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# Algebraic Central Limit Theorems in Noncommutative Probability

Thesis presented by

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*Ayman Alahmade*

# Abstract

Distributional symmetries and invariance principles in noncommutative probability theory provide sufficient conditions for the existence of central limit laws. In contrast to classical probability theory, there exist many different central limit laws for exchangeable sequences of noncommutative random variables and still little is known about their concrete form. This thesis goes one step further and investigates central limit laws for non-exchangeable spreadable sequences in the context of *-algebraic probability spaces. This provides first results on a new type of combinatorics underlying multivariate central limit theorems (CLTs).

The starting point of the thesis has been a quite simple family of spreadable sequences, which is parametrized by a unimodular complex parameter $\omega$. Each sequence of this family is spreadable, but not exchangeable for $\omega \neq \pm 1$. Moreover, the sequences from this family provide CLTs, which interpolate between the normal distribution ($\omega = 1$) and the symmetric Bernoulli distribution ($\omega = -1$), but differ from $q$-Gaussian distributions ($-1 < q < 1$). An algebraic structure, which underlies the considered family, is identified and used to define so-called '$\omega$-sequences of partial isometries'. These $\omega$-sequences encode all information, as it is relevant for computations of *-algebraic CLTs. Explicit combinatorial formulas are established for CLTs associated to such $\omega$-sequences, which involve the counting of oriented crossings of directed ordered pair partitions. The limiting distributions of certain multivariate CLTs associated to $\omega$-sequences show some features as they are defining for '$z$-circular systems' [MN01]. This similarity, as well as the well-known relation between $q$-circular systems and $q$-semicircular systems (for $-1 \leq q \leq 1$), guides the introduction of '$z$-semicircular systems' in this thesis. Finally, it is shown that the class of $z$-semicircular systems is stable under certain multivariate central limits. In other words, the moment formulas of $z$-semicircular systems are reproduced in large $N$-limit formulas of central limit type.

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# Chapter 1

# Introduction

*Nowadays, the central limit theorem is considered to be the unofficial sovereign of probability theory.* – Henk Tijms (2004)

The Central Limit Theorem (CLT) is considered to be among the most important results in classical probability. It tells that an appropriately scaled sum of $N$ identically distributed, independent random variables converges to a Gaussian (or normal) distributed random variable in the large $N$-limit. As classical physics needs to be replaced by quantum physics in many modern applications, the methods and results from classical theory need to be revised such that they apply to quantum phenomenon, where possible. This creates the subject of quantum probability or, in terms of a systematic algebraization of probability theory, the subject of noncommutative probability. Present thesis contributes to this programme of algebraization by investigating certain central limit laws in an algebraic framework of noncommutative probability.

Central limit theorems in quantum probability theory were established by mathematicians starting the early 1970s. To our knowledge, early versions of quantum CLTs were provided for certain bosonic and fermionic systems by Hudson et al. in [CH71, H73, CGH77], resulting in normal distributions as central limit laws. More algebraic versions of CLTs were established starting the end of the 1970s by von Waldenfels et al. in [GW78, W78, W86, W87]. Furthermore, algebraic versions of the CLT were also studied by Schürmann and von Waldenfels in [SW88, W90, S91].

Since then other algebraic versions of the CLT have been established and limit distributions beyond that of the normal distribution have been obtained. Quite general algebraic approaches to CLTs were given by Speicher in [Sp90, Sp92, Sp93] and further studies were done by Speicher and von Waldenfels in [SW94], as well as Bożejko and Speicher in [BS96]. Using mainly combinatorial arguments, these publications provide quite general conditions for the existence of large $N$-limits as they are considered for algebraic CLTs.

More concrete results and central limit laws were obtained in the context of $q$-Gaussian random variables $X_q$ for $-1 \leq q \leq 1$. The distribution of these random variables is given by the probability measure $\mu_q$ on $\mathbb{R}$ such that one has for the $n$-th moment

$$\mathbb{E}(X_q^n) = \int_{\mathbb{R}} t^n \mu_q(dt).$$

Here the measure $\mu_q$ has support $[\frac{-2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}]$ for $-1 < q < 1$ and support $\mathbb{R}$ for $q = 1$. Furthermore, $\mu_q$ is continuous for $-1 < q \leq 1$ with density function

$$f_q(t) = \frac{\sqrt{1-q}}{2\pi}\sqrt{4-(1-q)t^2}\sum_{k=1}^{\infty}(-1)^{k-1}q^{\frac{k(k-1)}{2}}U_{2k-2}\left(\frac{t\sqrt{1-q}}{2}\right),$$

where $U_k$ denotes the $k$-th Chebyshev polynomial of the second kind [Sz09] and $f_1(t) := \lim_{q\to 1} f_q(t)$ is the normal distribution. As special cases, one obtains the symmetric Bernoulli distribution $\mu_{-1} = \frac{1}{2}(\delta_{-1}+\delta_1)$, Wigner semicircle distribution $\mu_0$ with density $f_0(t) = \frac{\sqrt{4-t^2}}{2\pi}$ on the interval $[-2, 2]$, and the Gaussian distribution $\mu_1$ with density $f_1(t) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{t^2}{2})$ on $\mathbb{R}$. Alternatively, as shown in [LM95] for example, the $n$-th moment of a $q$-Gaussian random variable can be computed as

$$\mathbb{E}(X_q^n) = \sum_{\pi\in\mathcal{P}_2(n)} q^{\mathrm{cr}(\pi)}, \tag{1.1}$$

where $\mathcal{P}_2(n)$ is the set of pair partitions of the set $\{1, 2, \ldots, n\}$ and $\mathrm{cr}(\pi)$ denotes the number of crossings of the pair partition $\pi$ (see Section 2.1 for further information on set partitions etc.). We note that, for even $n = 2k \in \mathbb{N}$, the map $[-1, 1] \ni q \mapsto \mathbb{E}(X_q^{2k})$ is a polynomial of degree $\frac{k}{2}(k-1)$ with integer coefficients. Explicit computations of the first few even moments yield

$$\begin{aligned}
\mathbb{E}(X_q^2) &= 1,\\
\mathbb{E}(X_q^4) &= 2 + q,\\
\mathbb{E}(X_q^6) &= 5 + 6q + 3q^2 + q^3,\\
\mathbb{E}(X_q^8) &= 14 + 28q + 28q^2 + 20q^3 + 10q^4 + 4q^5 + q^6.
\end{aligned}$$

A mathematical rigorous realization of systems of $q$-Gaussian random variables is obtained by Bożejko and Speicher in [BS91, BS92] via their construction of generalized Brownian motions on $q$-Fock spaces. The classical and noncommutative aspects of $q$-Gaussian processes are further studied in [BKS97]. Moreover, it was shown by Speicher in [Sp92] that the distribution of $q$-Brownian motions (and thus any $q$-Gaussian random variable) can be obtained by algebraic central limit techniques via stochastic mixtures of commuting and anti-commuting noncommutative random variables. Thus the distribution of $q$-Gaussian random variables provides a family of central limit laws, which continuously interpolates between the normal distribution ($q = 1$), Wigner semicircle law ($q = 0$), and symmetric Bernoulli distribution ($q = -1$).

Building on the results for $q$-Gaussian random variables in [BS92, BKS97], Mingo and Nica in [MN01] show that certain averages of random unitaries in noncommutative tori converge to what they call $q$-circular systems, for some real parameter $q$ with $-1 < q < 1$. These $q$-circular systems can be seen to be closely related to so-called '$q$-semicircular systems' or, in other words, systems of $q$-Gaussian random variables. The main results of Mingo and Nica in [MN01] are around their introduction of $z$-circular systems for $z \in \mathbb{C}$ and $|z| < 1$. These systems are defined through moment formulas, which essentially count oriented crossings of certain pair partitions. Moreover, these systems reduce to $q$-circular systems whenever $z = \overline{z}$. It is shown by Mingo and Nica that these $z$-circular systems can be also obtained as certain averages of random unitaries in noncommutative tori. In contrast to the situation for $q$-circular systems, it is still an open problem to find a suitable deformed Fock space realization of $z$-circular systems in terms of annihilation and creation operators (compare also [MN01, Remark 1.12]).

As noticed in [Kö10], distributional symmetries and invariance principles in noncommutative probability theory provide sufficient conditions for the existence of noncommutative central limit laws. Already for exchangeable sequences of noncommutative random variables a huge variety of concrete central limit laws seems to exist and little is known about these laws. Essentially, depending on the underlying algebraic structure, there is the need to identify central limit laws in a case-by-case study. For example, recently Köstler and Nica have shown in [KN20] that the central limit law associated to certain characters of the infinite symmetric group is closely related to the distribution of certain GUE random matrices.

