CORA Cork Open Research Archive
Cartlann Taighde Oscailte Chorcaí

Title	Bounds on 2m/R for static spherical objects
Authors	Guven, Jemal;Ó Murchadha, Niall
Publication date	1999
Original Citation	Guven, J. and Ó Murchadha, N. (1999) 'Bounds on 2m/R for static spherical objects', Physical Review D, 60(8), 084020 (8pp). doi: 10.1103/PhysRevD.60.084020
Type of publication	Article (peer-reviewed)
Link to publisher's version	https://journals.aps.org/prd/abstract/10.1103/PhysRevD.60.084020 - 10.1103/PhysRevD.60.084020
Rights	© 1999, American Physical Society
Download date	2026-11-06 01:17:42
Item downloaded from	https://hdl.handle.net/10468/4580



UCC
University College Cork, Ireland
Coláiste na hOllscoile Corcaigh

Bounds on $2m/R$ for static spherical objects

Jemal Guven*
Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Apartado Postal 70-543, 04510 México, Distrito Federal, Mexico

Niall Ó Murchadha†
Physics Department, University College Cork, Cork, Ireland
(Received 17 March 1999; published 24 September 1999)

It is well known that a spherically symmetric constant density static star, modeled as a perfect fluid, possesses a bound on its mass m by its radial size R given by $2m/R \leqslant 8/9$ and that this bound continues to hold when the energy density decreases monotonically. The existence of such a bound is intriguing because it occurs well before the appearance of an apparent horizon at $m=R/2$. However, the assumptions made are extremely restrictive. They do not hold in a simple soap bubble and they certainly do not approximate any known topologically stable field configuration. In addition, such configurations will not generally be compact. We show that the 8/9 bound is robust by relaxing these assumptions. If the density is monotonically decreasing and the tangential stress is less than or equal to the radial stress we show that the 8/9 bound continues to hold through the entire bulk if m is replaced by the quasi-local mass. If the tangential stress is allowed to exceed the radial stress and/or the density is not monotonic we cannot recover the 8/9 bound. However, we can show that $2m/R$ remains strictly bounded away from unity by constructing an explicit upper bound which depends only on the ratio of the stresses and the variation of the density. [S0556-2821(99)09818-5]

PACS number(s): 04.20.Cv

I. INTRODUCTION

Consider any static solution of the Einstein equations with the matter satisfying the null energy condition. The Penrose singularity theorem shows that this system cannot have a trapped surface. In a spherically symmetric configuration, the first apparent horizon in the initial data occurs when the ratio of (twice) the mass to its radial extent, $2m/R$, is one. However, it is also well known that for a spherical star composed of ordinary matter with positive energy described as a perfect fluid with a monotonically decreasing energy profile, $2m/R$ cannot exceed 8/9, the constant density value [1–5]. Such a bound is particularly interesting because it occurs strictly before the appearance of an apparent horizon. Here we wish to examine this bound more closely when the assumptions on the distribution of matter underlying it are relaxed. Even in an astrophysical stellar object where matter is described phenomenologically, it is not clear that the perfect fluid assumption is justified. A humble soap bubble consisting of a film with a given (tangential) surface tension supported by the pressure of the enclosed perfect gas violates both the monotonicity of the energy density and the perfect fluid assumption. The approximation clearly also does not represent accurately the interior of topological defects such as monopoles [6]. The balance of forces providing the equilibrium typically turns out to be analogous to that holding a soap bubble together. In addition, in many extensions of Einstein gravity, the effective stress tensor describing a perfect fluid does not assume the perfect fluid form. Field configurations typically will not be compact, in which situation we require a generalization of the mass that holds throughout the bulk. This requires the replacement of m by a quasi-local mass.

While the 8/9 bound is not a universal one, it is robust in the sense that under physical conditions which are at least reasonable classically, the mass continues to be bounded by a value strictly below the apparent horizon value, $m=R/2$. It appears to be impossible physically, even in principle, to construct a static distribution which saturates it.

The idea to relax the isotropy (of the stresses) in order to overcome the $2m/R<8/9$ bound in static spheres is old, appearing already in a paper by Lemaitre [7]. The subject of anisotropic stresses in self-gravitating systems has been reviewed in Ref. [8].

In Secs. II and III, we establish our notation. We show in Sec. III that if the matter satisfies $\rho+S_r \geqslant 0$, where ρ is the energy density and S_r is the radial stress, the object cannot have an apparent horizon and thus $2m/R$ is strictly bounded away from 1. This condition, $\rho+S_r \geqslant 0$, is one of the so-called ''null energy'' or ''null convergence'' conditions [9–11]. It is interesting because we need no restriction on the tangential stress nor do we need to assume $\rho \geqslant 0$. In a spherically symmetric geometry, the tangential stress will generally differ from the radial one except at the center where the constraints dictate that they coincide. Consider the ratio, γ, of tangential to radial stress. In a perfect fluid $\gamma=1$. We further show, again in Sec. III, that static matter satisfying $\rho \geqslant 0$ together with $\gamma \leqslant 1$ must have positive radial pressure which monotonically decreases outwards. This guarantees that $\rho+S_r \geqslant 0$ which provides another way of excluding apparent horizons.

