1}\})$, i.e., $\mathit{Opp}(c_j) \subseteq \mathit{Supp}(\{c_1, \ldots, c_{j-1}\})$. These turn out to be just models of $\Gamma^{(\leq)}$.

Example 7. *Consider the HCLP structure as in Example 5 and preference statements $\Gamma = \{\varphi_1, \varphi_2\}$ with $\varphi_1 : \alpha < \beta$ and $\varphi_2 : \beta \leq \gamma$. Then $H = (h, f, e)$ is a Γ-allowed sequence since:*

- *$e \in \mathit{Next}(\{h, f\})$, i.e., $\mathit{Opp}(e) = \emptyset \subseteq \mathit{Supp}(\{h, f\}) = \{\varphi_1, \varphi_2\}$.*
- *$f \in \mathit{Next}(\{h\})$, i.e., $\mathit{Opp}(f) = \{\varphi_2\} \subseteq \mathit{Supp}(\{h\}) = \{\varphi_2\}$.*
- *$h \in \mathit{Next}(\emptyset)$, i.e., $\mathit{Opp}(h) = \emptyset \subseteq \mathit{Supp}(\emptyset) = \emptyset$.*

H satisfies both preference statements in Γ.

Proposition 6. *Consider an arbitrary sequence $H = (c_1, \ldots, c_k)$ of elements of $\mathcal{C}$. Then, $H \models \Gamma^{(\leq)}$ if and only H is a Γ-allowed sequence.*

Proof: Suppose that $H \not\models \Gamma^{(\leq)}$, so there exists some $\varphi \in \Gamma$ such that $H \not\models \alpha_\varphi \leq \beta_\varphi$. If all elements c_j of H were indifferent to φ (i.e., $c_j(\alpha_\varphi) = c_j(\beta_\varphi)$) then we would have $H \models \alpha_\varphi \leq \beta_\varphi$. Thus, some element c_j in H is not indifferent to φ; let c_i be the first such element in H. If it were the case that $c_i(\alpha_\varphi) < c_i(\beta_\varphi)$ then we would have $H \models \alpha_\varphi \leq \beta_\varphi$, so we must have $c_i(\alpha_\varphi) > c_i(\beta_\varphi)$, and thus, $\varphi \in \mathit{Opp}(c_i)$. Now, $\varphi \notin \mathit{Supp}(\{c_1, \ldots, c_{i-1}\})$, since $c_j(\alpha_\varphi) = c_j(\beta_\varphi)$ for all $j < i$, and hence, $\mathit{Opp}(c_i) \not\subseteq \mathit{Supp}(\{c_1, \ldots, c_{i-1}\})$. This shows that $c_i \notin \mathit{Next}(\{c_1, \ldots, c_{i-1}\})$, and so H is not a Γ-allowed sequence.

Conversely, suppose that for some $j \in \{1, \ldots, k\}$, $c_j \notin \mathit{Next}(\{c_1, \ldots, c_{j-1}\})$, and let c_i be the first such c_j. Then for all $j < i$, $c_j \in \mathit{Next}(\{c_1, \ldots, c_{j-1}\})$. Since $c_i \notin \mathit{Next}(\{c_1, \ldots, c_{i-1}\})$, there exists some $\varphi \in \Gamma - \mathit{Supp}(\{c_1, \ldots, c_{i-1}\})$ such that $\varphi \in$

Opp(c_i). so that $c_i(\alpha_\varphi) > c_i(\beta_\varphi)$. Let j be minimal such that $c_j(\alpha_\varphi) \neq c_j(\beta_\varphi)$. Since $\varphi \notin$ *Supp*$(\{c_1, \ldots, c_{i-1}\})$, we do not have $c_j(\alpha_\varphi) < c_j(\beta_\varphi)$, so we must have $c_j(\alpha_\varphi) > c_j(\beta_\varphi)$. This implies that $H \not\models \alpha_\varphi \leq \beta_\varphi$, where $\alpha_\varphi \leq \beta_\varphi$ is an element of $\Gamma^{(\leq)}$, and thus $H \not\models \Gamma^{(\leq)}$. $\square$

We also have the following property of Γ-allowed sequences.

Proposition 7. *Suppose that H is a Γ-allowed sequence. Then, for all $\varphi \in$ Supp(H), $H \models \alpha_\varphi < \beta_\varphi$, and for all $\varphi \in \Gamma -$ Supp(H), $\alpha_\varphi \equiv_H \beta_\varphi$, so, in particular $H \not\models \alpha_\varphi < \beta_\varphi$. Thus, for $\varphi \in \Gamma$, we have $H \models \alpha_\varphi < \beta_\varphi$ if and only if $\varphi \in$ Supp(H). Also, $H \models \Gamma$ if and only if every strict element of Γ is in Supp(H).*

Proof: First, consider any $\varphi \in$ *Supp*(H). Thus there exists $c_j \in \sigma(H)$ such that $c_j(\alpha_\varphi) < c_j(\beta_\varphi)$, so, in particular, $c_j(\alpha_\varphi) \neq c_j(\beta_\varphi)$. Let i be minimal such that $c_i(\alpha_\varphi) \neq c_i(\beta_\varphi)$. Proposition 6 implies that $H \models \alpha_\varphi \leq \beta_\varphi$, which implies that $c_i(\alpha_\varphi) \not> c_i(\beta_\varphi)$, and thus $c_i(\alpha_\varphi) < c_i(\beta_\varphi)$, proving that $H \models \alpha_\varphi < \beta_\varphi$.

Now, consider $\varphi \in \Gamma -$ *Supp*(H). If it were the case that there exists $c_j \in \sigma(H)$ such that $c_j(\alpha_\varphi) \neq c_j(\beta_\varphi)$, then the argument above implies that there exists i such that $c_i(\alpha_\varphi) < c_i(\beta_\varphi)$, and thus $\varphi \in$ *Supp*(H). Thus, for all $c_j \in \sigma(H)$, $c_j(\alpha_\varphi) = c_j(\beta_\varphi)$, and, hence, $\alpha_\varphi \equiv_H \beta_\varphi$.

For the last part, since, by Proposition 6, $H \models \Gamma^{(\leq)}$, we have: $H \models \Gamma$ if and only if for every strict element φ of Γ, $H \models \alpha_\varphi < \beta_\varphi$, i.e., $\varphi \in$ *Supp*(H). $\square$

4.2.2 Maximal Γ-allowed sequences, i.e., maximal models of $\Gamma^{(\leq)}$

We say that H is a maximal Γ-allowed sequence of $\mathcal{C}$ if H is a Γ-allowed sequence of $\mathcal{C}$ and no extension of H is a Γ-allowed sequence of $\mathcal{C}$, i.e., *Next*$(\sigma(H)) = \emptyset$. More generally, when talking about maximal models, with respect to some set of models $\mathcal{D}$, we mean maximality with respect to the extension relation, so a model in $\mathcal{D}$ is ($\mathcal{D}$-)maximal if there is no element of $\mathcal{D}$ that extends it.

Lemma 11. *Suppose that $H, H' \in \mathcal{C}$ and $H, H' \models \Gamma^{(\leq)}$, and that H' extends H. Then for all $\varphi \in \Gamma$, if $H \models \varphi$ then $H' \models \varphi$. In particular, if $H \models \Gamma$ then $H' \models \Gamma$.*

Proof: Assume that $H, H' \models \Gamma^{(\leq)}$, and H' extends H. Consider any $\varphi \in \Gamma$, and suppose that $H \models \varphi$. If φ is non-strict then $\varphi \in \Gamma^{(\leq)}$ and so $H' \models \varphi$. If φ is strict, then Lemma 8(i) implies that $H' \models \varphi$. $\square$

We use this in proving the next result, which shows that if we are interested in finding models of Γ it is sufficient to only consider maximal Γ-allowed sequences, i.e., maximal models of $\Gamma^{(\leq)}$.

Lemma 12. *If H is a Γ-allowed sequence, then either H is a maximal Γ-allowed sequence or there exists a maximal Γ-allowed sequence H' that extends H. Then, for all $\varphi \in \Gamma$, if $H \models \varphi$ then $H' \models \varphi$. In particular, if $H \models \Gamma$ then $H' \models \Gamma$.*

Proof: The *extends* relation on the finite set of Γ-allowed sequences is transitive and acyclic. It follows that for any Γ-allowed sequence H there exists a maximal Γ-allowed sequence extending H. The last part follows from previous result, Lemma 11 (using the equivalence stated by Proposition 6). $\square$

The following key lemma shows the close relationship between maximal Γ-allowed sequences and the maximal inconsistency base.

Lemma 13. *Suppose that H is a maximal Γ-allowed sequence. Then $(\Gamma - \mathit{Supp}(H), \mathcal{C} - \sigma(H))$ equals $\mathit{MIB}(\Gamma, \mathcal{C})$.*

Proof: We first check the two conditions in the definition of an inconsistency base (see Definition 1). Consider any element φ of $\Gamma - \mathit{Supp}(H)$. Proposition 7 implies that $\alpha_\varphi \equiv_H \beta_\varphi$, so that for all $c \in \sigma(H)$, $c(\alpha_\varphi) = c(\beta_\varphi)$, and so $\sigma(H) \subseteq \mathit{Ind}^\varphi$, showing that Condition (i) holds. Now, consider any evaluation c in $\mathcal{C} - \sigma(H)$. By definition of a maximal Γ-allowed sequence, $\mathit{Next}(\sigma(H)) = \emptyset$, so $c \notin \mathit{Next}(\sigma(H))$. Therefore, by Lemma 10, there exists $\varphi \in \Gamma - \mathit{Supp}(H)$ such that $c \notin \mathit{Supp}^\varphi \cup \mathit{Ind}^\varphi$, so $c \in \mathit{Opp}^\varphi$, showing that Condition (ii) of an inconsistency base holds.

Write $\mathit{MIB}(\Gamma, \mathcal{C})$ as $(\Gamma^\perp, C^\perp)$. Thus, by definition, $\Gamma - \mathit{Supp}(H) \subseteq \Gamma^\perp$ and $\mathcal{C} - \sigma(H) \subseteq C^\perp$. Proposition 6 implies that $H \models \Gamma^{(\leq)}$. Lemma 9 implies that $\mathit{MIB}(\Gamma^{(\leq)}, \mathcal{C}) = (\Gamma^\perp, C^\perp)$. Proposition 3 then implies that $C^\perp \cap \sigma(H) = \emptyset$, and so, $\mathcal{C} - \sigma(H) \supseteq C^\perp$. Thus, $\mathcal{C} - \sigma(H) = C^\perp$.