Actually, there is a hierarchy of distributional symmetries and invariance principles in noncommutative probability. In particular, it is shown in [GK09, Kö10] that exchangeability implies braidability, and that braidability implies spreadability. This motivates to study in more detail CLTs for exchangeable, braidable or spreadable sequences.

In this thesis we investigate CLTs for non-exchangeable spreadable sequences in the context of *-algebraic probability spaces, to provide the first results on the combinatorics of CLTs for certain non-exchangeable spreadable sequences. We emphasize that so far no central limit law is concretely identified in the wider context of non-exchangeable spreadable sequences, aside of those in the framework of Boolean or (anti-)monotone independence (see [Mu11, Wy08], for example).

The starting point of the investigations in this thesis has been the construction of a so-called braidable sequence $\mathbf{x} \equiv (x_n)_{n=1}^{\infty}$ in the infinite algebraic tensor product of complex $2 \times 2$ matrices $\mathcal{A} = \bigotimes_{n=1}^{\infty} \mathbb{M}_2(\mathbb{C})$ such that

$$x_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes 1_2^{\otimes \mathbb{N}}, \qquad x_{n+1} = u_n x_n u_n^* \qquad (n \in \mathbb{N}). \tag{1.2}$$

Here the unitary matrices $u_n \in \mathcal{A}$ are given by the amplifications

$$u_n = \underbrace{1_2 \otimes \cdots \otimes 1_2}_{(n-1)\text{-fold}} \otimes U \otimes 1_2^{\otimes \mathbb{N}}$$

of the unitary matrix

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix} \in \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C}) \qquad (\omega \in \mathbb{T}), \tag{1.3}$$

where $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$. It is elementary to verify that the unitaries $u_n$ satisfy the braid relations

$$\begin{aligned} u_i u_j u_i &= u_j u_i u_j && \text{for } |i-j| = 1, \\ u_i u_j &= u_j u_i && \text{for } |i-j| > 1. \end{aligned}$$

We remark here that a unitary matrix $U \in \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$ implements the commuting square

$$\begin{array}{ccc} U(\mathbb{M}_2(\mathbb{C}) \otimes 1_2)U^* & \subset & \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C}) \\ \cup & & \cup \\ \mathbb{C}\, 1_2 \otimes 1_2 & \subset & \mathbb{M}_2(\mathbb{C}) \otimes 1_2 \end{array}$$

if and only if $U$ is of the form (1.3) for some $\omega \in \mathbb{T}$, up to biunitary equivalence (see [KSV96, Proposition 12]). Thus the constructed sequence $\mathbf{x}$ is also of special interest as it represents a simple example of a braidable sequence in the context of Jones subfactor theory [GHJ89, JS97].

Throughout we consider the infinite tensor product algebra $\mathcal{A} = \bigotimes_{n=1}^{\infty} \mathbb{M}_2(\mathbb{C})$ to be equipped with the tensor product state $\mathrm{tr}\colon \mathcal{A} \to \mathbb{C}$ given by $\mathrm{tr} = \otimes_{\mathbb{N}} \mathrm{tr}_2$, where $\mathrm{tr}_2$ denotes the normalized trace on $\mathbb{M}_2(\mathbb{C})$. The pair $(\mathcal{A}, \mathrm{tr})$ is an example of a *-algebraic probability space such that the sequence $\mathbf{x} \subset \mathcal{A}$ can be interpreted as a sequence of quantum coin tosses. The starting point of this thesis is the following abstract CLT for the braidable sequence $\mathbf{x}$, which is proved in Theorem 4.3.6:

**Theorem 1.0.1.** *Let the sequence $\mathbf{x} \equiv (x_n)_{n=1}^{\infty} \subset \mathcal{A}$ be given as constructed above in (1.2) for some fixed $\omega \in \mathbb{T}$, and let $S_N := \frac{1}{\sqrt{N}}(x_1 + x_2 + \ldots + x_N)$. Then there exists a unique probability measure $\mu_\omega$ on $\mathbb{R}$ such that, for any $n \in \mathbb{N}$,*

$$M_n(\omega) := \lim_{N \to \infty} \mathrm{tr}(S_N^n) = \int_{\mathbb{R}} t^n \mu_\omega(dt).$$

One meets for $\omega = 1$ an algebraic reformulation of the classical central limit theorem for an infinite sequence of independent identically distributed coin tosses.

Consequently, $\mu_1$ is the probability measure of a centred Gaussian random variable with variance 1 and moments

$$M_n(1) = \begin{cases} 0 & \text{for } n \text{ odd,} \\ (n-1)!! & \text{for } n \text{ even.} \end{cases}$$

It is also known for $\omega = -1$ that one obtains the symmetric Bernoulli distribution $\mu_{-1} = \frac{1}{2}(\delta_1 + \delta_{-1})$ as central limit law such that

$$M_n(-1) = \begin{cases} 0 & \text{for } n \text{ odd,} \\ 1 & \text{for } n \text{ even.} \end{cases}$$

So far the probability measures $\mu_\omega$ seem to be unknown in the published literature for $\omega \in \mathbb{T}\backslash\{-1,1\}$. We aim at establishing combinatorial formulas for the moments $M_n(\omega)$, similar to those in (1.1) for $q$-Gaussian random variables. In other words, we aim at computing the large $N$-limit for all moments of order $n$ as explicitly as possible. It will follow from algebraic CLTs for spreadable sequences (see Theorem 3.4.9) that all odd moments vanish, i.e. one has

$$M_n(\omega) = 0$$

for any $\omega \in \mathbb{T}$ and odd $n \in \mathbb{N}$. Furthermore, brute force computation in matrices allows us to determine explicitly the first few moments of even order in terms of $q = \Re\omega$ as follows:

$$\begin{aligned}
M_2(\omega) &= \lim_{N\to\infty} \varphi(S_N^2) = 1, \\
M_4(\omega) &= \lim_{N\to\infty} \varphi(S_N^4) = 2 + q, \\
M_6(\omega) &= \lim_{N\to\infty} \varphi(S_N^6) = 5 + 6q + 3q^2 + q^3, \\
M_8(\omega) &= \lim_{N\to\infty} \varphi(S_N^8) = \frac{1}{3}(44 + 88q + 81q^2 + 52q^3 + 30q^4 + 16q^5 + 4q^6).
\end{aligned}$$

One can recognize that $M_2(\omega)$, $M_4(\omega)$ and $M_6(\omega)$ are the moments of a centred $q$-Gaussian random variable $X_q$ with variance 1. But $M_8(\omega)$ differs from the 8-th moment of $X_q$ for $\omega \in \mathbb{T}\backslash\{1,-1\}$ (and $q = \Re\omega$), as

$$\mathbb{E}(X_q^8) = 14 + 28q + 28q^2 + 20q^3 + 10q^4 + 4q^5 + q^6.$$

To be more precise, $M_8(\omega)$ is now a polynomial in the variable $q = \Re\omega$ of degree 6 with some non-integer-valued coefficients, in contrast to the moment formula (1.1) of $q$-Gaussian random variables. This difference becomes apparent for $\omega = \pm\,\mathrm{i}$ and thus $q = 0$, as the $2k$-th moment of a centred 0-Gaussian random variable with variance 1 is given by the Catalan number $C_k = \frac{1}{k+1}\binom{2k}{k}$:

$$\begin{aligned}
C_1 &= 1 = M_2(\pm\,\mathrm{i}), & C_3 &= 5 = M_6(\pm\,\mathrm{i}), \\
C_2 &= 2 = M_4(\pm\,\mathrm{i}), & C_4 &= 14 < 44/3 = M_8(\pm\,\mathrm{i}).
\end{aligned}$$

These observations rule out that the moment sequence $\big(M_n(\omega)\big)_{n=1}^{\infty}$ is that of a $q$-Gaussian random variable for $\omega \in \mathbb{T}\backslash\{1,-1\}$ with $q = \Re\omega$.

Key to the computation of higher even moments is the following quantum decomposition of the braidable sequence $\mathbf{x}$. Let

$$a_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes 1_2^{\otimes \mathbb{N}}, \qquad a_{n+1} = u_n a_n u_n^* \qquad (n \in \mathbb{N}).$$

Now each term $x_n$ of the sequence $\mathbf{x}$ can be written as $x_n = a_n + a_n^*$ with the sequence $\mathbf{a} \equiv (a_n)_{n=1}^{\infty} \subset \mathcal{A}$ enjoying the following algebraic properties:

$$\begin{aligned} a_i a_j &= \omega a_j a_i, & a_i a_j^* &= \bar{\omega} a_j^* a_i & &\text{for } 1 \le i < j < \infty, \\ a_i a_i &= 0, & a_i a_i^* + a_i^* a_i &= 1 & &\text{for } 1 \le i < \infty. \end{aligned}$$

We will see that sequences with such algebraic properties provide sufficient structure for establishing algebraic CLTs. We abstractly introduce them as $\omega$-sequences of partial isometries in Definition 4.2.1.