The rest of the article is devoted to investigating how close $2m/R$ can get to one, and generalizing the $2m/R<8/9$ bound noted above. We summarize very briefly the

*Email address: jemal@nuclecu.unam.mx
†Email address: niall@ucc.ie

 ©1999 The American Physical Society

simple constant density star in Sec. IV. In Sec. V we consider ''stars'' that are monotonic with positive density. If $\gamma \leqslant 1$, it is simple to show that the 8/9 inequality continues to hold, not only on the boundary but through the entire bulk. Indeed, the configuration need not be compact. If $\gamma \geqslant 1$ anywhere, however, we obtain a slightly weaker result. We construct a bound which shows that $2m/R$ is strictly bounded away from unity. If γ_{max} approaches 1 the bound smoothly approaches 8/9; as γ becomes unboundedly large the bound approaches 1. In particular, for a monotonic star with positive radial pressure and for which the transverse pressure is less than the density we can show that $2m/R < 0.974$. Finally, in Sec. VI, we relax the assumption of monotonicity and find essentially the same results, except that now the bound depends both on the variation of the matter as well as on γ_{max}.

II. STATIC LIMIT OF EINSTEIN EQUATIONS

The spacetime metric describing a static solution of the Einstein equations can always be written in the form

$$ds^2 = -N^2 dt^2 + g_{ab} dx^a dx^b, \tag{1}$$

where N is the lapse function and the shift vanishes. N is also the norm of the global timelike Killing vector, ∂_t, and so must satisfy $N > 0$. The spatial geometry at constant t is described by the metric tensor g_{ab}. Both the material current vector J^a and the extrinsic curvature tensor K_{ab} (describing the embedding of a hypersurface of fixed t in spacetime) vanish. In the canonical formulation of the theory, the momentum constraints of the theory are then vacuous. The Hamiltonian constraint reduces to the form [12,13] (see also the Appendix to [4]),

$$\mathcal{R} = 16\pi\rho, \tag{2}$$

where $\mathcal{R}$ is the scalar curvature constructed with the spatial metric g_{ab} and ρ is the material energy density. Given some specification of ρ, Eq. (2) is a constraint on the spatial geometry, g_{ab}. It does not involve the stresses operating on ρ. In the spherically symmetric case we will see that the intrinsic geometry is completely specified by ρ. The advantage of working within the canonical formulation is that this constraint is isolated explicitly.

Given that the time direction is Killing the evolution in this direction must be trivial. The dynamical Einstein equation reduces to $\dot{K}_{ab} = 0$, and now reads

$$-\nabla_a \nabla_b N + \mathcal{R}_{ab} N = 8\pi N \left(S_{ab} - \frac{1}{2} g_{ab} \operatorname{tr} S + \frac{1}{2} g_{ab} \rho \right), \tag{3}$$

where ∇_a is the covariant derivative compatible with g_{ab}, $\mathcal{R}_{ab}$ is the associated Ricci tensor, S_{ab} is the material pressure tensor and S is its trace. In a perfect fluid the stress is isotropic with $S_{ab} = P g_{ab}$. The other evolution equation, $\dot{g}_{ab} = 0$, is trivially satisfied.

For given ρ and g_{ab}, Eq. (3) consists of six partial differential equations (PDEs) for the seven functions, N and S_{ab}. This counting is not very precise, because if we had a realistic fluid/field theoretical model we would have to supplement these equations with an ''equation of state'' which would convert the equations from an underdetermined to an overdetermined system.

We will suppose that the spatial topology is R^3. For an object with an energy density of compact support (a star) or falling off sufficiently rapidly at infinity the spacetime will be asymptotically flat with $N \to 1$ at infinity. The appropriate boundary condition on S_{ab} in an object of compact support is that its normal component vanishes on the surface. For much of our discussion these boundary conditions are irrelevant.

Taking the trace of Eq. (3), and eliminating $\mathcal{R}$ in favor of ρ using Eq. (2), we obtain the linear elliptic equation for N,

$$\Delta N = 4\pi(\rho + \operatorname{tr} S) N. \tag{4}$$

If the strong energy condition is satisfied we have $\rho + \operatorname{tr} S \geqslant 0$ everywhere, and so $\Delta N \geqslant 0$ when $N > 0$. Thus the solution cannot possess an interior maximum. N falls towards the center. Even if the potential, $V = \rho + \operatorname{tr} S$, is large, N never falls to zero in the interior (e.g., [14]). The lapse can go to zero only when the density or pressure becomes unboundedly large.

Let us assume that N is positive in the exterior and negative on a compact region W. N vanishes on ∂W but will have a positive outward gradient. If we integrate ΔN over W we can turn it into a surface integral which must be positive. On the other hand, from Eq. (4) we see that $\Delta N \leqslant 0$ on W, so we have a contradiction. We also see that it is impossible for N to just touch zero at a point. At that point we would have that N, its first derivatives and its second derivatives all vanish. Thus the function could never grow away from zero [15].

One can deduce the conservation law,

$$\nabla_b S^{ab} = -(S^{ab} + \rho g^{ab}) \frac{\nabla_b N}{N}, \tag{5}$$

directly from the static Einstein equations, Eqs. (2) and (3). To do this, we simply take the divergence of Eq. (3). Exploiting the Ricci identities,

$$[\nabla_a, \nabla_b] V^b = R_{ab} V^b, \tag{6}$$

and the contracted Bianchi identity for $\mathcal{R}_{ab}$, $\nabla_a \mathcal{R}^{ab} = \nabla^b \mathcal{R}/2$, we reproduce Eq. (5).

It is clear from Eq. (4) that there are no non-trivial vacuum static solutions in the theory. We have $\Delta N = 0$ everywhere. If there is no internal boundary, the solution is $N = 1$ everywhere. Now $\mathcal{R}_{ab} = 0$, as well as $\mathcal{R} = 0$, so that the geometry is flat everywhere. There is a well known result that the only perfect fluid static equilibria are spherically symmetric [16–18]. This result implies that for perfect fluids the spherically symmetric analysis is complete. A discussion of the symmetries of equilibrium configurations is provided in [19].