Consider any $\varphi \in \Gamma^\perp$. By definition of an inconsistency base, $\mathcal{C} - C^\perp \subseteq \mathit{Ind}^\varphi$, i.e., $\sigma(H) \subseteq \mathit{Ind}^\varphi$, which implies $\alpha_\varphi \equiv_H \beta_\varphi$, and so, by Proposition 7, $\varphi \in \Gamma - \mathit{Supp}(H)$. Thus, $\Gamma^\perp \subseteq \Gamma - \mathit{Supp}(H)$, and hence, $\Gamma^\perp = \Gamma - \mathit{Supp}(H)$, completing the proof that $(\Gamma - \mathit{Supp}(H), \mathcal{C} - \sigma(H))$ equals $(\Gamma^\perp, C^\perp)$. $\square$

Different maximal Γ-allowed sequences satisfy the same subset of Γ and involve the same subset of $\mathcal{C}$:

Proposition 8. *Suppose that H is a maximal Γ-allowed sequence. Write $\mathit{MIB}(\Gamma, \mathcal{C})$ as $(\Gamma^\perp, C^\perp)$. Then $\Gamma^\perp = \Gamma - \mathit{Supp}(H)$ and $C^\perp = \mathcal{C} - \sigma(H)$. Thus, if H' is another maximal Γ-allowed sequence, then $\sigma(H') = \sigma(H)$ and $\mathit{Supp}(H') = \mathit{Supp}(H)$. Also, for all $\varphi \in \Gamma$, $H \models \varphi \iff H' \models \varphi$, which is if and only if φ is not a strict element of $\Gamma^\perp$. Hence, every maximal Γ-allowed sequence satisfies the same elements of Γ.*

Proof: By Lemma 13, $\Gamma^\perp = \Gamma - \mathit{Supp}(H)$ and $C^\perp = \mathcal{C} - \sigma(H)$. For any maximal Γ-allowed sequence H', $\sigma(H') = \mathcal{C} - C^\perp = \sigma(H)$, and $\mathit{Supp}(H') = \Gamma - \Gamma^\perp = \mathit{Supp}(H)$.

To prove the last part, suppose that $\varphi \in \Gamma$ is such that $H \not\models \varphi$. Proposition 6 implies that φ is strict. Proposition 7 implies that $\varphi \notin \mathit{Supp}(H)$ and thus $\varphi \in \Gamma^\perp$. Conversely, if φ is strict and $\varphi \in \Gamma^\perp$ then $\varphi \notin \mathit{Supp}(H)$, so $H \not\models \varphi$ by Proposition 7. We have shown, for $\varphi \in \Gamma$, that $H \models \varphi$ if and only if φ is not a strict element of $\Gamma^\perp$; the same argument applies to H', so $H \models \varphi \iff H' \models \varphi$. □

No model of $\Gamma^{(\leq)}$ satisfies any element of Γ that is not satisfied by a maximal Γ-allowed sequence H.

Proposition 9. *Consider any maximal Γ-allowed sequence H, and any $H' \in \mathcal{C}(1)$ such that $H' \models \Gamma^{(\leq)}$. For any $\varphi \in \Gamma$, if $H' \models \varphi$ then $H \models \varphi$.*

Proof: Suppose that $\varphi \in \Gamma$ and $H \not\models \varphi$, and so, by Proposition 7, φ is strict and $\varphi \in \Gamma - \mathit{Supp}(H)$. Consider any model $H' \models \Gamma^{(\leq)}$. By Proposition 6, H' is a Γ-allowed sequence. By Lemma 12, there exists some maximal Γ-allowed sequence H'' that extends or equals H'. We have $\mathit{Supp}(H') \subseteq \mathit{Supp}(H'')$. Proposition 8 implies that $\mathit{Supp}(H) = \mathit{Supp}(H'')$, so $\varphi \notin \mathit{Supp}(H')$. Since φ is strict, $H' \not\models \varphi$, again using Proposition 7. □

The theorem below shows that to test consistency, one just needs to generate a single maximal Γ-allowed sequence (i.e., maximal model of $\Gamma^{(\leq)}$), which can be easily done using an iterative algorithm.

Theorem 2. *Γ is $\mathcal{C}(1)$-consistent if and only if some maximal Γ-allowed sequence satisfies Γ, which is if and only if every maximal Γ-allowed sequence satisfies Γ.*

Proof: First assume that Γ is $\mathcal{C}(1)$-consistent, so there exists some HCLP model $H \in \mathcal{C}(1)$ such that $H \models \Gamma$. This trivially implies that $H \models \Gamma^{(\leq)}$ (since $H \models \alpha < \beta \Rightarrow H \models \alpha \leq \beta$), so by Proposition 6, H is a Γ-allowed sequence. By Lemma 12, there exists a maximal Γ-allowed sequence H' that extends or equals H, and $H' \models \Gamma$. We have proved that some maximal Γ-allowed sequence satisfies Γ. The converse is obvious: if some maximal Γ-allowed sequence satisfies Γ then Γ is $\mathcal{C}(1)$-consistent. The last part of Proposition 8 implies that some maximal Γ-allowed sequence satisfies Γ, if and only if every maximal Γ-allowed sequence satisfies Γ. □

This leads to a simple characterisation of $\mathcal{C}(1)$-consistency using the maximal inconsistency base: Γ is $\mathcal{C}(1)$-consistent if and only if no inconsistency base involves any strict element of Γ.

Corollary 1. *Write $\mathit{MIB}(\Gamma, \mathcal{C})$ as $(\Gamma^\perp, C^\perp)$. Γ is $\mathcal{C}(1)$-consistent if and only if $\Gamma^\perp \cap \mathcal{L}^{\mathcal{A}}_{<} = \emptyset$, which is if and only if $\Gamma^\perp$ is $\mathcal{C}(1)$-consistent. If Γ is $\mathcal{C}(1)$-inconsistent then there exists a* *finite set* $\Gamma' \subseteq \Gamma^{\perp}$ *such that* Γ' *is* $\mathcal{C}(1)$*-inconsistent, and* $(\Gamma', C^{\perp})$ *is an inconsistency base for* $(\Gamma, \mathcal{C})$.

Proof: Let $\Gamma_{<} = \Gamma \cap \mathcal{L}^{\mathcal{A}}_{<}$. First, suppose that Γ is $\mathcal{C}(1)$-consistent. Then, by Theorem 2, any maximal Γ-allowed sequence H satisfies Γ. By Proposition 7, $\Gamma_{<} \subseteq \mathit{Supp}(H)$, and thus, $\Gamma_{<} \subseteq \Gamma - \Gamma^{\perp}$, by Proposition 8. Hence, $\Gamma_{<} \cap \Gamma^{\perp} = \emptyset$, and so $\Gamma^{\perp} \cap \mathcal{L}^{\mathcal{A}}_{<} = \emptyset$.

Conversely, suppose that $\Gamma^{\perp} \cap \mathcal{L}^{\mathcal{A}}_{<} = \emptyset$. Proposition 8 implies that for any maximal Γ-allowed sequence H, $\Gamma - \Gamma^{\perp} = \mathit{Supp}(H)$ and thus, $\Gamma_{<} \subseteq \mathit{Supp}(H)$. Proposition 7 then implies that $H \models \Gamma$, and so Γ is $\mathcal{C}(1)$-consistent.

If Γ is $\mathcal{C}(1)$-consistent then $\Gamma^{\perp}$ is $\mathcal{C}(1)$-consistent, since $\Gamma^{\perp} \subseteq \Gamma$. Conversely, suppose that $\Gamma^{\perp}$ is $\mathcal{C}(1)$-consistent. Lemma 6 implies that $(\Gamma^{\perp}, C^{\perp})$ is an inconsistency base for $(\Gamma^{\perp}, \mathcal{C})$. Proposition 5 implies that $\Gamma^{\perp} \cap \mathcal{L}^{\mathcal{A}}_{<} = \emptyset$, which by the first part, implies that Γ is $\mathcal{C}(1)$-consistent.

Now suppose that Γ is $\mathcal{C}(1)$-inconsistent. The first part implies that $\Gamma^{\perp}$ contains a strict statement. By Lemma 6(i), there exists finite $\Gamma' \subseteq \Gamma^{\perp}$ such that (Γ', C') is an inconsistency base for $(\Gamma, \mathcal{C})$, and Γ' contains a strict statement. By Lemma 6(ii), (Γ', C') is an inconsistency base for $(\Gamma', \mathcal{C})$, and thus, by Proposition 5, Γ' is $\mathcal{C}(1)$-inconsistent, since it contains a strict statement. $\square$

The following result shows that this kind of preference inference is compact.

Corollary 2. *Consider any* $\Gamma \subseteq \mathcal{L}^{\mathcal{A}}$ *and* $\varphi \in \mathcal{L}^{\mathcal{A}}$.

(i) If Γ *is* $\mathcal{C}(1)$*-inconsistent then there exists finite* $\Gamma' \subseteq \Gamma$ *which is* $\mathcal{C}(1)$*-inconsistent.*

(ii) If $\Gamma \models_{\mathcal{C}(1)} \varphi$ *then there exists finite* $\Gamma' \subseteq \Gamma$ *such that* $\Gamma' \models_{\mathcal{C}(1)} \varphi$.

Proof: (i) Suppose that Γ is $\mathcal{C}(1)$-inconsistent. The last part of Corollary 1 implies that then there exists finite $\Gamma' \subseteq \Gamma$ which is $\mathcal{C}(1)$-inconsistent.

(ii) Suppose that $\Gamma \models_{\mathcal{C}(1)} \varphi$. Then $\Gamma \cup \{\neg\varphi\}$ is $\mathcal{C}(1)$-inconsistent, by Proposition 1. Part (i) implies that there exists finite $\mathcal{C}(1)$-inconsistent $\Delta \subseteq \Gamma \cup \{\neg\varphi\}$. If $\Delta \subseteq \Gamma$ then we can let $\Gamma' = \Delta$, since trivially $\Delta \models_{\mathcal{C}(1)} \varphi$. Otherwise, $\Delta \ni \varphi$, and we let $\Gamma' = \Delta - \{\varphi\}$. We have $\Gamma' \subseteq \Gamma$, and $\Gamma' \models_{\mathcal{C}(1)} \varphi$, again by Proposition 1. $\square$

4.2.3 The algorithm

The idea behind the algorithm is to build up a maximal $\Gamma^{(\leq)}$-satisfying sequence by repeatedly adding evaluations to the end; suppose that we have picked a sequence of elements of $\mathcal{C}$, C' being the set picked so far. We need to choose next an evaluation c such that, if c

opposes some φ in Γ, then φ is supported by some evaluation in C' (or else the generated sequence will not satisfy φ).

H is initialised as the empty sequence $()$ of evaluations. $H \leftarrow H + c$ means that evaluation c is added to the end of H.

Function *Cons-check*$(\Gamma, \mathcal{C})$

$H \leftarrow ()$
for $k = 1, \ldots, |\mathcal{C}|$ do
  if $\exists\, c \in \mathcal{C} - \sigma(H)$ such that $\mathit{Opp}(c) \subseteq \mathit{Supp}(H)$
    then choose some such c; $H \leftarrow H + c$
    else stop
  end for
return H

Note that at each stage, an element of $\mathit{Next}_\Gamma(\sigma(H))$ is chosen, so at each stage H is a Γ-allowed sequence. Also, the termination condition is equivalent to $\mathit{Next}_\Gamma(\sigma(H)) = \emptyset$, which implies that the returned H is a maximal Γ-allowed sequence.

The algorithm involves often non-unique choices. However, if we wish, the choosing of c can be done based on an ordering $c_1, \ldots, c_m$ of $\mathcal{C}$, where, if there exists more than one $c \in \mathcal{C} - \sigma(H)$ such that $\mathit{Opp}(c) \subseteq \mathit{Supp}(H)$, we choose the element c_i fulfilling this condition that has smallest index i. The algorithm then becomes deterministic, with a unique result following from the given inputs.