Our investigations reveal that certain mixed moments of $\omega$-sequences can be expressed in terms of *oriented crossings* of *directed ordered pair partitions*. Here our notion of an *oriented crossing* is inspired by the one which is used by Mingo and Nica in [MN01]. As detailed in Subsection 2.1.4, a directed ordered pair partition of the set $[2k] := \{1, 2, \ldots, 2k\}$ (with $k \in \mathbb{N}$) can be uniquely addressed by a triple $(\pi, \boldsymbol{\varepsilon}, \sigma)$, where $\pi$ is a pair partition of the set $[2k]$, and $\boldsymbol{\varepsilon}$ is a map which assigns a direction to each pair of $\pi$, and $\sigma \in S_k$ is a permutation encoding the order of the pairs of $\pi$. This provides all data as required for the notion of *oriented crossings*, such that one can talk about the number of positive oriented crossings $\operatorname{cr}_+(\pi, \boldsymbol{\varepsilon}, \sigma)$ and the number of negative oriented crossings $\operatorname{cr}_-(\pi, \boldsymbol{\varepsilon}, \sigma)$ of a directed ordered pair partition.

Our first main result is Theorem 4.2.9, which establishes properties of $\omega$-sequences as they are relevant for the computation of the moments $M_n(\omega)$. The informal notion of a 'balanced pair distribution' stipulates certain conditions on the pair $(\pi, \boldsymbol{\varepsilon})$ as stated in Theorem 4.2.9(iv).

**Theorem 1.0.2.** *An $\omega$-sequence of partial isometries $\mathbf{a} \equiv (a_n)_{n=1}^{\infty} \subset \mathcal{A}$ has the balanced pair distribution*

$$\varphi\big(a_{\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots a_{\mathbf{i}(2k)}^{\boldsymbol{\varepsilon}(2k)}\big) = \frac{1}{2^k}\, \omega^{\operatorname{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\, \overline{\omega}^{\operatorname{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}$$

*for $\mathbf{i} \colon [2k] \to \mathbb{N}$ with $\pi = \ker(\mathbf{i})$ and $\sigma \in S_k$. Here the direction map $\boldsymbol{\varepsilon} \colon [2k] \to \{*, 1\}$ and the permutation $\sigma \in S_k$ are as specified in Theorem 4.2.9(iv).*

Having established certain properties of $\omega$-sequences in Theorem 4.2.9, we prove in Theorem 4.3.1 an explicit combinatorial formula for the CLT associated to an $\omega$-sequence of partial isometries for a (tracial) *-algebraic probability space. In particular, this result applies to the concrete $\omega$-sequence $\mathbf{x}$ in the algebraic probability space $(\mathcal{A}, \varphi)$ as introduced above. We informally state this result for the convenience of the reader.

**Theorem 1.0.3.** *Let* $\mathbf{a} \equiv (a_n)_{n=1}^{\infty} \subset \mathcal{A}$ *be an* $\omega$*-sequence of partial isometries and let the sequence* $\mathbf{x} \equiv (x_n)_{n=1}^{\infty} \subset \mathcal{A}$ *be the sequence defined by* $x_n := a_n + a_n^*$*. Let* $k \in \mathbb{N}$*. Then one has* $M_{2k-1}(\omega) = 0$ *and*

$$M_{2k}(\omega) = \lim_{N\to\infty} \varphi(S_N^{2k}) = \frac{1}{k!}\frac{1}{2^k} \sum_{\sigma\in S_k} \sum_{\pi\in\mathcal{P}_2(2k)} \sum_{\substack{\boldsymbol{\varepsilon}\colon [2k]\to\{*,1\} \\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} \omega^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\overline{\omega}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}.$$

This result for the moment formula shares some features with the defining moment formulas for $z$-circular systems by Mingo and Nica (see Definition 5.3.1), as both formulas involve oriented crossings. This similarity, as well as the relation between $q$-circular systems and $q$-semicircular systems (see Chapter 5), guides us to introduce the notion of a *z-semicircular system.* The following definition is taken from Chapter 5 and its moment formula features a certain multivariate version of Theorem 1.0.3, as we have proven it in Theorem 4.3.8.

**Definition 1.0.4.** *Let* $(\mathcal{A}, \psi)$ *be a *-algebraic probability space and fix* $z \in \mathbb{C}$ *with* $|z| \leq 1$*. The family* $\widetilde{\mathcal{Y}} \equiv (\widehat{s}_r)_{r=1}^{s} \subseteq \mathcal{A}$ *with* $(s \geq 1)$ *is said to form a* $z$-semicircular system *in* $(\mathcal{A}, \psi)$ *if*

- $\widehat{s}_r = \widehat{s}_r^*$ *for all* $r \in [s] = \{1, 2, \ldots, s\}$*;*
- *for every odd* $n \geqslant 1$*,* $\mathbf{r}\colon [n] \to [s]$*,*

$$\psi(\widehat{s}_{\mathbf{r}(1)} \cdots \widehat{s}_{\mathbf{r}(n)}) = 0;$$

- *for every even* $n \geqslant 1$ *with* $n = 2k$*,* $\mathbf{r}\colon [n] \to [s]$*,*

$$\psi(\widehat{s}_{\mathbf{r}(1)} \cdots \widehat{s}_{\mathbf{r}(2k)}) = \frac{1}{k!}\frac{1}{2^k} \sum_{\sigma\in S_k} \sum_{\substack{\pi\in\mathcal{P}_2(2k) \\ \pi\leq\ker(\mathbf{r})}} \sum_{\substack{\boldsymbol{\varepsilon}\colon [2k]\to\{*,1\} \\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} z^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\overline{z}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}.$$

We show that the class of $z$-semicircular systems is stable under certain multivariate central limits. In other words, the moment formulas of $z$-semicircular systems are reproduced in large $N$-limit formulas of central limit type.

**Theorem 1.0.5.** *Suppose the sequence* $\widetilde{\mathcal{Y}} \equiv (\widehat{s}_r)_{r=1}^{\infty} \subseteq \mathcal{A}$ *forms a z-semicircular system in* $(\mathcal{A}, \psi)$*. Let* $s \in \mathbb{N}$ *and*

$$\widetilde{S}_{r,N} := \frac{1}{\sqrt{N}}\left(\widehat{s}_r + \widehat{s}_{s+r} + \ldots + \widehat{s}_{(N-1)s+r}\right)$$

*for* $r \in \{1, 2, \ldots, s\}$*. Then one has for all* $\mathbf{r}\colon [2k-1] \to \mathbb{N}$ *and* $k \in \mathbb{N}$*,*

$$\lim_{N\to\infty} \psi(\widetilde{S}_{\mathbf{r}(1),N} \cdots \widetilde{S}_{\mathbf{r}(2k-1),N}) = 0,$$

*and, for all* $\mathbf{r}\colon [2k] \to \mathbb{N}$ *and* $k \in \mathbb{N}$*,*

$$\lim_{N\to\infty} \psi(\widetilde{S}_{\mathbf{r}(1),N} \cdots \widetilde{S}_{\mathbf{r}(2k),N}) = \frac{1}{k!}\frac{1}{2^k} \sum_{\sigma\in S_k} \sum_{\substack{\pi\in\mathcal{P}_2(2k) \\ \pi\leq\ker(\mathbf{r})}} \sum_{\substack{\boldsymbol{\varepsilon}\colon [2k]\to\{*,1\} \\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} z^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\overline{z}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}.$$

We are left to outline the structure of this thesis. In Chapter 2, we introduce the necessary background of set partitions and ordered set partitions. Also, we introduce the notion of oriented crossings as they are relevant for the combinatorics of CLTs in the context of braided sequences. Moreover, we introduce the most common distributional symmetries such as exchangeability, spreadability, and braidability, to the extent as we will make use of them in the context of CLTs. In particular, we give a *-algebraic proof that braidability implies spreadability (see Theorem 2.3.22).

We start in Chapter 3 with reviewing the classical central limit theorem, including a multivariate version of it. Also, we present singleton vanishing properties (SVPs), as they are known in the literature to play a role for multivariate versions of *-algebraic CLTs. Additionally, we refine the notion of exchangeability/spreadability of sequences of random variables to that of $C$-jointly and $C$-separately exchangeable/spreadable families of random variables, for some 'color set' $C$. We provide multivariate CLTs, which correspond to these refined notions of distributional symmetries or invariance principles. So far $C$-separately exchangeable/spreadable families of random variables have not been addressed explicitly in the published results on *-algebraic CLTs. Related results will be used later for CLTs associated to $\omega$-sequences of partial isometries (which we introduce in Chapter 4). Also, we discuss how one can construct $C$-jointly and $C$-separately exchangeable/spreadable sequences from a single exchangeable or spreadable sequence. Furthermore, we present factorization properties of mixed moments in the context of distributional invariance principles. We will make use of these factorization properties for SVPs when establishing concrete moment formulas for CLTs associated to $\omega$-sequences of partial isometries.