III. SPHERICAL SYMMETRY

The line element describing the spatial part of a spherically symmetric geometry can always be written as

$$ds^2 = dl^2 + R^2 d\Omega^2, \tag{7}$$

l is the proper radial distance on the hypersurface; R is the areal radius. For R^3 topology, l has domain $[0,\infty)$. The appropriate boundary conditions on R are

$$R(0)=0, \quad dR/dl\big|_0 = R'(0) = 1. \tag{8}$$

The scalar curvature $\mathcal{R}$ is given by

$$\mathcal{R} = -\frac{2}{R^2}[2(RR')' - R'^2 - 1], \tag{9}$$

where primes denote derivatives with respect to l. The constraint equation can be cast in the form (see, for example, [20,21])

$$R'^2 = 1 - \frac{2m}{R}, \tag{10}$$

where the positive quasi-local mass is given by

$$m = 4\pi \int_0^l \rho R^2 R' dl = 4\pi \int_0^R \rho R^2 dR. \tag{11}$$

It is immediately clear that

$$m \leqslant R/2 \tag{12}$$

everywhere. In general, $R'^2 \leqslant 1$ in any regular geometry when the weak energy condition ($\rho \geqslant 0$) is satisfied, so that $R \leqslant l$ everywhere [21]. To show this we substitute Eq. (9) into Eq. (2) to get

$$2RR'' + R'^2 - 1 = -8\pi R^2 \rho. \tag{13}$$

At the center we have $R'=1$ and $m=0$. From Eq. (11) we see that m increases as soon as we meet matter and thus R' drops below 1. Let us assume that it later rises up to $+1$. However, from Eq. (13) we see that at this point $R'' \leqslant 0$ so it cannot be rising. On the other hand R' can drop below -1. We again get $R'' \leqslant 0$ which means that it cannot ever rise up again to the asymptotic $R' \approx 1$. Thus for any regular spherical geometry satisfying the weak energy condition $-1 < R' \leqslant +1$ and $R' = +1$ only at the origin and at infinity. This holds true for any solution of Eq. (13), no static assumption is required.

It is clear from Eq. (10) that m is positive everywhere in a regular geometry and vanishes only at the center and in any vacuum region surrounding it. In a static configuration, the extrinsic curvature vanishes so that an apparent horizon is a minimal surface with $R'=0$. Thus, if the geometry is free of an apparent horizon, it must also be free of singularities. In such a geometry $0 < R' \leqslant 1$ and m increases monotonically with l (or R). We emphasize that the spatial geometry and with it the Arnowitt-Deser-Misner (ADM) mass is completely determined by the source energy density. The material stresses play no role whatsoever.

At the surface of a compact object of radius $R=R_0$, the quasi-local mass coincides with the constant ADM mass, m_0. The exterior solution is given by Eq. (10):

$$R'^2 = 1 - \frac{2m_0}{R}. \tag{14}$$

In a spherically symmetric geometry, any symmetric tensor is completely characterized by two scalars. We have

$$\mathcal{R}_{ab} = \mathcal{R}_r n_a n_b + \mathcal{R}_t (g_{ab} - n_a n_b)$$

$$S_{ab} = S_r n_a n_b + S_t (g_{ab} - n_a n_b). \tag{15}$$

Here n^a is the outward pointing normal to a two sphere of fixed proper radius. The two scalars appearing in the Ricci tensor can be expressed in terms of $\mathcal{R}$, R and R' as follows [21]:

$$\mathcal{R}_r = \frac{1}{2}\mathcal{R} - \frac{1}{R^2}(1 - R'^2)$$

$$\mathcal{R}_t = \frac{1}{4}\mathcal{R} + \frac{1}{2R^2}(1 - R'^2). \tag{16}$$

Taking the two independent projections of Eq. (3), we therefore have in any spherically symmetric static equilibrium,

$$N'' = N\left\{4\pi(\rho - S_r + 2S_t) - \frac{2m}{R^3}\right\} \tag{17}$$

$$R'N' = RN\left\{4\pi S_r + \frac{m}{R^3}\right\}. \tag{18}$$

We can also combine Eqs. (18) and (13) to obtain

$$-\frac{R''}{R} + \frac{R'N'}{RN} = 4\pi(\rho + S_r). \tag{19}$$

These three equations are the complete set of equations satisfied by any static spherically symmetric system.

If the matter satisfies

$$\rho + S_r \geqslant 0, \tag{20}$$

we can immediately deduce from Eq. (19) that any spherical static configuration cannot have an apparent horizon. The apparent horizon coincides with a minimal surface, i.e., when $R'=0$. From Eq. (19) at a minimal surface we see that $R'' \leqslant 0$. However, at the outermost minimal surface we must have $R'' > 0$ since the area is increasing outwards. Thus we have a contradiction. Thus we have shown that, in a static star satisfying Eq. (20), $R' > 0$. From Eq. (10) this is equivalent to showing $m < R/2$. In [10] essentially the converse of this argument was given, showing that if a minimal surface existed (the throat of a static wormhole) then the matter cannot satisfy Eq. (20). This result has been recently proven in [22]. However, that proof requires that both ρ and S_r be positive while we only need to impose a condition on the combination.