A straight-forward implementation runs in $O(|\Gamma||\mathcal{C}|^2)$ time; however, a more careful implementation runs in $O(|\Gamma||\mathcal{C}|)$ time, which we now describe. Let H_k be the HCLP model after the k-th iteration of the for-loop. In every iteration of the for-loop, we update sets $\mathit{Opp}_k^\Delta(c) = \mathit{Opp}(c) - \mathit{Supp}(H_k)$ and $\mathit{Supp}_k^\Delta(c) = \mathit{Supp}(c) - \mathit{Supp}(H_k)$ for all $c \in \mathcal{C} - \sigma(H_k)$. This costs us $O(|\mathcal{C} - \sigma(H_k)| \times |\mathit{Supp}(H_k) \setminus \mathit{Supp}(H_{k-1})|) = O(|\mathcal{C} - \sigma(H_k)| \times |\mathit{Supp}_{k-1}^\Delta(c_k)|)$ more time for every iteration k in which we add evaluation c_k to H_{k-1}. However, the choice of the next evaluation c_k can be performed in constant time by marking evaluations c with $\mathit{Opp}_{k-1}^\Delta(c) = \emptyset$. Suppose the algorithm stops after $1 \leq l \leq |\mathcal{C}|$ iterations. Since all $\mathit{Supp}_{k-1}^\Delta(c_k)$ are disjoint, $\sum_{k=1}^{l} |\mathit{Supp}_{k-1}^\Delta(c_k)| = |\mathit{Supp}(H_l)| \leq |\Gamma|$. Altogether, the running time is $O(\sum_{k=1}^{l} |\mathcal{C} - \sigma(H_k)| \times |\mathit{Supp}_{k-1}^\Delta(c_k)|) \leq O(|\mathcal{C}| \times \sum_{k=1}^{l} |\mathit{Supp}_{k-1}^\Delta(c_k)|)$, and thus the running time is $O(|\mathcal{C}| \times |\Gamma|)$.

Properties of the Algorithm

The algorithm will always generate an HCLP model satisfying Γ if Γ is $\mathcal{C}(1)$-consistent. It can also be used for computing the maximal inconsistency base. The following result sums up some properties related to the algorithm.

Theorem 3. *Let H be a sequence returned by the algorithm with inputs Γ and $\mathcal{C}$, and write MIB$(\Gamma, \mathcal{C})$ as $(\Gamma^{\perp}, C^{\perp})$. Then $C^{\perp} = \mathcal{C} - \sigma(H)$ (i.e., the evaluations that don't appear in H), and $\Gamma^{\perp} = \Gamma - Supp(H)$. We have that $H \models \Gamma^{(\leq)}$. Also, Γ is $\mathcal{C}(1)$-consistent if and only if $Supp(H)$ contains all the strict elements of Γ, which is if and only if $\Gamma^{\perp} \cap \mathcal{L}^{\mathcal{A}}_{<} = \emptyset$. If Γ is $\mathcal{C}(1)$-consistent then $H \models \Gamma$.*

Proof: By the construction of the algorithm, H is a maximal Γ-allowed sequence, as observed earlier. Proposition 8 implies that $C^{\perp} = \mathcal{C} - \sigma(H)$ and $\Gamma^{\perp} = \Gamma - Supp(H)$. By Proposition 6, we have $H \models \Gamma^{(\leq)}$. Corollary 1 implies that Γ is $\mathcal{C}(1)$-consistent if and only if $\Gamma^{\perp} \cap \mathcal{L}^{\mathcal{A}}_{<} = \emptyset$. Theorem 2 implies that Γ is $\mathcal{C}(1)$-consistent if and only if $H \models \Gamma$. Proposition 7 implies that $H \models \Gamma$ if and only if $Supp(H)$ contains all the strict elements of Γ. $\square$

The algorithm therefore determines $\mathcal{C}(1)$-consistency, and hence $\mathcal{C}(1)$-deduction (because of Proposition 1), in polynomial time, and also generates the maximal inconsistency base.

4.2.4 The case of inconsistent Γ

For the case when Γ is not $\mathcal{C}(1)$-consistent, the output H of the algorithm is a model which, in a sense, comes closest to satisfying Γ: it always satisfies $\Gamma^{(\leq)}$, the non-strict version of Γ, and if any model $H' \in \mathcal{C}(1)$ satisfies $\Gamma^{(\leq)}$ and any element φ of Γ, then H also satisfies φ.

Proposition 10. *Let H be a sequence returned by the algorithm with inputs Γ and $\mathcal{C}$, and suppose that $H' \in \mathcal{C}(1)$ is such that $H' \models \Gamma^{(\leq)}$. Then, for all $\varphi \in \Gamma$, if $H' \models \varphi$ then $H \models \varphi$.*

Proof: Since H is a maximal Γ-allowed sequence, we have (by Proposition 6) that $H \models \Gamma^{(\leq)}$. Suppose that $H' \in \mathcal{C}(1)$ is such that $H' \models \Gamma^{(\leq)}$. Proposition 9 implies that if $H' \models \varphi$ then $H \models \varphi$. $\square$

These properties suggest the following way of reasoning with $\mathcal{C}(1)$-inconsistent Γ. Let us define Γ' to be equal to $(\Gamma - \Gamma^{\perp}) \cup \Gamma^{(\leq)}$. By Theorem 3, this is equal to $Supp(H) \cup \Gamma^{(\leq)}$, where H is a model generated by the algorithm, enabling easy computation of Γ'. Γ' is $\mathcal{C}(1)$-consistent, since it is satisfied by H. We might then (re-)define the (non-monotonic) deductions from $\mathcal{C}(1)$-inconsistent Γ to be the deductions from Γ'.

4.3 Strong consistency and max-model inference

In the set of models $\mathcal{C}(1)$, we allow models involving any subset of $\mathcal{C}$, the set of evaluations. We could alternatively consider a semantics where we only allow models H that involve all elements of $\mathcal{C}$, i.e., with $\sigma(H) = \mathcal{C}$.

Let $\mathcal{C}(1^*)$ be the set of elements H of $\mathcal{C}(1)$ with $\sigma(H) = \mathcal{C}$. Γ is defined to be *strongly* $\mathcal{C}(1)$*-consistent* if and only if there exists a model $H \in \mathcal{C}(1^*)$ such that $H \models \Gamma$. Let *MIB*$(\Gamma, \mathcal{C}) = (\Gamma^{\perp}, C^{\perp})$. Proposition 3 implies that, if Γ is strongly $\mathcal{C}(1)$-consistent then $C^{\perp}$ is empty, and $\Gamma^{\perp}$ consists of all the elements of Γ that are indifferent to all of $\mathcal{C}$, i.e., the set of $\varphi \in \Gamma$ such that $c(\alpha_\varphi) = c(\beta_\varphi)$ for all $c \in \mathcal{C}$.

There is an associated preference inference based on this restricted set of models. We write $\Gamma \models_{\mathcal{C}(1^*)} \varphi$ if $H \models \varphi$ holds for every $H \in \mathcal{C}(1^*)$ such that $H \models \Gamma$.

This form of deduction can be expressed in terms of strong consistency, as the following result shows.

Lemma 14. *If* Γ *is strongly* $\mathcal{C}(1)$*-consistent then* $\Gamma \models_{\mathcal{C}(1^*)} \varphi$ *holds if and only if* $\Gamma \cup \{\neg\varphi\}$ *is not strongly* $\mathcal{C}(1)$*-consistent.*

Proof: First suppose that $\Gamma \cup \{\neg\varphi\}$ is strongly $\mathcal{C}(1)$-consistent. Then there exists $H \in \mathcal{C}(1)$ such that $H \models \Gamma \cup \{\neg\varphi\}$ and $\sigma(H) = \mathcal{C}$. Thus $H \models \Gamma$ and $H \not\models \varphi$ (using Lemma 1), showing that $\Gamma \not\models_{\mathcal{C}(1^*)} \varphi$.

Now suppose that $\Gamma \not\models_{\mathcal{C}(1^*)} \varphi$. Then there exists $H \in \mathcal{C}(1)$ such that $H \models \Gamma$ and $\sigma(H) = \mathcal{C}$ and $H \not\models \varphi$. Then $H \models \Gamma \cup \{\neg\varphi\}$ (again using Lemma 1), so $\Gamma \cup \{\neg\varphi\}$ is strongly $\mathcal{C}(1)$-consistent. $\square$

In the next section we will consider a related (and, in a sense, more general) form of preference inference, where we only consider maximal models.

4.3.1 Max-model inference

For $\Gamma \subseteq \mathcal{L}^{\mathcal{A}}$, let $\mathcal{M}^{\max}_{\mathcal{C}(1)}(\Gamma)$ be the set of maximal models within $\mathcal{C}(1)$ of Γ, i.e., the set of $H \in \mathcal{C}(1)$ such that $H \models \Gamma$, and for all $H' \in \mathcal{C}(1)$ extending H, $H' \not\models \Gamma$. We define the max-model inference relation $\models^{\max}_{\mathcal{C}(1)}$ by:

$\Gamma \models^{\max}_{\mathcal{C}(1)} \varphi$ if and only if $H \models \varphi$ for all $H \in \mathcal{M}^{\max}_{\mathcal{C}(1)}(\Gamma)$.

The following result shows that maximal models of Γ involve the same set of evaluations. It also shows that, if Γ is $\mathcal{C}(1)$-consistent, the maximal models are the same as the maximal Γ-allowed sequences discussed earlier.

Proposition 11. *Suppose that* Γ *is* $\mathcal{C}(1)$*-consistent. Then, for* $H \in \mathcal{C}(1)$*, we have* $H \in \mathcal{M}^{\max}_{\mathcal{C}(1)}(\Gamma)$ *if and only if* H *is a maximal* Γ*-allowed sequence in* $\mathcal{C}$*. Thus, for all* $H, H' \in \mathcal{M}^{\max}_{\mathcal{C}(1)}(\Gamma)$*, we have* $\sigma(H) = \sigma(H') = \mathcal{C} - C^{\perp}$*, where MIB*$(\Gamma, \mathcal{C}) = (\Gamma^{\perp}, C^{\perp})$.

Proof: Consider any $H \in \mathcal{M}^{\max}_{\mathcal{C}(1)}(\Gamma)$. Since $H \models \Gamma$ we have $H \models \Gamma^{(\leq)}$, and so Proposition 6 implies that H is a Γ-allowed sequence. Suppose that H is not a maximal Γ-allowed sequence. Then, by Lemma 12, there exists a maximal Γ-allowed sequence H' extending H, and $H \models \Gamma$. This contradicts $H \in \mathcal{M}^{\max}_{\mathcal{C}(1)}(\Gamma)$.

Conversely, suppose that H is a maximal Γ-allowed sequence in $\mathcal{C}$. Theorem 2 implies that $H \models \Gamma$. To prove a contradiction, suppose that $H \notin \mathcal{M}^{\max}_{\mathcal{C}(1)}(\Gamma)$, so that there exists $H' \in \mathcal{M}^{\max}_{\mathcal{C}(1)}(\Gamma)$ with H' extending H. The argument above implies that H' is a maximal Γ-allowed sequence, which contradicts H being a maximal Γ-allowed sequence.