Chapter 4 is the main objective of this thesis. We construct a braidable sequence $\mathbf{x} \equiv (x_n)_{n=1}^{\infty}$ in the infinite algebraic tensor product of complex $2 \times 2$ matrices $\mathcal{A} = \bigotimes_{n=1}^{\infty} \mathbb{M}_2(\mathbb{C})$. Also, we extract algebraic properties of the constructed braidable sequence. In turn we use these algebraic properties to abstractly introduce $\omega$-sequences of partial isometries. We investigate some properties of $\omega$-sequences, as we will need them when establishing CLTs associated to $\omega$-sequences of partial isometries for a (tracial) *-algebraic probability space. In particular, we prove explicit combinatorial formulas for moments as they appear in the large $N$-limit of algebraic CLTs, including their multivariate versions, for $\omega$-sequences. These combinatorial formulas reveal that the moment formulas count oriented crossings of directed ordered pair partitions in the large $N$-limit and differ from those of $q$-Gaussian random variables starting the 8-th moment.

Chapter 5 starts with reviewing multivariate versions of CLTs for $q$-circular and $q$-semicircular systems. We show that such systems are exchangeable and thus yield CLTs. In particular, we show that certain multivariate CLTs associated to $q$-circular systems and $q$-semicircular systems have moment formulas which reproduce those of $q$-circular systems and $q$-semicircular systems, respectively. Inspired by the notion of a '*$z$-circular system*', defined and studied by Mingo and Nica in

[MN01], we introduce the notion of a '*$z$-semicircular system*'. These generalize the corresponding notions of $q$-circular and $q$-semicircular systems from the parameter $q \in [-1, 1]$ to the parameter $z \in \mathbb{C}$ with $|z| \leq 1$. We show that such systems are spreadable and satisfy SVPs. Thus $z$-circular systems and $z$-semicircular systems yield CLTs such that their moment formulas generalize those moment formulas obtained from CLTs associated to $\omega$-sequences of partial isometries. In particular, we show that certain multivariate CLTs for $z$-(semi)circular systems yield $z$-(semi)circular systems in the large $N$-limit.

# Chapter 2

# Preliminaries

We present definitions and notations for set partitions and ordered set partitions as they are relevant within the context of algebraic central limit theorems. In particular, this includes the notions of (oriented) crossings of (directed ordered) pair partitions. Also, we introduce some basics of *-algebraic probability spaces. Finally, we briefly discuss distributional symmetries such as exchangeability, spreadability, and braidability.

## 2.1 Partitions and Ordered Partitions

We introduce basic definitions and notations of *set partitions* and *ordered set partitions*, as we will make use of them in the context of central limit theorems for exchangeable and spreadable sequences. Furthermore, we introduce the crossing of a partition and an oriented crossing for ordered pair partitions, adapting the approach of [MN01, Subsection 1.4] as appropriate within our combinatorial treatment of *-algebraic CLTs.

### 2.1.1 Basics on Set Partitions

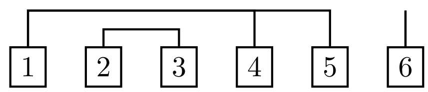
We start with providing the basics of set partitions as we will make use of them for CLTs which emerge from the distributional symmetry of exchangeability.

**Definition 2.1.1.** Let $A$ be a finite set.
(1) A *set partition* of $A$ is a set of mutually disjoint subsets $\pi = \{V_1, \ldots, V_k\}$ such that $\bigcup_{i=1}^{k} V_i = A$, $V_i \cap V_j = \emptyset$, for $1 \leq i, j \leq k$, where $k$ is called the *size of the partition* and $V_i$ is called a *block* of $\pi$.
(2) The *set of all partitions* of $A$ is denoted by $\mathcal{P}(A)$.
(3) A partition $\pi = \{V_1, \ldots, V_k\}$ of the set $A$ is called a *pair partition* if $|V_i| = 2$, for $i = 1, \ldots, k$. Here $|V_i|$ denotes the cardinality of the set $V_i$. In other words, each block $V_i$ contains exactly two elements.

(4) The *set of all pair partitions* of $A$ is denoted by $\mathcal{P}_2(A)$.
(5) A block $V_i$ of the partition $\pi \in \mathcal{P}(A)$ is called a *singleton* if $|V_i| = 1$.

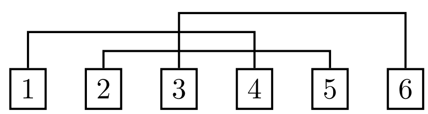
**Notation 2.1.2.** We write $[n]$ for the set $\{1, 2, \ldots, n\}$ for $n \in \mathbb{N}$. For $A = [n]$ we will also write $\mathcal{P}(n)$ instead of $\mathcal{P}([n])$.

**Example 2.1.3.** Consider the set $A = \{1, 2, 3, 4, 5, 6\}$. Then the partition $\pi = \{\{1, 4, 5\}, \{2, 3\}, \{6\}\}$ of $A$ is of size 3 and has the blocks $V_1 = \{1, 4, 5\}, V_2 = \{2, 3\}$, and $V_3 = \{6\}$. This partition represented by the figure below.




This partition $\pi$ contains a singleton since its block $V_3$ has only one element.

**Example 2.1.4.** The partition $\pi = \big\{\{1, 4\}, \{2, 5\}, \{3, 6\}\big\}$ of the set $[6]$ is a pair partition with 3 blocks. We visualize this pair partition also by the following diagram.




**Lemma 2.1.5.** *Let $k \in \mathbb{N}$. The set of all pair partitions $\mathcal{P}_2(2k)$ has*

$$|\mathcal{P}_2(2k)| = (2k-1) \cdot (2k-3) \cdot \ldots \cdot 1 = (2k-1)!! \tag{2.1}$$

*elements.*

*Proof.* Consider the first element of the set $[2k]$. There are $(2k-1)$ choices for the second element to obtain the first block. We keep repeating this procedure until all elements are paired. Thus there are a total of $(2k-1)!!$ choices. $\square$

We introduce next restrictions of partitions as we will meet them again in Theorem 3.3.13, a CLT for certain exchangeable families of random variables.

**Definition 2.1.6.** Let $W \subset [n]$ be non-empty. The *$W$-restriction of* $\pi = \{V_1, V_2, \ldots, V_k\} \in \mathcal{P}(n)$ is given by the partition

$$\pi_{|W} := \{W_{i_1}, W_{i_2}, \ldots, W_{i_\ell}\} \in \mathcal{P}(W),$$

where $W_i := V_i \cap W$ with $i \in I := \{i \in [k] \mid V_i \cap W \neq \emptyset\}$ and $1 \leq i_1 < i_2 < \cdots < i_\ell \leq k$.

Note that $\pi_{|W} \neq \emptyset$ is ensured for any $W$-restriction of a partition $\pi$. Let us also remind that, by definition, a block of a partition is a non-empty set. Thus, for a set $W \neq \emptyset$ and a partition $\pi$ as given above, the set $\{V_1 \cap W, \ldots, V_k \cap W\}$ is a partition of $W$ if and only if $V_i \cap W \neq \emptyset$ for all $i \in [k]$. In other words, any empty set $V_j \cap W$ needs to be removed from $\{V_1 \cap W, \ldots, V_k \cap W\}$, until all its elements are non-empty sets.

**Example 2.1.7.** Let $\pi = \{V_1, V_2, V_3, V_4\} = \big\{\{1,3,4\}, \{2,5,8,9\}, \{6,10\}, \{7\}\big\} \in \mathcal{P}(10)$ and $W = \{2,4,5,7,8,9\} \subset [10]$. Then one has

$$V_1 \cap W = \{4\}, \quad V_2 \cap W = \{2,5,8,9\}, \quad V_3 \cap W = \emptyset, \quad V_4 \cap W = \{7\}.$$

Thus the $W$-restriction of $\pi$ is given by $\pi_{|W} = \{V_1 \cap W, V_2 \cap W, V_4 \cap W\} \in \mathcal{P}(W)$.

**Definition 2.1.8.** Let $A$ and $B$ be non-empty sets. The *kernel set partition* of the function $\mathbf{f}\colon A \to B$, denoted by $\ker(\mathbf{f})$, is the partition of $A$ into the level sets of $\mathbf{f}$. That means two elements $a_1, a_2 \in A$ belong to the same block of $\ker(\mathbf{f})$ if and only if $\mathbf{f}(a_1) = \mathbf{f}(a_2)$.

Usually, we are interested in the level sets of a function $\mathbf{i}\colon [n] \to \mathbb{N}$. In this case, $[n]$ is partitioned into finitely many level sets which are also called blocks of the kernel set partition $\ker(\mathbf{i})$.