The energy condition Eq. (20) is a single component of what is called the ''null convergence'' or ''null energy'' condition [9,11]. It is a consequence of each of the three

standard energy conditions (the ''strong,'' ''weak,'' and ''dominant''). Consider the outgoing radial null vector

$$\xi^\mu = (1/N, 1, 0, 0), \tag{21}$$

and multiply it into the spacetime Ricci tensor ${}^{(4)}\mathcal{R}_{\mu\nu}$ to get

$${}^{(4)}\mathcal{R}_{\mu\nu}\xi^\mu\xi^\nu = {}^{(4)}\mathcal{G}_{\mu\nu}\xi^\mu\xi^\nu = 8\pi(\rho + S_r), \tag{22}$$

where ${}^{(4)}\mathcal{G}_{\mu\nu}$ is the spacetime Einstein tensor. The equality above is to be expected because the Einstein tensor only differs from the Ricci tensor by a trace and the trace term, when dotted twice with a null vector, vanishes. The positivity of ${}^{(4)}\mathcal{R}_{\mu\nu}\xi^\mu\xi^\nu$ implies Eq. (20). Choosing ξ^μ to be an outgoing tangential null vector we obtain $\rho + S_r + 2S_t \geqslant 0$.

If both $S_r \geqslant 0$ and $\rho \geqslant 0$ hold independently, as supposed in [22], it is clear that Eq. (20) is satisfied which guarantees $R' > 0$. We also now have that the right hand side of Eq. (18) is positive so that $N' \geqslant 0$ everywhere. The lapse function, the length of the Killing vector, for any regular solution must grow monotonically out from the center to its asymptotic value one. With positive radial stress and positive ρ we do not need to assume the strong energy condition. Spherical symmetry is very restrictive. Compare this to the spherically symmetric statement of the maximum principle which was applied earlier to the trace equation, Eq. (4).

For completeness, we note that the lapse is evaluated in the exterior of a compact object as follows. Using Eq. (18), we have

$$R'N' = N\frac{m_0}{R^2}, \tag{23}$$

so that using Eq. (14),

$$\frac{N'}{R'} = \frac{m_0}{R^2}\left(1 - \frac{2m_0}{R}\right)^{-1}. \tag{24}$$

The boundary condition at infinity, $N \to 1$, fixes

$$N = \left(1 - \frac{2m_0}{R}\right)^{1/2}. \tag{25}$$

Equation (25) together with Eq. (14) reproduce the exterior Schwarzschild form of the spacetime metric.

The conservation of the stress tensor reduces to the single equation,

$$S_r' + 2\frac{R'}{R}(S_r - S_t) = -(S_r + \rho)\frac{N'}{N}. \tag{26}$$

We deduce immediately that at $l = 0$ in a non-singular geometry

$$S_r = S_t. \tag{27}$$

The perfect fluid form of the stress tensor is the only one consistent with the symmetry at the origin. This is exactly as in Newtonian theory.

While we only needed a condition on the radial stress to eliminate apparent horizons, the transverse stress does play a role in establishing the equilibrium. This can be seen in simple mechanical models. For example, in a soap bubble, the surface tension, which is effectively a negative transverse stress, is the object which balances the positive outward pressure difference between the inside and outside. On the other hand, if we had an evacuated spherical metal shell, with a vacuum inside and positive pressure outside, the outside pressure forces the metal shell to contract setting up a positive transverse stress. The radial stress obviously increases outwards and is balanced by the positive transverse stress. In a self-gravitating system, we expect the radial pressure to decrease outwards. However, it need not if there are large positive transverse stresses to support the external pressure.

In a spherically symmetric geometry it is possible to exploit the first order Einstein equation, Eq. (18), to reduce the dependence on the stress tensor appearing in Eq. (17) to a dependence on the ratio of the tangential to the radial stress,

$$\gamma = \frac{S_t}{S_r}. \tag{28}$$

We have

$$N'' + \frac{R'}{R}(1 - 2\gamma)N' = \left\{4\pi\rho - \frac{m}{R^3}(1 + 2\gamma)\right\}N. \tag{29}$$

In particular, if the stress is isotropic then $\gamma = 1$ and Eq. (29) is independent of S_{ab}.

Alternatively, we can exploit Eq. (18) to eliminate the lapse from the conservation equation, Eq. (26):

$$S_r' + 2\frac{R'}{R}(S_r - S_t) = -\frac{R}{R'}(S_r + \rho)\left(4\pi S_r + \frac{m}{R^3}\right). \tag{30}$$

In the isotropic limit, Eq. (30) is the Tolman-Oppenheimer-Volkov equation. In the Newtonian limit, the right hand side (RHS) of Eq. (30) is replaced by $-\rho m/r^2$.

In the simple mechanical models provided above, it is clear that positive transverse stresses reduce the internal pressure and *vice versa*. We can exploit Eq. (26) [or Eq. (30)] to show that, in general, if the transverse pressure is smaller than the radial pressure the radial pressure builds up inside. We will give two slightly different versions of this result.

First, let us assume $S_r - S_t \geqslant 0$, $\rho + S_r \geqslant 0$ and $\rho + S_r + 2S_t \geqslant 0$ (both of the latter coming from the ''strong energy'' condition). From the trace equation, Eq. (4), we have that $N' > 0$ and from Eq. (19) we have $R' > 0$. When these are substituted into Eq. (26) we get $S_r' < 0$ so the pressure monotonically increases inward. The object does not need to be compact.