The last part follows from Proposition 8. $\square$

The next result shows that the same non-strict preference statements are inferred for the max-model inference relation $\models^{\max}_{\mathcal{C}(1)}$ as for the inference relation $\models_{\mathcal{C}(1)}$.

Proposition 12. *Consider any $\Gamma \subseteq \mathcal{L}^{\mathcal{A}}$, and any preference statement $\alpha \leq \beta$ in $\mathcal{L}^{\mathcal{A}}$.*

(i) Γ is $\mathcal{C}(1)$-consistent if and only if $\mathcal{M}^{\max}_{\mathcal{C}(1)}(\Gamma) \neq \emptyset$.

(ii) $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha \leq \beta \iff \Gamma \models_{\mathcal{C}(1)} \alpha \leq \beta$.

Proof: (i) follows easily: if Γ is $\mathcal{C}(1)$-consistent, then there exists some $H \in \mathcal{C}(1)$ with $H \models \Gamma$, so there exists $H' \in \mathcal{M}^{\max}_{\mathcal{C}(1)}(\Gamma)$ extending or equalling H. The converse is immediate: if there exists $H \in \mathcal{M}^{\max}_{\mathcal{C}(1)}(\Gamma)$ then $H \in \mathcal{C}(1)$ and $H \models \Gamma$, so Γ is $\mathcal{C}(1)$-consistent.

(ii) If Γ is not $\mathcal{C}(1)$-consistent then by part (i), $\mathcal{M}^{\max}_{\mathcal{C}(1)}(\Gamma) = \emptyset$, so $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha \leq \beta$ and $\Gamma \models_{\mathcal{C}(1)} \alpha \leq \beta$ both hold vacuously. Let us thus now assume that Γ is $\mathcal{C}(1)$-consistent.

$\Rightarrow$: Assume $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha \leq \beta$, and consider any $H \in \mathcal{C}(1)$ such that $H \models \Gamma$. We need to show that $H \models \alpha \leq \beta$. Since $H \models \Gamma$, we have $H \models \Gamma^{(\leq)}$, and so H is a Γ-allowed $\mathcal{C}$-sequence, by Proposition 6. Choose, by Lemma 12, any maximal Γ-allowed sequence H' extending or equalling H, and we have $H' \models \Gamma$. By, Proposition 11, $H' \in \mathcal{M}^{\max}_{\mathcal{C}(1)}(\Gamma)$. Then, $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha \leq \beta$ implies that $H' \models \alpha \leq \beta$. Lemma 8(ii) then implies that $H \models \alpha \leq \beta$.

$\Leftarrow$: Assume $\Gamma \models_{\mathcal{C}(1)} \alpha \leq \beta$, and consider any $H \in \mathcal{M}^{\max}_{\mathcal{C}(1)}(\Gamma)$. This implies that $H \in \mathcal{C}(1)$ and $H \models \Gamma$, so $H \models \alpha \leq \beta$ showing that $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha \leq \beta$. $\square$

We write $\Gamma \models_{\mathcal{C}(1)} \alpha \equiv \beta$ as an abbreviation of the conjunction of $\Gamma \models_{\mathcal{C}(1)} \alpha \leq \beta$ and $\Gamma \models_{\mathcal{C}(1)} \beta \leq \alpha$; and similarly for other inference relations. The last result can be used to prove that inferred equivalences are the same for max-model inference, and have a simple form.

Proposition 13. *Consider any $\mathcal{C}(1)$-consistent $\Gamma \subseteq \mathcal{L}^{\mathcal{A}}$, and any $\mathcal{C}$. Let MIB$(\Gamma, \mathcal{C})$ equal $(\Gamma^{\perp}, C^{\perp})$. Consider any $\alpha, \beta \in \mathcal{A}$. Then, $\Gamma \models_{\mathcal{C}(1)} \alpha \equiv \beta$ if and only if $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha \equiv \beta$ if and only if for all $c \in \mathcal{C} - C^{\perp}$, $c(\alpha) = c(\beta)$.*

Proof: First assume that $\Gamma \models_{\mathcal{C}(1)} \alpha \equiv \beta$. This trivially implies that $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha \equiv \beta$, since $\models^{\max}_{\mathcal{C}(1)} \subseteq \models_{\mathcal{C}(1)}$.

Now assume that $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha \equiv \beta$. Γ is $\mathcal{C}(1)$-consistent so $\mathcal{M}^{\max}_{\mathcal{C}(1)}(\Gamma) \neq \emptyset$, by Proposition 12(i). Consider any $H \in \mathcal{M}^{\max}_{\mathcal{C}(1)}(\Gamma)$. Then $\alpha \equiv_H \beta$, which implies that for all $c \in \sigma(H)$, $c(\alpha) = c(\beta)$, and thus, by Proposition 11, for all $c \in \mathcal{C} - C^{\perp}$, $c(\alpha) = c(\beta)$.

Finally, let us assume that for all $c \in \mathcal{C} - C^{\perp}$, $c(\alpha) = c(\beta)$. Consider any $H \in \mathcal{C}(1)$ such that $H \models \Gamma$. Proposition 3 implies that $\sigma(H) \cap C^{\perp} = \emptyset$, i.e., $\sigma(H) \subseteq \mathcal{C} - C^{\perp}$. So, for all $c \in \sigma(H)$, $c(\alpha) = c(\beta)$, and thus $\alpha \equiv_H \beta$, and hence $\Gamma \models_{\mathcal{C}(1)} \alpha \equiv \beta$. This completes the proof that the three statements are equivalent. $\square$

The following result shows that the strict inferences with $\models^{\max}_{\mathcal{C}(1)}$ are closely tied with the non-strict inferences.

Proposition 14. $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha \leq \beta$ *if and only if either* $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha \equiv \beta$ *or* $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha < \beta$. *Also, if* Γ *is* $\mathcal{C}(1)$*-consistent then* $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha < \beta$ *holds if and only if* $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha \leq \beta$ *and* $\Gamma \not\models^{\max}_{\mathcal{C}(1)} \alpha \equiv \beta$.

Proof: If Γ is not $\mathcal{C}(1)$-consistent then, by Proposition 12(i), $\mathcal{M}^{\max}_{\mathcal{C}(1)}(\Gamma) = \emptyset$, so $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha \leq \beta$ and $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha \equiv \beta$ (and $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha < \beta$) hold vacuously, and therefore the equivalence holds. Let us thus now assume that Γ is $\mathcal{C}(1)$-consistent. One direction holds easily: suppose that $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha \equiv \beta$ or $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha < \beta$, and consider any $H \in \mathcal{M}^{\max}_{\mathcal{C}(1)}(\Gamma)$. We have either $\alpha \equiv_H \beta$ or $H \models \alpha < \beta$, so either $\alpha \equiv_H \beta$ or $\alpha \prec_H \beta$, and thus $\alpha \preccurlyeq_H \beta$, and $H \models \alpha \leq \beta$, showing that $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha \leq \beta$.

Now, let us assume that $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha \leq \beta$, and that it is not the case that $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha \equiv \beta$. It is sufficient to show that $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha < \beta$. Consider any $H \in \mathcal{M}^{\max}_{\mathcal{C}(1)}(\Gamma)$. Since, $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha \leq \beta$, we have $H \models \alpha \leq \beta$. Proposition 13 implies that there exists $c \in \mathcal{C} - C^{\perp}$ such that $c(\alpha) \neq c(\beta)$, where $MIB(\Gamma, \mathcal{C}) = (\Gamma^{\perp}, C^{\perp})$. By, Proposition 11, $\sigma(H) = \mathcal{C} - C^{\perp}$, so there exists some $c \in \sigma(H)$ such that $c(\alpha) \neq c(\beta)$; let c be earliest such element of $\sigma(H)$. Since $H \models \alpha \leq \beta$, we have $c(\alpha) < c(\beta)$, so $H \models \alpha < \beta$. This shows that $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha < \beta$, as required.

Assume that Γ is $\mathcal{C}(1)$-consistent. Suppose that $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha < \beta$ holds. Then clearly, $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha \leq \beta$. Consider any $H \models \Gamma$. Then we have $\alpha \prec_H \beta$, so we do not have $\alpha \equiv_H \beta$, which implies that $\Gamma \models_{\mathcal{C}(1)} \alpha \equiv \beta$ does not hold. Conversely, suppose that $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha \leq \beta$ and $\Gamma \not\models^{\max}_{\mathcal{C}(1)} \alpha \equiv \beta$. The first part then implies that $\Gamma \models^{\max}_{\mathcal{C}(1)} \alpha < \beta$. $\square$

4.3.2 Properties of strong consistency and of the associated inference

The following result shows that the consequences of Γ with respect to $\models_{\mathcal{C}(1^*)}$ are the same as those with respect to $\models^{\max}_{\mathcal{C}(1)}$, when Γ is strongly $\mathcal{C}(1)$-consistent. (Of course, if Γ is not strongly $\mathcal{C}(1)$-consistent then all φ in $\mathcal{L}^{\mathcal{A}}$ are consequences of $\models_{\mathcal{C}(1^*)}$.)

Lemma 15. *If Γ is strongly $\mathcal{C}(1)$-consistent then, for any $\varphi \in \mathcal{L}^{\mathcal{A}}$, $\Gamma \models_{\mathcal{C}(1^*)} \varphi \iff \Gamma \models^{\max}_{\mathcal{C}(1)} \varphi$.*

Proof: Assume that Γ is strongly $\mathcal{C}(1)$-consistent, so there exists a model H' with $\sigma(H') = \mathcal{C}$. By definition of $\models_{\mathcal{C}(1^*)}$ and $\models^{\max}_{\mathcal{C}(1)}$ it is sufficient to show that $\mathcal{M}^{\max}_{\mathcal{C}(1)}(\Gamma)$ is equal to the set $\mathcal{H}$ of all $H \in \mathcal{C}(1)$ such that $H \models \Gamma$ and $\sigma(H) = \mathcal{C}$. It immediately follows that $\mathcal{M}^{\max}_{\mathcal{C}(1)}(\Gamma) \supseteq \mathcal{H}$. Conversely, consider any $H \in \mathcal{M}^{\max}_{\mathcal{C}(1)}(\Gamma)$. Since $H' \in \mathcal{H}$, we have $H' \in \mathcal{M}^{\max}_{\mathcal{C}(1)}(\Gamma)$. Proposition 11 implies that $\sigma(H) = \sigma(H') = \mathcal{C}$, proving that $H \in \mathcal{H}$. $\square$

The next result shows that the non-strict $\models_{\mathcal{C}(1^*)}$ inferences are the same as the non-strict $\models_{\mathcal{C}(1)}$ inferences, and that (in contrast to the case of $\models_{\mathcal{C}(1)}$) the strict $\models_{\mathcal{C}(1^*)}$ inferences almost correspond with the non-strict ones. The result also implies that the algorithm in Section 4.2 can be used to efficiently determine the $\models_{\mathcal{C}(1^*)}$ inferences.