**Lemma 2.1.9.** *The following are equivalent for two functions* $\mathbf{i}, \mathbf{j}\colon [n] \to \mathbb{N}$*:*

*(a)* $\mathbf{i}(r) = \mathbf{i}(s) \Longleftrightarrow \mathbf{j}(r) = \mathbf{j}(s)$ *for all* $r, s \in [n]$*;*

*(b)* $\ker(\mathbf{i}) = \ker(\mathbf{j})$*;*

*(c)* $\mathbf{i} = \sigma \circ \mathbf{j}$ *for some* $\sigma \in S_\infty$*.*

*Here* $S_\infty$ *denotes the group of all bijections* $\sigma\colon \mathbb{N} \to \mathbb{N}$ *which permute only finitely many elements of* $\mathbb{N}$*.*

**Definition 2.1.10.** We say that the two functions $\mathbf{i}, \mathbf{j}\colon [n] \to \mathbb{N}$ are *equivalent*, in symbols: $\mathbf{i} \sim \mathbf{j}$, if one (and thus all) of the conditions of Lemma 2.1.9 are satisfied.

*Proof of Lemma 2.1.9.* '(a) $\Longleftrightarrow$ (b)' is evident.
'(b) $\Longrightarrow$ (c)': We infer from $\ker(\mathbf{i}) = \ker(\mathbf{j})$ that $\mathbf{i}(k) = \mathbf{i}(\ell)$ if and only if $\mathbf{j}(k) = \mathbf{j}(\ell)$. Thus the map $\operatorname{Ran}\mathbf{j} \ni \mathbf{j}(k) \mapsto \mathbf{i}(k) \in \operatorname{Ran}\mathbf{i}$ is bijective and extends to a bijective map $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that $\sigma(m) = m$ for $m > \max\{\operatorname{Ran}\mathbf{i} \cup \operatorname{Ran}\mathbf{j}\}$.
'(c) $\Longrightarrow$ (b)': We infer from the bijectivity of the map $\sigma \in S_\infty$ that $\ker(\mathbf{j}) = \ker(\sigma \circ \mathbf{j}) = \ker(\mathbf{i})$. □

### 2.1.2 Basics on Ordered Set Partitions

We continue with providing the basics of ordered set partitions as we will need them for CLTs which emerge from the distributional invariance principle of spreadability. We closely follow notations and conventions as introduced in [HL19].

**Definition 2.1.11.** Let $A$ be a finite set and suppose $\{V_1, \ldots, V_k\}$ is a set partition of $A$.
(1) An *ordered set partition* $\pi$ of $A$ is a sequence $(V_1, \ldots, V_k)$. In other words, it is a set partition of $A$ where each block is decorated with label to track the

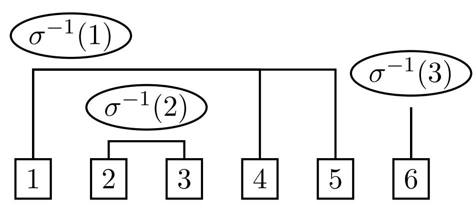
order. The *size of the ordered partition* $|\pi|$ is the size of the underlying partition $\{V_1, \ldots, V_k\}$.
(2) The set of *ordered set partitions* of $A$ is denoted by $\mathcal{OP}(A)$. If $A = [n]$, then we also write $\mathcal{OP}(A)$ as $\mathcal{OP}(n)$.
(3) The ordered partition $\pi = (V_1, \ldots, V_k) \in \mathcal{OP}(A)$ is called an ordered pair partition if $|V_i| = 2$ for all $1 \leq i \leq k$.
(4) The *set of all ordered set pair partitions* is denoted by $\mathcal{OP}_2(A)$.
(5) The map $\mathcal{OP}(A) \ni \pi \mapsto \overline{\pi} \in \mathcal{P}(A)$ is defined as $\pi = (V_1, \ldots, V_k) \mapsto \{V_1, \ldots, V_k\} = \overline{\pi}$.

Note that, in general, the set $A$ may not be equipped with an order and an order is only given for the blocks of a partition of this set $A$. Thus, if $\pi = (V_1, \ldots, V_k) \in \mathcal{OP}(A)$, then its blocks are labelled by the ordered set $[k]$. Whenever there is no risk of confusion, we will say that an ordered partition $\pi \in \mathcal{OP}(A)$ has the 'Property A' if the corresponding partition $\overline{\pi} \in \mathcal{P}(A)$ has the 'Property A'. For example, $V \in \pi$ denotes the block $V \in \overline{\pi}$, or a block $V_i$ of the ordered set partition $\pi$ is called a *singleton* if $V_i$ is a singleton of $\overline{\pi}$.

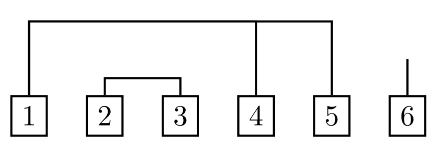
**Example 2.1.12.** We further discuss Example 2.1.3 in the context of ordered set partitions. Consider the set partition $\{\{1,4,5\},\{2,3\},\{6\}\}$ of the set $A = \{1,2,3,4,5,6\}$ and denote its three blocks by $V_1 = \{1,4,5\}$, $V_2 = \{2,3\}$, and $V_3 = \{6\}$. Then there are 3! ordered set partitions $\pi_\sigma \in \mathcal{OP}(6)$ of the explicit form

$$\pi_\sigma = (V_{\sigma(1)}, V_{\sigma(2)}, V_{\sigma(3)}) \qquad (\sigma \in S_3).$$

Here $S_3$ denotes the permutation group on the set $[3]$. The ordered partition $\pi_\sigma$ are represented by the figure below, where $\sigma^{-1}$ denotes the inverse of the permutation $\sigma \in S_3$.

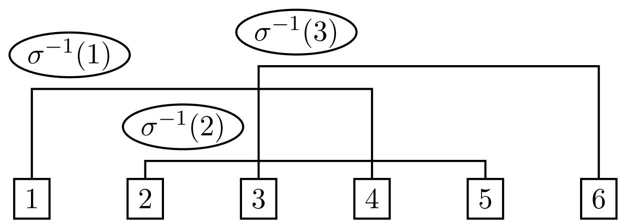



Note that the map $\mathcal{OP}(6) \ni \pi_\sigma \mapsto \overline{\pi_\sigma} \in \mathcal{P}(6)$, loosely phrasing, drops the 'decoration' of the partition, such that above figure becomes the following diagram:




**Example 2.1.13.** Consider the pair partition $\{\{1,4\},\{2,5\},\{3,6\}\} \in \mathcal{P}_2(6)$ and denote its three blocks by $V_1 = \{1,4\}$, $V_2 = \{2,5\}$, and $V_3 = \{3,6\}$. For $\sigma \in S_3$, the six ordered pair partitions $(V_{\sigma(1)}, V_{\sigma(2)}, V_{\sigma(3)})$ are visualized again by the

following diagram:




**Lemma 2.1.14.** *Let $k \in \mathbb{N}$. The set of all ordered pair partitions $\mathcal{OP}_2(2k)$ has*

$$|\mathcal{OP}_2(2k)| = k!\,|\mathcal{P}_2(2k)| = k! \cdot (2k-1)!!$$

*elements.*

*Proof.* We already know $|\mathcal{P}_2(2k)| = (2k-1)!!$ from Lemma 2.1.5, and there are $k!$ possibilities to order the blocks of each pair partition in $\mathcal{P}(2k)$. □

We introduce next restrictions of ordered partitions as we will meet them again in Theorems 3.4.9 and 3.4.13, a CLT for certain spreadable families of random variables.

**Definition 2.1.15.** Let $W \subset [n]$ be non-empty. The *$W$-restriction of* $\pi = (V_1, V_2, \dots, V_k) \in \mathcal{OP}(n)$ is given by the ordered partition

$$\pi_{|W} := (W_{i_1}, W_{i_2}, \dots, W_{i_\ell}) \in \mathcal{P}(W),$$

where $W_i := V_i \cap W$ with $i \in I := \{i \in [k] \mid V_i \cap W \neq \emptyset\}$ and $1 \leq i_1 < i_2 < \cdots < i_\ell \leq k$.

Note again that, as already discussed after Definition 2.1.6, $\pi_{|W} \neq \emptyset$ is ensured for any $W$-restriction of an ordered partition $\pi$.

We repeat Example 2.1.7 in the context of ordered partitions for the convenience of the reader.