Alternatively, even more simply, let us assume $S_r - S_t \geqslant 0$ and $\rho \geqslant 0$. The Hamiltonian constraint guarantees $m > 0$. Suppose first that the object is compact. We cannot have an apparent horizon outside so we have $R' > 0$ on the boundary. At the boundary of a compact object $S_r = 0$ and so from Eq. (30) we get $S_r' < 0$ so that S_r is decreasing outwards and therefore must be positive near the boundary. However a positive S_r makes the right hand side of Eq. (30) even more

negative so S_r becomes ever larger as one moves inwards. Thus we have shown that $S_r>0$ and monotonically decreases as one travels out. In turn this guarantees both $\rho+S_r>0$ and $4\pi S_r+m/R^3>0$. Hence $R'>0$ and $N'>0$. Note that we need not assume that S_t vanishes on the boundary but it cannot be positive there. In other words, surface tension is good. If the object is not compact, the argument we have just presented is valid in the region bounded by any sphere with $S_r\geqslant 0$.

In this section, using very weak assumptions, we have demonstrated that in a spherical static star we have $0<R'\leqslant 1$. From Eq. (10) we now get that both $m>0$ and $2m/R<1$. However, we have to date no information on how close R' can get to zero or how close $2m/R$ can get to 1. This will be discussed in the following sections, where, by imposing various restrictions on the matter, we get extra control on the behavior of $2m/R$.

IV. CONSTANT DENSITY PERFECT FLUID STAR

It is our good fortune that for a perfect fluid constant density star, Eq. (30) is exactly solvable. Equations (10) and (11) reduce to

$$R'^2+\left(\frac{8\pi\rho_0}{3}\right)R^2=1. \tag{31}$$

We then have

$$\frac{dP}{dR}=-4\pi\frac{R}{1-\frac{8\pi}{3}\rho_0R^2}(P+\rho_0)\left(P+\frac{1}{3}\rho_0\right), \tag{32}$$

with the well known solution,

$$P(R)=\rho_0\left(\frac{\left(1-\frac{2m_0R^2}{R_0^3}\right)^{1/2}-\left(1-\frac{2m_0}{R_0}\right)^{1/2}}{3\left(1-\frac{2m_0}{R_0}\right)^{1/2}-\left(1-\frac{2m_0R^2}{R_0^3}\right)^{1/2}}\right). \tag{33}$$

The pressure always exceeds the Newtonian value. In fact, in an isotropic uniform Newtonian fluid ball of radius R_0, the central pressure is given by $P=P_c$, where

$$P_c=\frac{2\pi}{3}\rho_0^2R_0^2. \tag{34}$$

The pressure given by Eq. (33) diverges at the center $R=0$ when $m_0=4R_0/9$. This occurs when the surface lapses, $N_0=1/3$. If $m_0>4R_0/9$, it diverges at some finite value of R. As $2m_0$ is increased up to R_0, the divergence moves out to R_0.

Let us now examine the lapse. In a constant density perfect fluid, Eq. (29) assumes the very simple form

$$\left(\frac{N'}{R}\right)'=0. \tag{35}$$

We exploit the continuity of the lapse and its first derivative across R_0 which follow from Eqs. (17) and (18). We first integrate out from some interior point to the surface at $R_0=R(l_0)$:

$$\left(\frac{N'}{R}\right)_{R<R_0}=\left(\frac{N'}{R}\right)_{R=R_0}=\frac{m_0}{R_0^3}, \tag{36}$$

where we have exploited Eq. (25) to evaluate the RHS. We integrate again over the same domain. We find for the surface lapse,

$$N_0=N_c+\frac{m_0}{R_0^3}\int_0^{l_0}dlR(l), \tag{37}$$

where N_c is the value of the lapse at the center. We note that generally

$$\int_l^{l_0}dlR(l)=\int_R^{R_0}RdR\left(1-\frac{2m}{R}\right)^{-1/2}. \tag{38}$$

Thus, in a constant density star,

$$\begin{aligned}\int_l^{l_0}dlR(l)&=\int_R^{R_0}RdR\left(1-\frac{2m_0R^2}{R_0^3}\right)^{-1/2}\\&=\frac{1}{2}\left(1-\left(1-\frac{2m_0R^2}{R_0^3}\right)^{1/2}\right)\frac{R_0^3}{m_0},\end{aligned} \tag{39}$$

and we get

$$N_0=\left(1-\frac{2m_0}{R_0}\right)^{1/2}=N_c+\frac{1}{2}\left(1-\left(1-\frac{2m_0}{R_0}\right)^{1/2}\right). \tag{40}$$

We require $N_c\geqslant 0$. Eq. (40) then implies

$$0\leqslant\frac{3}{2}\left(1-\frac{2m_0}{R_0}\right)^{1/2}-\frac{1}{2}, \tag{41}$$

or

$$m_0\leqslant\frac{4}{9}R_0, \tag{42}$$

exactly as before. This route, however, has the advantage that Eq. (29) is linear in N unlike Eq. (30) which is nonlinear in S_r.

It is worth noting that $N\to 0$ as $m\to 4R/9$ should not be viewed as the Killing vector going null. It is another version of the ''collapse of the lapse'' phenomenon, in this case, driven by the fact that the pressure is becoming unboundedly large.

V. MONOTONIC STARS

Buchdahl [2] demonstrated that if the energy density profile in a star is monotonically decreasing, and it is modeled as a perfect fluid, this 4/9 bound continues to hold. The constant density star saturates the bound within this class of systems. In this section we follow Buchdahl in only consid-

ering objects with monotonically decreasing densities but we will push the calculations much further. We start off with a perfect fluid assumption and rederive the 4/9 bound. We then weaken this to the dominant radial pressure assumption ($S_r \geqslant S_t$) that we used in Sec. III and prove that the 4/9 bound is still valid. We next extend the inequality to interior points. We finally consider the situation where S_t may be larger than S_r. We no longer can recover the 4/9 bound; however, if the ratio of the pressures is bounded we show that m/R is strictly bounded away from 1/2.