To illustrate the difference between the $\models_{\mathcal{C}(1)}$ inferences and the $\models_{\mathcal{C}(1^*)}$ inferences for the case of strict statements, consider some strongly $\mathcal{C}(1)$-consistent Γ which only includes non-strict statements. Then, for every strict preference statement $\alpha < \beta$, we will have $\Gamma \not\models_{\mathcal{C}(1)} \alpha < \beta$ since the empty sequence satisfies Γ but not $\alpha < \beta$. However, we will have $\Gamma \models_{\mathcal{C}(1^*)} \alpha < \beta$ if $\Gamma \models_{\mathcal{C}(1)} \alpha \leq \beta$ and $\Gamma \not\models_{\mathcal{C}(1)} \beta \leq \alpha$. For example, if Γ is just $\{\alpha \leq \beta\}$, where for some $c \in \mathcal{C}$, $c(\alpha) < c(\beta)$, then we will have $\Gamma \models_{\mathcal{C}(1^*)} \alpha < \beta$ but not $\Gamma \models_{\mathcal{C}(1)} \alpha < \beta$.

Proposition 15. *Let MIB$(\Gamma, \mathcal{C}) = (\Gamma^{\perp}, C^{\perp})$. Γ is strongly $\mathcal{C}(1)$-consistent if and only if $C^{\perp} = \emptyset$ and $\Gamma \cap \mathcal{L}^{\mathcal{A}}_{<} \subseteq$ Supp$(\mathcal{C})$.*

Suppose that Γ is strongly $\mathcal{C}(1)$-consistent. Then,

(i) $\Gamma \models_{\mathcal{C}(1)} \alpha \leq \beta \iff \Gamma \models_{\mathcal{C}(1^*)} \alpha \leq \beta$;

(ii) $\Gamma \models_{\mathcal{C}(1^*)} \alpha \equiv \beta$ *if and only if α and β agree on all of $\mathcal{C}$, i.e., for all $c \in \mathcal{C}$, $c(\alpha) = c(\beta)$;*

(iii) $\Gamma \models_{\mathcal{C}(1^*)} \alpha < \beta$ *if and only if $\Gamma \models_{\mathcal{C}(1)} \alpha \leq \beta$ and α and β differ on some element of $\mathcal{C}$, i.e., there exists $c \in \mathcal{C}$ such that $c(\alpha) \neq c(\beta)$.*

Proof: First, suppose that Γ is strongly $\mathcal{C}(1)$-consistent. Then there exists $H' \in \mathcal{C}(1)$ such that $H' \models \Gamma$ and $\sigma(H') = \mathcal{C}$. Since $H' \models \Gamma^{(\leq)}$, by Proposition 6, H' is a Γ-allowed sequence. By Lemma 12, there exists a maximal Γ-allowed sequence H extending or equalling H', so, since $\sigma(H') = \mathcal{C}$, we must have $H = H'$. Proposition 8 implies that $C^{\perp} = \emptyset$ and $\Gamma^{\perp} = \Gamma - \textit{Supp}(H) = \Gamma - \textit{Supp}(\mathcal{C})$, and Corollary 1 shows then that $(\Gamma - \textit{Supp}(\mathcal{C})) \cap \mathcal{L}^{\mathcal{A}}_{<} = \emptyset$, which implies that $\Gamma \cap \mathcal{L}^{\mathcal{A}}_{<} \subseteq \textit{Supp}(\mathcal{C})$.

Conversely, suppose that $C^{\perp} = \emptyset$ and $\Gamma \cap \mathcal{L}^{\mathcal{A}}_{<} \subseteq \mathit{Supp}(\mathcal{C})$. Let H be a maximal Γ-allowed sequence. Proposition 8 implies that $\sigma(H) = \mathcal{C}$. Then $\mathit{Supp}(H) = \mathit{Supp}(\mathcal{C})$, and Proposition 7 implies that $H \models \Gamma$, showing that Γ is strongly $\mathcal{C}(1)$-consistent.

Now suppose that Γ is strongly $\mathcal{C}(1)$-consistent. Lemma 15 implies that for any $\varphi \in \mathcal{L}^{\mathcal{A}}$, $\Gamma \models_{\mathcal{C}(1^*)} \varphi \iff \Gamma \models^{\max}_{\mathcal{C}(1)} \varphi$. Part (i) then follows by Proposition 12(ii). Part (ii) follows from Proposition 13, using the fact that $C^{\perp}$ is empty. Part (iii) follows from part (ii) and Proposition 14. □

The next result shows that $\models_{\mathcal{C}(1)}$ inference is not affected if one removes the evaluations in the MIB.

Proposition 16. *Suppose that Γ is $\mathcal{C}(1)$-consistent, let MIB$(\Gamma, \mathcal{C}) = (\Gamma^{\perp}, C^{\perp})$, and let $C' = \mathcal{C} - C^{\perp}$. Then Γ is strongly $C'(1)$-consistent, and $\Gamma \models_{\mathcal{C}(1)} \varphi$ if and only if $\Gamma \models_{C'(1)} \varphi$.*

Proof: By Theorem 3, any output of the algorithm is in $C'(1^*)$ and satisfies Γ. Thus Γ is strongly $C'(1)$-consistent. Let $\mathcal{H}' = \{H \in C'(1) \;:\; H \models \Gamma\}$ and $\mathcal{H} = \{H \in \mathcal{C}(1) \;:\; H \models \Gamma\}$. Then $\mathcal{H}' \subseteq \mathcal{H}$, because $C'(1) \subseteq \mathcal{C}(1)$. By Proposition 3, for every $H \in \mathcal{H}$, we have $\sigma(H) \cap C^{\perp} = \emptyset$, and hence $H \in \mathcal{H}'$. Thus $\mathcal{H}' = \mathcal{H}$ and $\Gamma \models_{\mathcal{C}(1)} \varphi$ if and only if $\Gamma \models_{C'(1)} \varphi$. □

4.4 Orderings on evaluations

The preference logic defined here is closely related to a logic based on disjunctive ordering statements. Given set of evaluations $\mathcal{C}$, we consider the set of statements $\mathcal{O}_{\mathcal{C}}$ of the form $C_1 < C_2$, and of $C_1 \leq C_2$, where C_1 and C_2 are disjoint subsets of $\mathcal{C}$.

We say that $H \models C_1 < C_2$ if some evaluation in C_1 appears in H before every element of C_2, that is, there exists some element of C_1 in H (i.e., $C_1 \cap \sigma(H) \neq \emptyset$) and the earliest element of $C_1 \cup C_2$ to appear in H is in C_1.

We say that $H \models C_1 \leq C_2$ if either $H \models C_1 < C_2$ or no element of C_1 or C_2 appears in H: $(C_1 \cup C_2) \cap \sigma(H) = \emptyset$. By Lemma 5 we have that

$H \models \alpha_{\varphi} < \beta_{\varphi} \iff H \models \mathit{Supp}^{\varphi}_{\mathcal{C}} < \mathit{Opp}^{\varphi}_{\mathcal{C}}$,

and $H \models \alpha_{\varphi} \leq \beta_{\varphi} \iff H \models \mathit{Supp}^{\varphi}_{\mathcal{C}} \leq \mathit{Opp}^{\varphi}_{\mathcal{C}}$.

This shows that the language $\mathcal{O}_{\mathcal{C}}$ can express anything that can be expressed in $\mathcal{L}^{\mathcal{A}}$. It can be shown, conversely, that for any statement τ in $\mathcal{O}_{\mathcal{C}}$, one can define α_{φ} and β_{φ}, and the values of elements of $\mathcal{C}$ on these, such that for all $H \in \mathcal{C}(1)$, $H \models \tau$ if and only if $H \models \varphi$ (where φ is strict if and only if τ is strict). For instance, if τ is the statement $C_1 < C_2$, we can define $c(\alpha_{\varphi}) = 1$ for all $c \in C_2$, and $c(\alpha_{\varphi}) = 0$ for $c \in \mathcal{C} - C_2$; and define $c(\beta_{\varphi}) = 1$ for all $c \in C_1$, and $c(\beta_{\varphi}) = 0$ for $c \in \mathcal{C} - C_1$.

The algorithm adapts in the obvious way to the case where we have Γ consisting of (or including) elements in $\mathcal{O}_{\mathcal{C}}$. When viewed in this way, the algorithm can be seen as a simple extension of a topological sort algorithm; the standard case corresponds to when the ordering statements only involve singleton sets.

5 Proof theory for simple lexicographic inference

Preference inference has been defined semantically, and we have an efficient algorithm for the simple lexicographic case. From a logical perspective, it is natural to consider if we can construct an equivalent syntactical definition of inference via a proof theory; this can give another view of the assumptions being made by the logic. In this section we construct such a proof theory for preference inference based on simple lexicographic models, involving an axiom schema and a number of fairly simple inference rules. We consider a fixed set of evaluations $\mathcal{C}$ here, and we abbreviate $\models_{\mathcal{C}(1)}$ to just $\models$.

We make use of a form of Pareto (pointwise) ordering on alternatives, and we define a kind of addition and rescaling operation on alternatives and thus on preference statements.

We define the following pointwise (or weak Pareto) ordering on alternatives. For $\alpha, \beta \in \mathcal{A}$, $\alpha \preceq_{par} \beta \iff$ for all $c \in \mathcal{C}$, $c(\alpha) \leq c(\beta)$. We also define the Pareto Difference relation between elements of $\mathcal{L}^{\mathcal{A}}$. For $\psi, \theta \in \mathcal{L}^{\mathcal{A}}$, we say that $\psi \preceq_{parD} \theta$ holds if and only if (i) ψ and θ are either both strict or both non-strict; and (ii) for all $c \in \mathcal{C}$, $c(\beta_\psi) - c(\alpha_\psi) \leq c(\beta_\theta) - c(\alpha_\theta)$. Thus, if $\psi \preceq_{parD} \theta$ and $c(\alpha_\psi) \leq c(\beta_\psi)$ then $c(\alpha_\theta) \leq c(\beta_\theta)$. If $\psi \preceq_{parD} \theta$ and $H \models \psi$ then $H \models \theta$ (see Lemma 16(vi) below).

Pointwise multiplication of alternatives and preference statements: Let F be the set of functions from $\mathcal{C}$ to the strictly positive rational numbers. For $f \in F$, we define $\frac{1}{f} \in F$ in the obvious way, by, for $c \in \mathcal{C}$, $\frac{1}{f}(c) = \frac{1}{f(c)}$. Let f be an arbitrary element of F.

- For $\alpha, \gamma \in \mathcal{A}$, we say that $\alpha \doteq f\gamma$ if for all $c \in \mathcal{C}$, $c(\alpha) = f(c) \times c(\gamma)$ (where $\times$ is the standard multiplication).
- For $\varphi, \psi \in \mathcal{L}^{\mathcal{A}}$, we say that $\varphi \doteq f\psi$ if (i) $\alpha_\varphi \doteq f\alpha_\psi$ and $\beta_\varphi \doteq f\beta_\psi$, and (ii) φ is strict if and only if ψ is strict.

Note that if $\varphi \doteq f\psi$ then for all $c \in \mathcal{C}$, $c(\alpha_\varphi) \leq c(\beta_\varphi) \iff c(\alpha_\psi) \leq c(\beta_\psi)$. It is then easy to show that if $H \in \mathcal{C}(1)$ and $\varphi \doteq f\psi$ then $H \models \varphi$ if and only if $H \models \psi$: see Lemma 16(iv).