**Example 2.1.16.** Let $\pi = (V_1, V_2, V_3, V_4) = \big(\{1,3,4\}, \{2,5,8,9\}, \{6,10\}, \{7\}\big) \in \mathcal{OP}(10)$ and $W = \{2,4,5,7,8,9\} \subset [10]$. Then one has

$$V_1 \cap W = \{4\}, \quad V_2 \cap W = \{2,5,8,9\}, \quad V_3 \cap W = \emptyset, \quad V_4 \cap W = \{7\}.$$

Thus the $W$-restriction of the ordered partition $\pi$ is given by $\pi_{|W} = (V_1 \cap W, V_2 \cap W, V_4 \cap W) \in \mathcal{OP}(W)$.

**Notation 2.1.17.** Let $(B, <)$ be an (totally) ordered set and suppose $B_1, B_2$ are subsets of $B$. We will write $B_1 < B_2$ if $b_1 < b_2$ for all $b_1 \in B_1$ and $b_2 \in B_2$.

**Definition 2.1.18.** Let $A$ be a set and $(B, <)$ be an (totally) ordered set. The *ordered kernel set partition* of the function $\mathbf{i}\colon A \to B$, denoted by $\ker_{\mathcal{O}}(\mathbf{i})$, is the ordered partition $\pi = (V_1, \dots, V_k) \in \mathcal{OP}(A)$ such that $\{V_1, \dots, V_k\} = \ker(\mathbf{i})$ for some $1 \leq k \leq n$ with $n = |A|$ and $\mathbf{i}(V_i) < \mathbf{i}(V_j)$ for all $1 \leq i < j \leq k$.

**Remark 2.1.19.** We will be mainly interested in the sets $A = [n]$ and $B = \mathbb{N}$ in Definition 2.1.18. This is also the setting for which Hasebe and Lehner introduce ordered kernel set partitions in [HL19, Definition 3.3]. The ordered kernel set partition $\ker_{\mathcal{O}}(\mathbf{i})$ of the function $\mathbf{i}\colon [n] \to \mathbb{N}$ can be constructed as follows. Pick the smallest element in the image of $\mathbf{i}$, say $\mathbf{i}(r_1)$, and then define the first block $V_1 = \{r \in [n] | \mathbf{i}(r) = \mathbf{i}(r_1)\}$. Then choosing the second smallest element, say $\mathbf{i}(r_2)$, define $V_2 = \{r \in [n] | \mathbf{i}(r) = \mathbf{i}(r_2)\}$. Repeating this procedure until all elements in $[n]$ are associated to a block, we obtain the ordered set partition $\ker_{\mathcal{O}}(\mathbf{i}) := (V_1, \dots, V_k)$, where $k$ is the size of the ordered partition.

**Example 2.1.20.** Let $A = [10]$ and $B = \mathbb{N}$ (equipped with its natural order). Consider the function $\mathbf{i}\colon [10] \to \mathbb{N}$ with kernel set partition

$$\ker(\mathbf{i}) = \{\{1, 3, 4\}, \{2, 5, 8, 9\}, \{6, 10\}, \{7\}\}.$$

Note that the specific order of listing these blocks does not matter for $\ker(\mathbf{i})$. If $\mathbf{i}(2) < \mathbf{i}(7) < \mathbf{i}(1) < \mathbf{i}(6)$, then $\mathbf{i}$ has the ordered set kernel partition

$$\ker_{\mathcal{O}}(\mathbf{i}) = (\{2, 5, 8, 9\}, \{7\}, \{1, 3, 4\}, \{6, 10\}).$$

If $\mathbf{i}(1) < \mathbf{i}(6) < \mathbf{i}(7) < \mathbf{i}(2)$, then $\mathbf{i}$ has the ordered set kernel partition

$$\ker_{\mathcal{O}}(\mathbf{i}) = (\{1, 3, 4\}, \{6, 10\}, \{7\}, \{2, 5, 8, 9\}).$$

There are 4! possibilities of how the four values $\mathbf{i}(1)$, $\mathbf{i}(2)$, $\mathbf{i}(3)$, and $\mathbf{i}(4)$ can be ordered. Defining the blocks

$$V_1 := \{1, 3, 4\}, \quad V_2 := \{2, 5, 8, 9\}, \quad V_3 := \{6, 10\}, \quad V_4 := \{7\},$$

there is a bijective correspondence between ordered kernel set partitions $\ker_{\mathcal{O}}(\mathbf{i})$ with kernel set partition $\ker(\mathbf{i}) = \{V_1, V_2, V_3, V_4\}$ and permutations $\sigma \in S_4$ such that $\ker_{\mathcal{O}}(\mathbf{i}) = (V_{\sigma(1)}, V_{\sigma(2)}, V_{\sigma(3)}, V_{\sigma(4)})$. Note that we did not make use of that the set $A = [10]$ is actually an ordered set. Also, we did not make use of the fact that the set $[4]$ (labelling the blocks $V_1$ to $V_4$) is an ordered set.

Let $(A, <)$ and $(B, <)$ be ordered sets. A function $f\colon A \to B$ is said to be order preserving if $a < a'$ implies $f(a) < f(a')$ for all $a, a' \in A$.

**Lemma 2.1.21.** *The following are equivalent for two functions* $\mathbf{i}, \mathbf{j}\colon [n] \to \mathbb{N}$*:*

*(a)* $\mathbf{i}(r) \le \mathbf{i}(s) \iff \mathbf{j}(r) \le \mathbf{j}(s)$ *for all* $r, s \in [n]$*;*

*(b)* $\ker_{\mathcal{O}}(\mathbf{i}) = \ker_{\mathcal{O}}(\mathbf{j})$*;*

*(c)* $\tau \circ \mathbf{i} = \sigma \circ \mathbf{j}$ *for some* $\sigma, \tau \in S_\infty$ *with order preserving restrictions* $\sigma|_{\operatorname{Ran} \mathbf{j}}$ *and* $\tau|_{\operatorname{Ran} \mathbf{i}}$*.*

Actually, one can choose in Lemma 2.1.21 (c) either $\tau$ or $\sigma$ to be the trivial permutation.

**Definition 2.1.22.** We say that the two functions $\mathbf{i}, \mathbf{j} \colon [n] \to \mathbb{N}$ are *order equivalent*, in symbols: $\mathbf{i} \sim_{\mathcal{O}} \mathbf{j}$, if one (and thus all) of the conditions of Lemma 2.1.21 are satisfied.

*Proof of Lemma 2.1.21.* '(a) $\Longrightarrow$ (b)': Since (a) implies $\mathbf{i}(r) = \mathbf{i}(s) \iff \mathbf{j}(r) = \mathbf{j}(s)$ for all $r, s \in [n]$, it follows that $\ker(\mathbf{i}) = \ker(\mathbf{j}) = \{V_1, V_2, \dots V_k\}$ for some $V_1, V_2, \dots, V_k$ which we may choose such that $\mathbf{i}(V_i) < \mathbf{i}(V_j)$ for $1 \le i < j \le k$. Doing so we conclude from the order relations in (a) that one also has $\mathbf{j}(V_i) < \mathbf{j}(V_j)$ for $1 \le i < j \le k$. But this shows $\ker_{\mathcal{O}}(\mathbf{i}) = \ker_{\mathcal{O}}(\mathbf{j})$.
'(b) $\Longrightarrow$ (a)': Suppose $\ker_{\mathcal{O}}(\mathbf{i}) = \ker_{\mathcal{O}}(\mathbf{j}) = (V_1, V_2, \dots, V_k)$. It is immediate from the definition of an ordered kernel set partition that $\mathbf{i}(r) \le \mathbf{i}(s)$ implies $r \in V_i$ and $s \in V_j$ with $1 \le i \le j \le k$. But the latter implies $\mathbf{j}(r) \le \mathbf{j}(s)$. Exchanging the roles of $\mathbf{i}$ and $\mathbf{j}$, the same argument ensures that $\mathbf{j}(r) \le \mathbf{j}(s)$ implies $\mathbf{i}(r) \le \mathbf{i}(s)$.
'(a) $\Longrightarrow$ (c)': Consider the two order equivalent $n$-tuples $(\mathbf{i}(1), \dots, \mathbf{i}(n))$ and $(\mathbf{j}(1), \dots, \mathbf{j}(n))$, and let $(\mathbf{k}(1), \dots, \mathbf{k}(n))$ be another order equivalent tuple such that $\min\{\mathbf{k}(\ell) \mid 1 \le \ell \le n\} > \max\{\mathbf{i}(\ell), \mathbf{j}(\ell) \mid 1 \le \ell \le n\}$. Clearly, there exist two permutations $\sigma, \tau \in S_\infty$ with $\tau \circ \mathbf{i} = \mathbf{k}$ and $\sigma \circ \mathbf{j} = \mathbf{k}$ such that $\tau|_{\operatorname{Ran} \mathbf{i}}$ and $\sigma|_{\operatorname{Ran} \mathbf{j}}$ are order preserving.
'(c) $\Longrightarrow$ (a)': Clearly $\tau \circ \mathbf{i}(r) \le \tau \circ \mathbf{i}(s)$ if and only if $\sigma \circ \mathbf{j}(r) \le \sigma \circ \mathbf{j}(s)$. Since the restrictions of $\tau$ and $\sigma$ to the ranges of $\mathbf{i}$ and $\mathbf{j}$, respectively, are order preserving we also know that $\tau \circ \mathbf{i}(r) \le \tau \circ \mathbf{i}(s)$ if and only if $\mathbf{i}(r) \le \mathbf{i}(s)$, as well as $\sigma \circ \mathbf{j}(r) \le \sigma \circ \mathbf{j}(s)$ if and only if $\mathbf{j}(r) \le \mathbf{j}(s)$. As the considered relations are transitive, we arrive at the equivalence claimed in (a). $\square$