Let us define

$$\frac{4\pi}{3}\langle\rho\rangle := \frac{m}{R^3}, \tag{43}$$

so that

$$\langle\rho\rangle = \frac{\int_0^R \rho R^2 dR}{\int_0^R R^2 dR}, \tag{44}$$

is an average of $\rho(R)$ (not to be confused with the physical average) within a Euclidean ball. Thus if $\rho' \leqslant 0$, it is clear that $\langle\rho\rangle \geqslant \rho$ and

$$\left(\frac{m}{R^3}\right)' = \frac{4\pi}{3}\langle\rho\rangle' \leqslant 0. \tag{45}$$

In particular, one can deduce that $m/R \geqslant m_0 R^2/R_0^3$, so that

$$\left(1-\frac{2m}{R}\right)^{-1/2} \geqslant \left(1-\frac{2m_0R^2}{R_0^3}\right)^{-1/2}, \tag{46}$$

a lower bound is always provided by a constant density star with the same m_0 and R_0.

We mimic the constant density star calculation. This is essentially the Buchdahl derivation, however we allow for a non-perfect fluid. We combine Eqs. (17) and (18) to give

$$\left(\frac{N'}{R}\right)' = \frac{N''}{R} - \frac{N'R'}{R^2} = \frac{4\pi N}{R}[(\rho - \langle\rho\rangle) + 2(S_t - S_r)]. \tag{47}$$

Both terms on the RHS of Eq. (47) are negative when $\rho' \leqslant 0$ and $S_r - S_t \geqslant 0$. Thus we have

$$\left(\frac{N'}{R}\right)' \leqslant 0, \tag{48}$$

with equality only in a constant density star supported by isotropic pressure. The remainder of the calculation in this case mimics that for a constant density star.

As before, we first integrate Eq. (48) out from some interior point to the surface at $R = R_0$:

$$\left(\frac{N'}{R}\right)_0 \geqslant \frac{m_0}{R_0^3}. \tag{49}$$

We follow this by integrating out from the center at $l = 0$ to the surface. We find

$$N_0 \geqslant N_c + \frac{m_0}{R_0^3}\int_0^{l_0} dl R(l). \tag{50}$$

We require a lower bound on the integral appearing in the second term on the RHS. We cast it, as before, in the form (38). Using Eq. (46), it is clear that

$$\int_0^{l_0} dl R(l) \geqslant \frac{1}{2}\left(1-\left(1-\frac{2m_0}{R_0}\right)^{1/2}\right)\frac{R_0^3}{m_0}. \tag{51}$$

When we substitute Eq. (51) and Eq. (25) into Eq. (50) together with the requirement that $N_c \geqslant 0$ we recover Eq. (41) and so we have that $2m_0/R_0 \leqslant 8/9$.

This gives only a bound at the boundary. If the configuration has a ''thin'' atmosphere with $3\rho < \langle\rho\rangle$, m/R is decreasing so the maximum value of m/R occurs somewhere in the interior and not on the boundary. In such a scenario the above result is not very useful. Happily, the argument can be tweaked to show that $2m/R \leqslant 8/9$ through the whole system.

Let us assume $\rho \geqslant 0$, $\rho' \leqslant 0$ and $S_r \geqslant S_t$. The argument at the end of Sec. III shows us that $S_r \geqslant 0$. From Eq. (18) we have

$$\frac{N'}{R} = \frac{N}{R'}\left(4\pi S_r + \frac{m}{R^3}\right) \geqslant \frac{N}{R'}\left(\frac{m}{R^3}\right). \tag{52}$$

Let us assume that $2m/R$ possesses a maximum at a point a distance l_1 from the center. The monotonicity of N'/R [Eq. (48)] and Eq. (52) gives [in contrast Eq. (49)]

$$\frac{N'}{R} \geqslant \left(\frac{N}{R'}\frac{m}{R^3}\right)_1 \quad \forall l \leqslant l_1. \tag{53}$$

Integrate from the center to l_1 to get

$$N_1 \geqslant N_c + \left(\frac{N}{R'}\frac{m}{R^3}\right)_1 \int_0^{l_1} R dl \tag{54}$$

$$= N_c + \left(\frac{N}{R'}\frac{m}{R^3}\right)_1 \int_0^{R_1} \frac{R dR}{(1-2m/R)^{1/2}} \tag{55}$$

$$\geqslant N_c + \left(\frac{N}{R'}\frac{m}{R^3}\right)_1 \int_0^{R_1} \frac{R dR}{(1-2m_1R^2/R_1^3)^{1/2}}, \tag{56}$$

where the last line follows from the monotonicity of m/R^3. This can be integrated to give

$$N_1 \geqslant N_c + \frac{m_1 N_1}{R_1^3\left(1-\frac{2m_1}{R_1}\right)^{1/2}}\frac{R_1^3}{2m_1}\left[1-\left(1-\frac{2m_1}{R_1}\right)^{1/2}\right]. \tag{57}$$

Requiring that $N_c \geqslant 0$ allows us to cancel the N_1 on both sides and we immediately get that $2m/R \leqslant 8/9$.

To deal with the situation where S_r can be less than S_t we need a somewhat more complicated argument. We add now as one of our assumptions that $S_r \geqslant 0$.