Addition of alternatives and preference statements:

- For $\alpha, \beta, \gamma \in \mathcal{A}$, we say that $\gamma \doteq \alpha + \beta$ if for all $c \in \mathcal{C}$, $c(\gamma) = c(\alpha) + c(\beta)$.

- For $\varphi, \psi, \chi \in \mathcal{L}^{\mathcal{A}}$, we say that $\varphi \doteq \psi + \chi$ if (i) $\alpha_\varphi \doteq \alpha_\psi + \alpha_\chi$, and $\beta_\varphi \doteq \beta_\psi + \beta_\chi$; and (ii) φ is non-strict if both ψ and χ are non-strict, and otherwise, φ is strict.

5.1 Syntactic deduction $\vdash$ and soundness of inference rules

As usual the proof theory is constructed from axioms and inference rules.

Axioms:

$\alpha \leq \beta$ for all $\alpha, \beta \in \mathcal{A}$ with $\alpha \preceq_{par} \beta$.

Inference rules schemata:

(1) From Strict to Non-Strict: For any $\alpha, \beta \in \mathcal{A}$ the following rule:
From $\alpha < \beta$ deduce $\alpha \leq \beta$.

(2) Addition: For $\chi \in \mathcal{L}^{\mathcal{A}}$ such that $\chi \doteq \varphi + \psi$ the following inference rule
From φ and ψ deduce χ.

(3) Pointwise Multiplication: For any $f \in F$ and $\varphi \in \mathcal{L}^{\mathcal{A}}$ such that $\varphi \doteq f\psi$ the following rule
From ψ deduce φ.

(4) Inconsistent Statement: For any $\alpha \in \mathcal{A}$ and any $\varphi \in \mathcal{L}^{\mathcal{A}}$,
From $\alpha < \alpha$ deduce φ.

(5) Pareto Difference: For any $\psi, \theta \in \mathcal{L}^{\mathcal{A}}$ such that $\psi \preceq_{parD} \theta$:
From ψ deduce θ.

Defining syntactic deduction $\vdash$: Let Γ be a subset of $\mathcal{L}^{\mathcal{A}}$ and $\varphi \in \mathcal{A}$. We say that φ can be proved from Γ, written $\Gamma \vdash \varphi$, if there exists a sequence $\varphi_1, \ldots, \varphi_k$ of elements of $\mathcal{L}^{\mathcal{A}}$ such that $\varphi_k = \varphi$ and for all $i = 1, \ldots, k$, either $\varphi_i \in \Gamma$ or φ_i is an axiom, or there exists an instance of one of the inference rules with consequent φ_i and such that the antecedents are in $\{\varphi_1, \ldots, \varphi_{i-1}\}$. Relation $\vdash$ depends strongly on the set of alternatives $\mathcal{A}$; e.g., $\{\varphi, \psi\} \vdash \varphi + \psi$ (if and) only if $\varphi + \psi \in \mathcal{L}^{\mathcal{A}}$, i.e., only if $\alpha_\varphi + \alpha_\psi$ and $\beta_\varphi + \beta_\psi$ are in $\mathcal{A}$. We write $\vdash$ as $\vdash_{\mathcal{A}}$ if we want to emphasise this dependency. It can happen that for $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathcal{A}} \subseteq \mathcal{L}^{\mathcal{B}}$, we have $\Gamma \vdash_{\mathcal{B}} \varphi$, but $\Gamma \not\vdash_{\mathcal{A}} \varphi$. (We could also write $\models_{\mathcal{A}}$ to emphasise the dependency on $\mathcal{A}$; however, it isn't usually important to do so, since for $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathcal{A}} \subseteq \mathcal{L}^{\mathcal{B}}$, we have $\Gamma \models_{\mathcal{B}} \varphi \iff \Gamma \models_{\mathcal{A}} \varphi$.)

Any given set of alternatives may not be closed under addition (for instance), and there may be $\alpha, \beta \in \mathcal{A}$ with no $\gamma \in \mathcal{A}$ such that $\gamma \doteq \alpha + \beta$. We assume that we can augment $\mathcal{A}$ with additional alternatives, and for any function $g : \mathcal{C} \to \mathbb{Q}^+$, we can construct an alternative α with, for all $c \in \mathcal{C}$, $c(\alpha) = g(c)$.

Next we state a lemma showing soundness of the axioms and inference rules, which is used to prove soundness of the associated syntactic deduction (Proposition 17).

Lemma 16. *Consider any $H \in \mathcal{C}(1)$, any $\alpha, \beta \in \mathcal{A}$, and any $\varphi, \psi, \chi, \theta \in \mathcal{L}^{\mathcal{A}}$.*

(i) If $\alpha \preceq_{par} \beta$ then $H \models \alpha \leq \beta$.

(ii) If $H \models \alpha < \beta$ then $H \models \alpha \leq \beta$.

(iii) If $\chi \doteq \varphi + \psi$, and $H \models \varphi$ and $H \models \psi$ then $H \models \chi$.

(iv) If $\varphi \doteq f\psi$ then $H \models \varphi \iff H \models \psi$.

(v) $H \not\models \alpha < \alpha$.

(vi) If $H \models \psi$ and $\psi \preceq_{parD} \theta$ then $H \models \theta$.

Proof: Write H as $(c_1, \ldots, c_k)$. For $\varphi \in \mathcal{L}^{\mathcal{A}}$ we define i^φ to be $k+1$ if for all $i = 1, \ldots, k$, $c_i(\alpha_\varphi) = c_i(\beta_\varphi)$; otherwise, we define i^φ to be the minimum i such that $c_i(\alpha_\varphi) \neq c_i(\beta_\varphi)$. Then $\alpha_\varphi \equiv_H \beta_\varphi \iff i^\varphi = k+1$, and $H \models \alpha_\varphi < \beta_\varphi \iff i^\varphi \leq k$ and $c_{i^\varphi}(\alpha_\varphi) < c_{i^\varphi}(\beta_\varphi)$.

(i): Assume that $\alpha \preceq_{par} \beta$, so that for all $c \in \mathcal{C}$, we have $c(\alpha) \leq c(\beta)$. This implies $\alpha \preccurlyeq_H \beta$ and thus $H \models \alpha \leq \beta$.

(ii): Assume that $H \models \alpha < \beta$, so that $\alpha \prec_H \beta$. This implies $\alpha \preccurlyeq_H \beta$ and hence $H \models \alpha \leq \beta$.

(iii): Assume that $\chi \doteq \varphi + \psi$, and $H \models \varphi$ and $H \models \psi$.

Case (I): $i^\varphi = i^\psi = k+1$. Then for all $i = 1, \ldots, k$, $c_i(\alpha_\varphi) = c_i(\beta_\varphi)$ and $c_i(\alpha_\psi) = c_i(\beta_\psi)$. Then, $c_i(\alpha_\chi) = c_i(\alpha_\varphi) + c_i(\alpha_\psi) = c_i(\beta_\varphi) + c_i(\beta_\psi) = c_i(\beta_\chi)$, so $i^\chi = k+1$, which implies that $\alpha_\chi \equiv_H \beta_\chi$. We have $\alpha_\varphi \equiv_H \beta_\varphi$, and also $H \models \varphi$, so φ is non-strict. Similarly, ψ is non-strict. Thus χ is non-strict, and so $H \models \chi$.

Case (II): $i^\varphi = i^\psi \leq k$. Because $c_{i^\varphi}(\alpha_\varphi) \neq c_{i^\varphi}(\beta_\varphi)$ and $H \models \varphi$, we have $c_{i^\varphi}(\alpha_\varphi) < c_{i^\varphi}(\beta_\varphi)$. The same argument implies that $c_{i^\varphi}(\alpha_\psi) < c_{i^\varphi}(\beta_\psi)$. We then have $c_{i^\varphi}(\alpha_\chi) < c_{i^\varphi}(\beta_\chi)$, and $i^\chi = i^\varphi$. This implies that $H \models \alpha_\chi < \beta_\chi$, and thus, $H \models \chi$, whether χ is strict or non-strict.

Case (III): $i^\varphi < i^\psi$. Arguing as in Case (II), we have $c_{i^\varphi}(\alpha_\varphi) < c_{i^\varphi}(\beta_\varphi)$. We also have $c_{i^\varphi}(\alpha_\psi) = c_{i^\varphi}(\beta_\psi)$. We then have $c_{i^\varphi}(\alpha_\chi) < c_{i^\varphi}(\beta_\chi)$, and $i^\chi = i^\varphi$. Again we have $H \models \chi$, whether χ is strict or non-strict.

Case (IV): $i^\varphi > i^\psi$. This is similar to Case (III), but with the roles of φ and ψ reversed.

(iv): Assume that $\varphi \doteq f\psi$, and consider any $c \in \mathcal{C}$. Because $f(c) > 0$, we have $c(\alpha_\varphi) = c(\beta_\varphi)$ if and only if $c(\alpha_\psi) = c(\beta_\psi)$; and $c(\alpha_\varphi) < c(\beta_\varphi)$ if and only if $c(\alpha_\psi) < c(\beta_\psi)$. This shows that $H \models \varphi \iff H \models \psi$.

(v): $H \not\models \alpha < \alpha$ follows since $\alpha \equiv_H \alpha$ and so $\alpha \not\prec_H \alpha$.

(vi): Suppose that $H \models \psi$ and $\psi \preceq_{parD} \theta$, so that ψ and θ are either both strict or both non-strict; and for all $c \in \mathcal{C}$, $c(\beta_\psi) - c(\alpha_\psi) \leq c(\beta_\theta) - c(\alpha_\theta)$. If it were the case that $i^\psi < i^\theta$ then, because $H \models \psi$, we would have that $c_{i^\psi}(\alpha_\psi) < c_{i^\psi}(\beta_\psi)$ and $c_{i^\psi}(\alpha_\theta) = c_{i^\psi}(\beta_\theta)$, and thus, $c_{i^\psi}(\beta_\psi) - c_{i^\psi}(\alpha_\psi) > 0 = c_{i^\psi}(\beta_\theta) - c_{i^\psi}(\alpha_\theta)$, which contradicts $\psi \preceq_{parD} \theta$. Thus we must have that $i^\psi \geq i^\theta$.

First consider the case when $i^\theta = k + 1$. Then $i^\psi = k + 1$, and so $\alpha_\theta \equiv_H \beta_\theta$ and $\alpha_\psi \equiv_H \beta_\psi$. The latter implies that ψ is non-strict, since $H \models \psi$. Then θ is non-strict and thus, $H \models \theta$.

Now consider the case when $i^\theta \leq k$, and thus $c_{i^\theta}(\alpha_\theta) \neq c_{i^\theta}(\beta_\theta)$. We showed earlier that $i^\theta \leq i^\psi$. If $i^\theta = i^\psi$ then $H \models \psi$ implies that $c_{i^\theta}(\alpha_\psi) < c_{i^\theta}(\beta_\psi)$. If $i^\theta < i^\psi$ then $c_{i^\theta}(\alpha_\psi) = c_{i^\theta}(\beta_\psi)$. So, in either case we have $c_{i^\theta}(\alpha_\psi) \leq c_{i^\theta}(\beta_\psi)$, i.e., $c_{i^\theta}(\beta_\psi) - c_{i^\theta}(\alpha_\psi) \geq 0$. The assumption $\psi \preceq_{parD} \theta$ then implies that $c_{i^\theta}(\beta_\theta) - c_{i^\theta}(\alpha_\theta) \geq 0$, and so, $c_{i^\theta}(\alpha_\theta) \leq c_{i^\theta}(\beta_\theta)$. Since $i^\theta \leq k$ we have $c_{i^\theta}(\alpha_\theta) < c_{i^\theta}(\beta_\theta)$, showing that $H \models \alpha_\theta < \beta_\theta$, and therefore $H \models \theta$ whether θ is strict or non-strict. $\square$

We are now ready to state and prove the soundness result.