**Example 2.1.23.** Let the functions $\mathbf{i}, \mathbf{j} \colon [5] \to \mathbb{N}$ be given by the two tuples $(1, 3, 4, 1, 3)$ and $(1, 2, 5, 1, 2)$, respectively. We find two permutations $\sigma, \tau \in S_\infty$ such that their restrictions $\sigma|_{\operatorname{Ran} \mathbf{j}}$ and $\tau|_{\operatorname{Ran} \mathbf{i}}$ are order preserving maps such that $\tau \circ \mathbf{i} = \sigma \circ \mathbf{j}$. To see this consider the tuple $(1, 10, 100, 1, 10)$ which corresponds to a function $\mathbf{k} \colon [n] \to \mathbb{N}$. Clearly, there exist permutations $\sigma, \tau \in S_\infty$ such that $\sigma|_{\operatorname{Ran} \mathbf{j}}$ and $\tau|_{\operatorname{Ran} \mathbf{i}}$ are order preserving such that $\tau \circ \mathbf{i} = \mathbf{k}$ and $\sigma \circ \mathbf{j} = \mathbf{k}$.

Alternatively, one could have taken for $\mathbf{k} \colon [5] \to \mathbb{N}$ the function which is determined by the tuple $(1, 2, 3, 1, 2)$, as both other tuples can be obtained from this one by 'spreading'. Now the argument is that there exist permutations $\sigma, \tau \in S_\infty$ such that $\mathbf{i} = \tau \circ \mathbf{k}$ and $\mathbf{j} = \sigma \circ \mathbf{k}$ with order preserving restrictions $\sigma|_{\operatorname{Ran} \mathbf{k}}$ and $\tau|_{\operatorname{Ran} \mathbf{k}}$.

So far we have considered ordered set partitions $\mathcal{P}(A)$ where $A$ was assumed to be a set.

**Definition 2.1.24.** Let $(A, <)$ be an ordered finite set. The partition $\pi = \{V_1, \dots, V_k\} \in \mathcal{P}(A)$ or the ordered partition $\pi = (V_1, \dots, V_k) \in \mathcal{OP}(A)$ are said to be in *standard order* if $\min V_1 < \min V_2 < \dots < \min V_k$.

We will frequently make use of the fact that the ordered set partitions of an ordered finite set $(A, <)$ are in a bijective correspondence with pairs consisting of a set partition and a permutation.

**Notation 2.1.25.** Let $(A, <)$ be a finite ordered set and suppose $1 \leq k \leq |A|$. We write

$$\mathcal{P}(A, k) := \{\pi \in \mathcal{P}(A) \mid \pi \text{ is standard ordered with } |\pi| = k\ \}$$

and

$$\mathcal{OP}(A, k) := \{\pi \in \mathcal{OP}(A) \mid |\pi| = k\}$$

for set partitions and ordered set partitions of length $k$, respectively. If $A = [n]$, then $\mathcal{P}([n], k)$ and $\mathcal{OP}([n], k)$ will be also written as $\mathcal{P}(n, k)$ and $\mathcal{OP}(n, k)$, respectively.

Note that $\mathcal{P}(A, k) = \{\pi \in \mathcal{P}(A) \mid |A| = k\}$, as the standard ordering only effects the labeling of the blocks of the partition $\pi$. Insisting on writing down the partition $\pi$ in standard order is of advantage for the formulation of the following bijective correspondence between ordered partitions, pairs of partitions, and permutations.

**Lemma 2.1.26.** *Let $A$ be a finite ordered set. The map*

$$\mathcal{P}(A, k) \times S_k \ni (\{V_1, \ldots, V_k\}, \sigma) \mapsto (V_{\sigma(1)}, \ldots, V_{\sigma(k)}) \in \mathcal{OP}(A, k)$$

*is bijective.*

*Proof.* We show that the map, as stated in the lemma, is both injective and surjective. Suppose $\{V_1, \ldots, V_k\}, \{\widetilde{V}_1, \ldots, \widetilde{V}_k\} \in \mathcal{P}(A, k)$ and the permutations $\sigma, \widetilde{\sigma} \in S_k$ satisfy $(V_{\sigma(1)}, \ldots, V_{\sigma(k)}) = (\widetilde{V}_{\widetilde{\sigma}(1)}, \ldots, \widetilde{V}_{\widetilde{\sigma}(k)})$. Thus

$$\begin{aligned} \{V_1, \ldots, V_k\} &= \{V_{\sigma(1)}, \ldots, V_{\sigma(k)}\} \\ &= \{\widetilde{V}_{\widetilde{\sigma}(1)}, \ldots, \widetilde{V}_{\widetilde{\sigma}(k)}\} = \{\widetilde{V}_1, \ldots, \widetilde{V}_k\}. \end{aligned}$$

We conclude from this that $V_i = \widetilde{V}_i$ for all $1 \leq i \leq k$, since both partitions $\{V_1, \ldots, V_k\}$ and $\{\widetilde{V}_1, \ldots, \widetilde{V}_k\}$ are in standard order. We conclude from this on the level of ordered partitions that

$$(\widetilde{V}_{\sigma(1)}, \ldots, \widetilde{V}_{\sigma(k)}) = (V_{\sigma(1)}, \ldots, V_{\sigma(k)}) = (\widetilde{V}_{\widetilde{\sigma}(1)}, \ldots, \widetilde{V}_{\widetilde{\sigma}(k)}),$$

and thus $\sigma = \widetilde{\sigma}$. This insures the injectivity of the map. We are left to prove the surjectivity of the map. So let $\pi \in \mathcal{OP}(A, k)$ of the form $\pi = (W_1, \ldots, W_k)$ which may not be in standard order. Since $A$ is an ordered set, the blocks $W_1$ to $W_k$ can be reordered by a permutation $\sigma \in S_k$ such that $(W_{\sigma(1)}, \ldots, W_{\sigma(k)})$ is in standard order. Now put $V_i := W_{\sigma(i)}$ for $1 \leq i \leq k$. Then we have

$$(\{V_1, \ldots, V_k\}, \sigma^{-1}) \mapsto (V_{\sigma^{-1}(1)}, \ldots, V_{\sigma^{-1}(k)}) = (W_1, \ldots, W_k),$$

since $V_{\sigma^{-1}(i)} = W_{\sigma(\sigma^{-1}(i))} = W_i$. Thus the map is also surjective. $\square$

**Corollary 2.1.27.** *Let $A$ be a finite ordered set with even cardinality $|A| = 2k$. The map*

$$\mathcal{P}_2(A) \times S_k \ni (\{V_1, \ldots, V_k\}, \sigma) \mapsto (V_{\sigma(1)}, \ldots, V_{\sigma(k)}) \in \mathcal{OP}_2(A)$$

*is bijective.*

*Proof.* Clearly $\mathcal{P}_2(A) \subset \mathcal{P}(A, k)$ and $\mathcal{OP}_2(A) \subset \mathcal{OP}(A, k)$. Clearly, the bijection

$$\mathcal{P}(A, k) \times S_k \ni (\{V_1, \ldots, V_k\}, \sigma) \mapsto (V_{\sigma(1)}, \ldots, V_{\sigma(k)}) \in \mathcal{OP}(A, k)$$

restricts to a bijection from $\mathcal{P}_2(A) \times S_k$ onto $\mathcal{OP}_2(A)$. □

**Remark 2.1.28.** It can be seen that, for an ordered finite set $A$, the map $\mathcal{OP}(A) \ni \pi \mapsto \overline{\pi} \in \mathcal{P}(A)$ restricts to a bijection from the subset of standard ordered partitions in $\mathcal{OP}(A)$ to the set of partitions $\mathcal{P}(A)$.

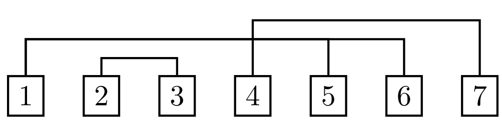
### 2.1.3 Crossings

We introduce next the notation of crossing partitions. This requires the set $A$ to be a equipped with an order relation $<$, to obtain an ordered set $(A, <)$. When considering for $A$ the sets $\mathbb{N}$ or $[n]$, we assume throughout that they are equipped with the canonical order relation. For brevity, we will address the ordered sets $(\mathbb{N}, <)$ and $([n], <)$ just as $\mathbb{N}$ or $[n]$, respectively.