If we divide Eq. (47) by Eq. (18) we can get

$$\left(\frac{N'}{R}\right)' = \frac{N'}{R}\frac{R'}{R}\frac{(\rho-\langle\rho\rangle)+2(S_t-S_r)}{S_r+\dfrac{\langle\rho\rangle}{3}}. \tag{58}$$

Let us assume that the term on the right hand side of Eq. (58) which depends on the sources is bounded. In other words we assume

$$\frac{(\rho-\langle\rho\rangle)+2(S_t-S_r)}{S_r+\dfrac{\langle\rho\rangle}{3}} \leqslant \beta. \tag{59}$$

It is clear that β cannot be negative because the numerator vanishes at the center. We have $\beta=0$ for a monotonic star with $S_t \leqslant S_r$. In general, it will be some positive number. Eq. (58) now reads

$$\left(\frac{N'}{R}\right)' \leqslant \beta\frac{N'}{R}\frac{R'}{R}. \tag{60}$$

Find the point where $2m/R$ is a maximum (call it l_1 as before) and integrate Eq. (60) out to it to give

$$\ln\left(\frac{(N'/R)_1}{(N'/R)}\right) \leqslant \beta \ln R_1/R, \tag{61}$$

so that

$$\frac{N'}{R} \geqslant \left(\frac{N'}{R}\right)_1\left(\frac{R}{R_1}\right)^\beta. \tag{62}$$

As before, we integrate this equation from the center out to l_1 to get

$$N_1 \geqslant N_c + \left(\frac{N'}{R}\right)_1\int_0^{l_1}\left(\frac{R}{R_1}\right)^\beta R dl \tag{63}$$

$$N_1 \geqslant N_c + \left(\frac{N}{R'}\left[S_r+\frac{m}{R^3}\right]\right)_1\int_0^{l_1}\left(\frac{R}{R_1}\right)^\beta R dl \tag{64}$$

$$\geqslant N_c + \left(\frac{N}{R'}\frac{m}{R^3}\right)_1\int_0^{R_1}\left(\frac{R}{R_1}\right)^\beta \frac{RdR}{(1-2m/R)^{1/2}} \tag{65}$$

$$\geqslant N_c + \left(\frac{N}{R'}\frac{m}{R^3}\right)_1\int_0^{R_1}\left(\frac{R}{R_1}\right)^\beta \frac{RdR}{\left(1-\dfrac{2m_1R^2}{R_1^3}\right)^{1/2}}. \tag{66}$$

In going from Eq. (63) to Eq. (64) we use Eq. (18) and in going from Eq. (64) to Eq. (65) we use $S_r(l_1)\geqslant 0$. It is clear that the integral in Eq. (66) is finite and well behaved for any finite β. Thus we get a bound on $2m/R$ which is strictly bounded away from 1. Only in the limit as $\beta\to\infty$ does the integral go to zero. In this case the bound on $2m/R\to 1$. In the other limit, when $\beta\to 0$, we recover Eq. (56) and so we get $2m/R\to 8/9$. In special cases where $\beta=2,4,6,\ldots$ the integral in Eq. (66) can be done simply. This includes one especially interesting case.

Let us assume we are given a monotonic star with positive radial pressure (these assumptions can be justified by stability criteria). Let us further assume that the transverse pressure is bounded. More precisely let us assume $S_t \leqslant \rho$. This can be justified on some kind of speed of sound argument. From the monotonicity we get $S_t \leqslant \langle\rho\rangle$. From these we immediately get $\beta\leqslant 6$. Now we can do the integration and get $2m/R \leqslant 0.974$.

Alternatively, if the material was approximately a perfect fluid we could use the ratio of the pressures, γ, that we introduced earlier. It is clear that $\beta \leqslant 2(\gamma_{Max}-1)$.

VI. COMPLETELY GENERAL SPHERICAL CONFIGURATION

Let us now consider a general static spherical ball. We no longer wish to assume either monotonicity or a perfect fluid. The only constraints we place are that both $\rho\geqslant 0$ and $S_r \geqslant 0$. We also assume that β as defined by Eq. (59) exists. We are less interested in obtaining the tightest bound on $2m/R$ than in establishing that such a bound exists.

All the equations, starting from Eq. (59) up to and including Eq. (65) continue to hold. However, in going from Eq. (65) to Eq. (66) we used the monotonicity. One way of avoiding that difficulty is by replacing Eq. (66) with

$$N_1 \geqslant N_c + \left(\frac{N}{R'}\frac{m}{R^3}\right)_1\int_0^{R_1}\left(\frac{R}{R_1}\right)^\beta \frac{RdR}{\left(1-\dfrac{8\pi\langle\rho\rangle_{\rm Min}}{3}R^2\right)^{1/2}}. \tag{67}$$

This uses $2m/R = 8\pi\langle\rho\rangle R^2/3 \geqslant 8\pi\langle\rho\rangle_{\rm Min}R^2/3$. Equation (67) can be simplified by introducing a new variable $x^2 = 8\pi\langle\rho\rangle_{\rm Min}R^2/3$ and $x_1^2 = 8\pi\langle\rho\rangle_{\rm Min}R_1^2/3 = 2\bar{m}_1/R_1 \leqslant 2m_1/R_1$ where $\bar{m}=4\pi\langle\rho\rangle_{\rm Min}R^3/3 \leqslant m$. We then get from $N_c>0$

$$\sqrt{1-\frac{2m_1}{R_1}} \geqslant \frac{\langle\rho\rangle}{2\langle\rho\rangle_{\rm Min}}\int_0^{x_1}\left(\frac{x}{x_1}\right)^\beta\frac{xdx}{\sqrt{1-x^2}}. \tag{68}$$