Proposition 17. *For* $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathcal{A}}$*, and any* $\mathcal{B} \supseteq \mathcal{A}$*, if* $\Gamma \vdash_{\mathcal{B}} \varphi$ *then* $\Gamma \models_{\mathcal{A}} \varphi$*.*

Proof: First note that if Γ is $\mathcal{C}(1)$-inconsistent, then there is nothing to prove, since $\Gamma \models_{\mathcal{A}} \varphi$ follows trivially. So, let us assume now that Γ is $\mathcal{C}(1)$-consistent. We use an inductive proof based on Lemma 16. Suppose that $\Gamma \vdash_{\mathcal{B}} \varphi$. Consider any $H \in \mathcal{C}(1)$ such that $H \models \Gamma$. We need to show that $H \models \varphi$. Since $\Gamma \vdash_{\mathcal{B}} \varphi$ there exists a sequence $\varphi_1, \ldots, \varphi_k$ of elements of $\mathcal{L}^{\mathcal{B}}$ such that $\varphi_k = \varphi$ and for all $i = 1, \ldots, k$, either $\varphi_i \in \Gamma$ or φ_i is an axiom, or there exists an instance of one of the inference rules with consequent φ_i and such that the antecedents are in $\{\varphi_1, \ldots, \varphi_{i-1}\}$. Consider any $i \in \{1, \ldots, k\}$. We will prove that, if for all $j < i$, $H \models \varphi_j$ then $H \models \varphi_i$. This then implies that for all $i = 1, \ldots, k$, we have $H \models \varphi_i$, and thus $H \models \varphi_k$, as required.

Therefore, let i be some arbitrary element in $\{1, \ldots, k\}$, and assume that for all $j < i$, $H \models \varphi_j$. We will prove that $H \models \varphi_i$. Let us abbreviate φ_i to be θ. One of the cases (1)–(7) below applies. We consider each case in turn.

(1): θ equals $\alpha \leq \beta$ for some $\alpha, \beta \in \mathcal{B}$, and there exists some $j < i$ with φ_j equalling $\alpha < \beta$. Since $H \models \varphi_j$, by Lemma 16(ii), we have $H \models \alpha \leq \beta$, i.e., $H \models \theta$.

(2): θ equals χ for some $\chi \in \mathcal{L}^{\mathcal{B}}$ such that $\chi \doteq \varphi + \psi$, and for some $j, l < i$ we have $\varphi = \varphi_j$ and $\psi = \varphi_l$. Since $H \models \varphi_j, \varphi_k$, Lemma 16(iii) implies that $H \models \theta$.

(3): There exists $j < i$ and $f \in F$ such that $\theta \doteq f\varphi_j$. Lemma 16(iv) implies that $H \models \theta$.

(4): There exists $\alpha \in \mathcal{B}$ and $j < i$ such that φ_j equals $\alpha < \alpha$, so we have $H \models \alpha < \alpha$. However, by Lemma 16(v), this is impossible, so Case (4) cannot arise.

(5): There exists $j < i$ such that $\psi = \varphi_j \in \mathcal{L}^{\mathcal{B}}$ and $\psi \preceq_{parD} \theta$. Lemma 16(vi) implies $H \models \theta$.

(6): $\theta \in \Gamma$. Then $H \models \theta$.

(7): θ is equal to $\alpha \leq \beta$ for some $\alpha, \beta \in \mathcal{B}$ such that $\alpha \preceq_{par} \beta$. Lemma 16(i) implies $H \models \theta$.

□

5.2 Completeness of proof theory

We now give a pair of technical lemmas which we will use in the completeness proof.

Lemma 17. *Consider any $\mathcal{C}(1)$-inconsistent $\Gamma \subseteq \mathcal{L}^{\mathcal{A}}$, and suppose that $(\{\varphi_1, \ldots, \varphi_k\}, C')$ is an inconsistency base for $(\Gamma, \mathcal{C})$, with $\{\varphi_1, \ldots, \varphi_k\}$ being inconsistent. Then there exist strictly positive functions $f_1, \ldots, f_k \in F$, set of alternatives $\mathcal{B} \supseteq \mathcal{A}$ with $\mathcal{B} - \mathcal{A}$ finite, preference statement $\rho \in \mathcal{L}^{\mathcal{B}}$ and strict preference statement ψ in $\mathcal{L}^{\mathcal{B}}$ such that $\rho \doteq f_1\varphi_1 + \cdots + f_{k-1}\varphi_{k-1}$ and $\psi \doteq f_1\varphi_1 + \cdots + f_k\varphi_k$, and $\Gamma \vdash_{\mathcal{B}} \rho$ and $\Gamma \vdash_{\mathcal{B}} \psi$, and $\beta_\psi \preceq_{par} \alpha_\psi$.*

Proof: Let $T = \{|c(\alpha_{\varphi_i}) - c(\beta_{\varphi_i})| : c \in \mathcal{C}, i \in \{1, \ldots, k\}\} - \{0\}$. If $T = \emptyset$ then set $a = b = 1$, and if $T \neq \emptyset$ let $a = \min T$ and let $b = \max T$, so $0 < a \leq b$. For $i = 1, \ldots, k$ and $c \in \mathcal{C}$, we define $f_i(c) = 1$ if $c(\alpha_{\varphi_i}) > c(\beta_{\varphi_i})$, and otherwise, we define $f_i(c) = d$ where $d = a/(kb) > 0$.

For $i = 1, \ldots, k$ we include elements $\gamma_i, \delta_i, \epsilon_i, \lambda_i$ in $\mathcal{B}$, where $\gamma_i \doteq f_i\alpha_{\varphi_i}$, and $\delta_i \doteq f_i\beta_{\varphi_i}$; and we let $\epsilon_1 = \gamma_1$ and $\lambda_1 = \delta_1$, and for $i = 2, \ldots, k$, $\epsilon_i \doteq \epsilon_{i-1} + \gamma_i$, and $\lambda_i \doteq \lambda_{i-1} + \delta_i$.

There exists $\psi_1 \in \mathcal{L}^{\mathcal{B}}$ with $\psi_1 \doteq f_1\varphi_1$, and $\alpha_{\psi_1} = \gamma_1 = \epsilon_1$ and $\beta_{\psi_1} = \delta_1 = \lambda_1$. Similarly, for $i = 2, \ldots, k$, there exists $\psi_i \in \mathcal{L}^{\mathcal{B}}$ with $\psi_i \doteq \psi_{i-1} + f_i\varphi_i$, and $\alpha_{\psi_i} = \epsilon_i$ and $\beta_{\psi_i} = \lambda_i$.

By the Addition and Pointwise Multiplication rules, for each $i = 1, \ldots, k$, we have $\Gamma \vdash_{\mathcal{B}} \psi_i$. Abbreviate ψ_k to ψ and ψ_{k-1} to ρ. We have $\Gamma \vdash_{\mathcal{B}} \psi$ and $\psi \doteq f_1\varphi_1 + \cdots + f_k\varphi_k$, and $\Gamma \vdash_{\mathcal{B}} \rho$ and $\rho \doteq f_1\varphi_1 + \cdots + f_{k-1}\varphi_{k-1}$. Since $\{\varphi_1, \ldots, \varphi_k\}$ is inconsistent, some φ_i is strict (else the empty model satisfies them all), and therefore, ψ is a strict preference statement.

Consider any $c \in \mathcal{C} - C'$. By Definition 1(i), $c(\alpha_{\varphi_i}) = c(\beta_{\varphi_i})$ for all $i = 1, \ldots, k$. Thus $c(\alpha_\psi) = c(\beta_\psi)$.

Now consider any $c \in C'$. For any $j \in \{1, \ldots, k\}$, $c(\alpha_{\varphi_j}) - c(\beta_{\varphi_j}) \geq -b$, and so $c(\gamma_j) - c(\delta_j) \geq -bd = -a/k$. By Definition 1(ii), there exists some $i \in \{1 \ldots, k\}$ such that $c(\alpha_{\varphi_i}) > c(\beta_{\varphi_i})$. This implies that $T \neq \emptyset$. We have $c(\alpha_{\varphi_i}) - c(\beta_{\varphi_i}) \geq a$, and thus

$c(\gamma_i) - c(\delta_i) \geq a > 0$. Now, $c(\alpha_\psi) = \sum_{j=1}^{k} c(\gamma_j)$ and $c(\beta_\psi) = \sum_{j=1}^{k} c(\delta_j)$. Therefore, $c(\alpha_\psi) - c(\beta_\psi) \geq a - (k-1)a/k > 0$. We have shown that for all $c \in \mathcal{C}$, $c(\alpha_\psi) \geq c(\beta_\psi)$, so $\beta_\psi \preceq_{par} \alpha_\psi$. □

Lemma 18. *Suppose $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathcal{A}}$, and that Γ is $\mathcal{C}(1)$-consistent and $\Gamma \models \varphi$. Then there exists $\mathcal{B} \supseteq \mathcal{A}$ (with $\mathcal{B} - \mathcal{A}$ finite), and $\chi, \theta \in \mathcal{L}^{\mathcal{B}}$ such that $\Gamma \vdash_{\mathcal{B}} \chi$, and θ is strict and $\theta \doteq \chi + \neg\varphi$, and $\beta_\theta \preceq_{par} \alpha_\theta$.*

Proof: By Lemma 1, $\Gamma \cup \{\neg\varphi\}$ is $\mathcal{C}(1)$-inconsistent. By Corollary 1 there exists an inconsistency base (Δ, C') for $(\Gamma \cup \{\neg\varphi\}, \mathcal{C})$ with Δ being a finite and $\mathcal{C}(1)$-inconsistent subset of $\Gamma \cup \{\neg\varphi\}$, and $C' \subseteq \mathcal{C}$. Now, Δ contains $\neg\varphi$, since Δ is $\mathcal{C}(1)$-inconsistent and Γ is $\mathcal{C}(1)$-consistent. We write Δ as $\{\varphi_1, \ldots, \varphi_k\}$ with $\varphi_k = \neg\varphi$.

By Lemma 17, there exist strictly positive functions $f_1, \ldots, f_k \in F$, set of alternatives $\mathcal{B} \supseteq \mathcal{A}$ with $\mathcal{B} - \mathcal{A}$ finite, preference statement $\rho \in \mathcal{L}^{\mathcal{B}}$ and strict preference statement ψ in $\mathcal{L}^{\mathcal{B}}$ such that $\rho \doteq f_1\varphi_1 + \cdots + f_{k-1}\varphi_{k-1}$ and $\psi \doteq f_1\varphi_1 + \cdots + f_k\varphi_k$, $\Gamma \vdash_{\mathcal{B}} \rho$ and $\Gamma \vdash_{\mathcal{B}} \psi$, and $\beta_\psi \preceq_{par} \alpha_\psi$.

Let $\mathcal{B}' = \mathcal{B} \cup \{\alpha_\chi, \beta_\chi, \alpha_\theta, \beta_\theta\}$, where $\alpha_\chi \doteq \frac{1}{f_k}\alpha_\rho$ and $\beta_\chi \doteq \frac{1}{f_k}\beta_\rho$, and $\alpha_\theta \doteq \alpha_\chi + \beta_\varphi$ and $\beta_\theta \doteq \beta_\chi + \alpha_\varphi$, and χ, θ (which are thus in $\mathcal{L}^{\mathcal{B}'}$) are such that $\chi \doteq \frac{1}{f_k}\rho$ and $\theta \doteq \chi + \neg\varphi$, i.e., $\theta \doteq \chi + \varphi_k$. We have $f_k\theta \doteq f_k\chi + f_k\varphi_k \doteq \rho + f_k\varphi_k$ and thus $\psi \doteq f_k\theta$. This implies that θ is a strict statement and that $\beta_\theta \preceq_{par} \alpha_\theta$. Now, $\Gamma \vdash_{\mathcal{B}} \rho$ implies that $\Gamma \vdash_{\mathcal{B}'} \rho$ (because $\mathcal{B}' \subseteq \mathcal{B}$). Since $\chi \doteq \frac{1}{f_k}\rho$, we have $\Gamma \vdash_{\mathcal{B}'} \chi$, using the Pointwise Multiplication inference rule, completing the proof. □

These lemmas lead to the completeness theorems.

Theorem 4. *Consider any $\Gamma \subseteq \mathcal{L}^{\mathcal{A}}$ and any $\varphi \in \mathcal{L}^{\mathcal{A}}$. Then there exists $\mathcal{B} \supseteq \mathcal{A}$, with $\mathcal{B} - \mathcal{A}$ finite such that $\Gamma \models \varphi \iff \Gamma \vdash_{\mathcal{B}} \varphi$.*

Proof: $\Leftarrow$ follows by Proposition 17. To prove the converse, let us assume that $\Gamma \models \varphi$; we will show that $\mathcal{A}$ can be extended to $\mathcal{B}$ such that $\Gamma \vdash_{\mathcal{B}} \varphi$.

First let us consider the case when Γ is $\mathcal{C}(1)$-inconsistent. By Corollary 1 there exists $C' \subseteq \mathcal{C}$ and a $\mathcal{C}(1)$-inconsistent subset $\{\varphi_1, \ldots, \varphi_k\}$ of Γ, such that $(\{\varphi_1, \ldots, \varphi_k\}, C')$ is an inconsistency base for $(\Gamma, \mathcal{C})$. By Lemma 17, there exist strictly positive functions $f_1, \ldots, f_k \in F$, set of alternatives $\mathcal{B} \supseteq \mathcal{A}$ with $\mathcal{B} - \mathcal{A}$ finite, and strict preference statement ψ in $\mathcal{B}$ such that $\psi \doteq f_1\varphi_1 + \cdots + f_k\varphi_k$, and $\Gamma \vdash_{\mathcal{B}} \psi$ and $\beta_\psi \preceq_{par} \alpha_\psi$. Consider any $\gamma \in \mathcal{A}$. Then $\beta_\psi \preceq_{par} \alpha_\psi$ implies for all $c \in \mathcal{C}$, $c(\beta_\psi) - c(\alpha_\psi) \leq 0 = c(\gamma) - c(\gamma)$. The Pareto Difference inference rule then implies that $\Gamma \vdash_{\mathcal{B}} \gamma < \gamma$, since ψ is strict, and hence, by the Inconsistent Statement inference rule, $\Gamma \vdash_{\mathcal{B}} \varphi$, as required.

Now we consider the case when Γ is $\mathcal{C}(1)$-consistent. By Lemma 18, we have that there exists set of alternatives $\mathcal{B} \supseteq \mathcal{A}$ with $\mathcal{B} - \mathcal{A}$ finite, and $\chi, \theta \in \mathcal{L}^{\mathcal{B}}$ such that $\Gamma \vdash_{\mathcal{B}} \chi$, and θ is strict, $\theta \doteq \chi + \neg\varphi$, and $\beta_\theta \preceq_{par} \alpha_\theta$. Then, by definition of $\neg\varphi$, we have $\alpha_\theta \doteq \alpha_\chi + \beta_\varphi$ and $\beta_\theta \doteq \beta_\chi + \alpha_\varphi$. This implies that for all $c \in \mathcal{C}$, $c(\beta_\chi) + c(\alpha_\varphi) \leq c(\alpha_\chi) + c(\beta_\varphi)$, and thus, for all $c \in \mathcal{C}$, $c(\beta_\chi) - c(\alpha_\chi) \leq c(\beta_\varphi) - c(\alpha_\varphi)$. Now, since $\theta \doteq \chi + \neg\varphi$ and θ is strict, if χ is non-strict then $\neg\varphi$ must be strict and so φ is non-strict. The Pareto Difference inference rule then implies that $\Gamma \vdash_\mathcal{B} \varphi$. If, on the other hand, χ is strict then the Pareto Difference inference rule implies that $\Gamma \vdash_\mathcal{B} \alpha_\varphi < \beta_\varphi$, and thus $\Gamma \vdash_\mathcal{B} \alpha_\varphi \leq \beta_\varphi$, using the From Strict to Non-Strict rule. Therefore $\Gamma \vdash_\mathcal{B} \varphi$ whether φ is strict or non-strict. □

Let $\mathcal{A}^*$ be a set of alternatives including for each function $g : \mathcal{C} \to \mathbb{Q}^+$, an alternative α with, for all $c \in \mathcal{C}$, $c(\alpha) = g(c)$, and let $\mathcal{A}' = \mathcal{A} \cup \mathcal{A}^*$. Consider any $\Gamma \subseteq \mathcal{L}^\mathcal{A}$ and any $\varphi \in \mathcal{L}^\mathcal{A}$. Then $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^{\mathcal{A}'}$. If we use $\mathcal{A}'$ instead of $\mathcal{A}$ in the proofs of Lemma 17 and 18, and Theorem 4, we can use $\mathcal{B} = \mathcal{A}'$ in each case. This leads, for arbitrary Γ and φ, to: $\Gamma \models_{\mathcal{A}'} \varphi \iff \Gamma \vdash_{\mathcal{A}'} \varphi$, which since $\Gamma \models_{\mathcal{A}'} \varphi$ holds if and only if $\Gamma \models_\mathcal{A} \varphi$ holds, gives the following version of the completeness result.

Theorem 5. *For any $\mathcal{A}$, there exists $\mathcal{A}' \supseteq \mathcal{A}$ such that for any $\Gamma \subseteq \mathcal{L}^\mathcal{A}$ and any $\varphi \in \mathcal{L}^\mathcal{A}$, $\Gamma \models \varphi \iff \Gamma \vdash_{\mathcal{A}'} \varphi$.*

Discussion of related preference inference based on weighted sum

Another natural notion of preference inference, which is similar to that defined e.g., in [13, 12], is based on weighted sums. In each model a non-negative weight is assigned to each evaluation, and the overall desirability of an alternative is the weighted sum of the evaluations on the alternative. More precisely, let the set of models be the set of functions e from $\mathcal{C}$ to $\mathbb{Q}^+$. We say that e *satisfies* $\alpha \leq \beta$ if $\sum_{c\in\mathcal{C}} e(c)c(\alpha) \leq \sum_{c\in\mathcal{C}} e(c)c(\beta)$. Similarly, we say that e *satisfies* $\alpha < \beta$ if $\sum_{c\in\mathcal{C}} e(c)c(\alpha) < \sum_{c\in\mathcal{C}} e(c)c(\beta)$. As for the other kinds of preference inference, we say, for $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^\mathcal{A}$, that Γ entails φ if e satisfies φ for every e satisfying Γ. This preference inference satisfies the above axiom schema, and all the inference rules except for (3) Pointwise Multiplication (and thus is weaker than $\models_{\mathcal{C}(1)}$). Instead a weaker form of (3) holds, based on using only constant functions f. The Pointwise Multiplication inference rule might thus be considered as characteristic of preference inference based on simple lexicographic models.

6 Discussion and conclusions

We defined a class of relatively simple preference logics based on hierarchical models. These generate an adventurous form of inference, which can be helpful if there is only relatively sparse input preference information. We showed that the complexity of preference deduction is coNP-complete in general, and polynomial for the case where the criteria are assumed to be totally ordered (the simple lexicographic models case, Section 4).

The latter logic has strong connections with the preference inference formalism described in [18]. To clarify the connection, for each evaluation $c \in \mathcal{C}$ we can generate a variable X_c, and let V be the set of these variables. For each alternative $\alpha \in \mathcal{A}$ we generate a complete assignment α^* on the variables V (i.e., an outcome as defined in [18]) by $\alpha^*(X_c) = c(\alpha)$ for each $X_c \in V$. Note that values of $\alpha^*(X_c)$ are non-negative numbers, and thus have a fixed ordering, with zero being the best value. A preference statement $\alpha \leq \beta$ in $\mathcal{L}^{\mathcal{A}}_{\leq}$ then corresponds with a basic preference formula $\alpha^* \geq \beta^*$ in [18]. Each model $H \in \mathcal{C}(1)$ corresponds to a sequence of evaluations, and thus has an associated sequence of variables; this sequence together with the fixed value orderings, generates a lexicographic model as defined in [18].

In contrast with the lexicographic inference system in [18], the logic developed in this paper allows strict (as well as non-strict) preference statements, and allows more than one variable at the same level. However, the lexicographic inference logic from [18] does not assume a fixed value ordering (which, translated into the current formalism, corresponds to not assuming that the values of the evaluation function are known); it also allows a richer language of preference statements, where a statement can be a compact representation for a (possibly exponentially large) set of basic preference statements of the form $\alpha \leq \beta$. Many of the results of Section 4 immediately extend to richer preference languages (by replacing a preference statement by a corresponding set of basic preference statements). In future work we will determine under what circumstances deduction remains polynomial when extending the language, and when removing the assumption that the evaluation functions are known.

The coNP-hardness result for the general case (and for the $\models^{\oplus}_{\mathcal{C}(t)}$ systems with $t \geq 2$) is notable and perhaps surprising, since these preference logics are relatively simple ones. The result obviously extends to more general systems. The preference inference system described in [16] is based on much more complex forms of lexicographic models, allowing conditional dependencies, as well as having local orderings on sets of variables (with bounded cardinality). Theorem 1 implies that the (polynomial) deduction system in [16] is not more general than the system described here (assuming P $\neq$ NP). It also implies that if one were to extend the system from [16] to allow a richer form of equivalence, generalising e.g., the $\models^{\oplus}_{\mathcal{C}(2)}$ system, then the preference inference will no longer be polynomial.

Acknowledgements

This publication has emanated from research conducted with the financial support of Science Foundation Ireland (SFI) under Grant Number SFI/12/RC/2289. Nic Wilson was also supported by the School of EEE&CS, Queen's University Belfast. We are grateful to the reviewers for their helpful comments.

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Received xx Month 20xx