It is clear that the right hand side of Eq. (68) is finite and bounded away from zero as long as $\langle\rho\rangle_{\rm Min}$ is non-zero. This is very misleading because of the dependence of x_1 on $\langle\rho\rangle_{\rm Min}$. If we return to Eq. (67) we can see that the integral has a lower bound of $R_1^2/(\beta+2)$ and this is achieved when $\langle\rho\rangle_{\rm Min}=0$. In this case we get a bound on $2m_1/R_1$ given by

$$2(\beta+2)\sqrt{1-\frac{2m_1}{R_1}} \geqslant \frac{2m_1}{R_1}. \tag{69}$$

Thus we get the following bound:

$$\frac{2m_1}{R_1} \leqslant \frac{1}{2}(\beta+2)^2\left[\sqrt{1+\frac{4}{(\beta+2)^2}}-1\right] \approx 1-\frac{2}{(\beta+2)^2}. \tag{70}$$

The approximation given in Eq. (70) holds only in the limit as β becomes large. In general, we can show that the expres-

sion in Eq. (70) is always less than 1 and monotonically increases with β. For example, when $\beta=0$ we get $2m_1/R_1 \leqslant 0.944$.

We know that $0 \leqslant \langle\rho\rangle_{\rm Min} \leqslant \langle\rho\rangle_1$. For any fixed value of β, if $\langle\rho\rangle_{\rm Min} \approx \langle\rho\rangle_1$ the bound we get on $2m_1/R_1$ agrees with the monotonic bound as given implicitly by Eq. (66). As $\langle\rho\rangle_{\rm Min}$ reduces below $\langle\rho\rangle_1$ the bound on $2m_1/R_1$ increases monotonically and in the limit $\langle\rho\rangle_{\rm Min} \approx 0$ it reaches the bound given in Eq. (70) and so is always strictly bounded away from 1.

VII. CONCLUSIONS

We have examined the ratio of the quasi-local mass to the circumferential radius, $2m/R$, for physically reasonable spherically symmetric isolated static configurations in general relativity. We have demonstrated how the theory always places an upper bound on this ratio which lies strictly below the value it assumes when a horizon forms. This extends considerably earlier work on this question.

The bounds we have derived do not take into account the stability of these static equilibria. It would be interesting to know if and how these bounds get tightened when only stable configurations are considered [23].

A direction we have not followed is to exploit specific physical properties of the configuration to place stronger limits on $2M/R$. In this context, Lindblom placed a limit below 8/9 on this value for a realistic neutron star [24].

In [10] Morris and Thorne addressed the problem of constructing a static wormhole. If one has a spherical static wormhole one must have a minimal surface and thus $\rho+S_r<0$ somewhere. This raises an interesting question. Assume one has a spherical static wormhole and assume $\rho \geqslant 0$. How much ''exotic material'' (violating the strong energy condition) does one need [25]?

ACKNOWLEDGMENTS

We gratefully acknowledge support from CONACyT-NSF to J.G. and Forbairt Grant SC/96/750 to N.Ó.M. We would like to thank Luis Herrera for drawing our attention to Refs. [7] and [8].

[1] J. R. Oppenheimer and G. Volkov, Phys. Rev. **55**, 374 (1939).
[2] H. A. Buchdahl, Phys. Rev. **116**, 1027 (1959).
[3] H. Bondi, Proc. R. Soc. London **A282**, 303 (1964); in *Lectures on General Relativity*, Brandeis Summer Institute in Theoretical Physics, Vol. 1 (Prentice Hall, New Jersey, 1964).
[4] R. Wald, *General Relativity* (Chicago University Press, Chicago, 1984).
[5] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
[6] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Other Topological Defects* (Cambridge University Press, Cambridge, England, 1994).
[7] G. Lemaitre, Ann. Soc. Sci. Bruxelles **A53**, 51 (1933).
[8] L. Herrera and N. O. Santos Phys. Rep. **286**, 53 (1997).
[9] S. Hawking and G. Ellis, *The Large Scale Structure of Space-time* (Cambridge University Press, Cambridge, England, 1973).
[10] M. Morris and K. Thorne, Am. J. Phys. **56**, 295 (1988).
[11] J. Friedman, K. Schleich, and D. Witt, Phys. Rev. Lett. **71**, 1486 (1993).
[12] C. Misner, K. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1972).
[13] Y. Choquet-Bruhat and J. York, in *General Relativity and Gravitation*, edited by A. Held (Plenum, New York, 1980).
[14] N. Ó Murchadha, Phys. Lett. **60A**, 177 (1977).
[15] M. Cantor, J. Math. Phys. **20**, 1741 (1979).
[16] A. K. M. Masood-ul-alam, Class. Quantum Grav. **5**, 409 (1988).
[17] R. Beig and W. Simon, Lett. Math. Phys. **21**, 245 (1991).
[18] The most recent installment and the strongest current proof of the static implies spherical symmetry conjecture is L. Lindblom and A. K. M. Masood-ul-Alam, Commun. Math. Phys. **162**, 123 (1994).
[19] L. Lindblom, Philos. Trans. R. Soc. London **A340**, 353 (1992).
[20] J. Guven and N. Ó Murchadha, Phys. Rev. D **52**, 758 (1995). An extensive list of references is provided here.
[21] J. Guven and N. Ó Murchadha, Phys. Rev. D **52**, 776 (1995).
[22] T. W. Baumgarte and A. D. Rendall, Class. Quantum Grav. **10**, 327 (1993).
[23] The issue of stability is discussed in [8].
[24] L. Lindblom, Astrophys. J. **278**, 364 (1984).
[25] Since the manuscript was completed this question has been addressed by D. Ida and S. Hayward, ''How much negative energy does a wormhole need,'' gr-qc/9905033.