**Definition 2.1.29.** (1) A partition $\pi = \{V_1, \ldots, V_k\}$ of the finite (ordered) set $[n]$ is said to be a *crossing partition* if there exist elements $p_1 < q_1 < p_2 < q_2$ in $[n]$ such that $p_1, p_2 \in V_p$ and $q_1, q_2 \in V_q$ for some distinct $p, q \in [k]$.
(2) A partition $\pi = \{V_1, \ldots, V_k\}$ of the (ordered) set $[n]$ is said to be *non-crossing* if it is not a crossing partition. The *set of non-crossing partitions* of the (ordered) set $[n]$ is denoted by $\mathcal{NP}(n)$.
(3) Suppose $\pi = \{V_1, \ldots, V_k\} \in \mathcal{P}_2(2k)$ is a pair partition which has listed its blocks $V_1, V_2, \ldots, V_k$ in increasing order of their minimal elements. We denote by $\mathrm{cr}(\pi)$ the total number of crossings of the pair partition $\pi$ which is given by the explicit formula

$$\mathrm{cr}(\pi) = \mathrm{card}\{(i, j) \mid 1 \le i < j \le k,\ V_i \text{ and } V_j \text{ are crossing }\}.$$

**Example 2.1.30.** Consider the partition $\pi = \big\{\{1, 5, 6\}, \{2, 3\}, \{4, 7\}\big\}$ of the ordered set $[7]$. Then $\pi$ is a crossing partitions since $1 < 4 < 5 < 7$ and $1, 5 \in \{1, 5, 6\}$ and $4, 7 \in \{4, 7\}$. This is visualized in the following diagram.



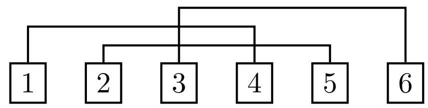



Figure 2.1: Pair partition $\pi \in \mathcal{P}(6)$ with $\operatorname{cr}(\pi) = 3$.

**Example 2.1.31.** The pair partition $\pi = \big\{\{1,4\},\{2,5\}\{3,6\}\big\} \in \mathcal{P}_2(6)$ has $\operatorname{cr}(\pi) = 3$ crossings, as it can be read off from Figure 2.1, as already shown in Example 2.1.4: We use this example to illustrate an alternative method to visualize pair partitions of an ordered finite set. So consider $\pi \in \mathcal{P}_2(6)$ as given above. We arrange the elements of the ordered set [6] counterclockwise on the circumference of circle and connect the two elements of each pair by an arc which drawn inside of the circle. This results for the pair partition $\pi$ in three crossings as shown in Figure 2.2.

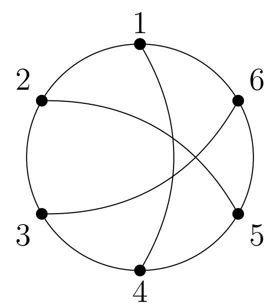



Figure 2.2: Pair partition $\pi \in \mathcal{P}(6)$ with $\operatorname{cr}(\pi) = 3$.

If the underlying set $A$ is an ordered set, then the notion of a crossing partition transfers to ordered partitions. We will make use of this only when the ordered set $A$ is given by $[n]$.

**Definition 2.1.32.** *An* ordered set partition $\pi \in \mathcal{OP}(n)$ *is* non-crossing *if the partition* $\overline{\pi}$ *has this property. The set of* non-crossing ordered partitions *is denoted by* $\mathcal{NOP}(n)$.

### 2.1.4 Oriented Crossings

The concept of an oriented crossing is well-studied for braids and is also underlying the definition of so-called $z$-circular systems by Mingo and Nica in [MN01]. Independently, we have identified that oriented crossings are of relevance for the combinatorics which emerges from certain CLTs of braidable sequences. We introduce next the idea of an oriented crossing for ordered pair partitions, adapting the approach of Mingo and Nica in [MN01, Subsection 1.4] as appropriate within our combinatorial treatment of *-algebraic CLTs.

Let $C, D, E, F$ be distinct points in the plane such that the line segment $\overline{CD}$ crosses the line segment $\overline{EF}$. To formulate the idea of an oriented crossing one needs additional data about

($\alpha$) the direction of each of the two line segments,

($\beta$) the order of the two line segments,

such that one can determine an orientation of the crossing according to the right-hand rule. Thus one has to consider the two directed line segments $\overrightarrow{CD}$ and $\overrightarrow{EF}$. Furthermore, these two directed line segments need to be decorated by labels to specify the order of $\overrightarrow{CD}$ and $\overrightarrow{EF}$ in their vector product. Thus, if the vector $\vec{w} := \overrightarrow{CD} \times \overrightarrow{EF}$ is oriented upwards of the plane spanned by the two vectors $\overrightarrow{CD}$ and $\overrightarrow{EF}$, then $\overrightarrow{CD}$ and $\overrightarrow{EF}$ are said to have a positive crossing. If $\vec{w}$ is oriented downwards of this plane, then we say that $\overrightarrow{CD}$ and $\overrightarrow{EF}$ are said to have a negative crossing. These conventions can be remembered by the so called right-hand rule, see Figure 2.3. To be more precise, in the left diagram of Figure 2.3, the thumb represents the line segment with the smaller label '1' and points into the direction of the point 'D'. Furthermore, the index finger represents the line segment with the larger label '2' and points into the direction of 'F'. Consequently, the middle finger points 'upwards', and thus the crossing needs to be decorated with a plus sign.

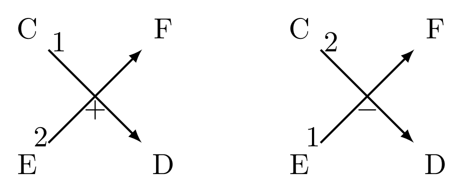



Figure 2.3: Positive crossing (left) and negative crossing (right).

Equally well the order of the two line segments may be encoded topologically by drawing the first line segment above the second one, as it is done for geometric braids and shown in Figure 2.4.

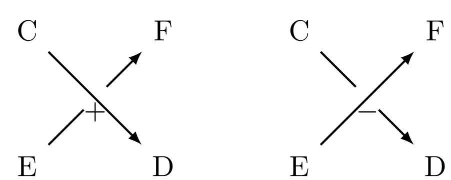



Figure 2.4: Positive crossing (left) and negative crossing (right).

Alternatively, we may display the direction of the line segments by drawing an asterix at the point from which the directed line segment emerges. This convention allows us to draw an oriented crossing as shown in Figure 2.5.

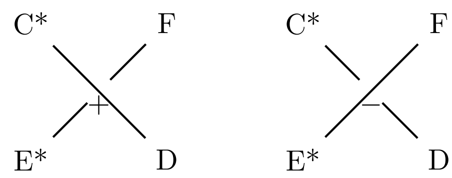



Figure 2.5: Positive crossing (left) and negative crossing (right).

Quite in parallel to displaying pair partitions for ordered finite sets, as shown in Figure 2.1 and Figure 2.2, ordered pair partitions can be displayed as depicted for the standard ordered pair partition $\pi = (\{1,4\},\{2,5\},\{3,6\}) \in \mathcal{OP}_2(6)$ in Figure 2.6. The elements of the ordered set [6] are again displayed in linear order (Figure 2.6 (i)) or counterclockwise on the circumference of a circle (Figure 2.6 (ii) & (iii)). Additionally, one needs now to keep track of the order of the blocks of the partition, which is done graphically by labelling each of the three blocks (see Figure 2.6 (i) & (ii)) or, alternatively, by topologically specifying the order of the blocks (as shown in Figure 2.6 (iii)). To be more precise, the counterclockwise oriented circumference of the circle is embedded into three-dimensional space. Now an arc connecting two points on the circumference is thought of to be a strand connecting these two points. The counterclockwise orientation of the circumference allows us to specify what is considered topologically to be 'on top' or 'below'. So, the 1-labelled strand runs on top of all strands, the 2-labelled strand runs below the 1-labelled strand and, more generally, the $(k+1)$-labelled strand runs below the $k$-labelled strand (as shown for $k = 1, 2$ in Figure 2.6 (iii)).

Next we need to take care about having 'directed pairs' or, addressing this more geometrically, directed line segments or arcs. Essentially, repeating Figure 2.6, this is illustrated in Figure 2.7 by drawing arced arrows instead of arcs.

Algebraically it will be more convenient to encode the direction of line segments or arcs by decorating the 'source' (from which the arc emerges) by the asterix '*' and the 'target'(where the arc arrives) by the symbol '1' (which is suppressed in writing to maintain a light notation).

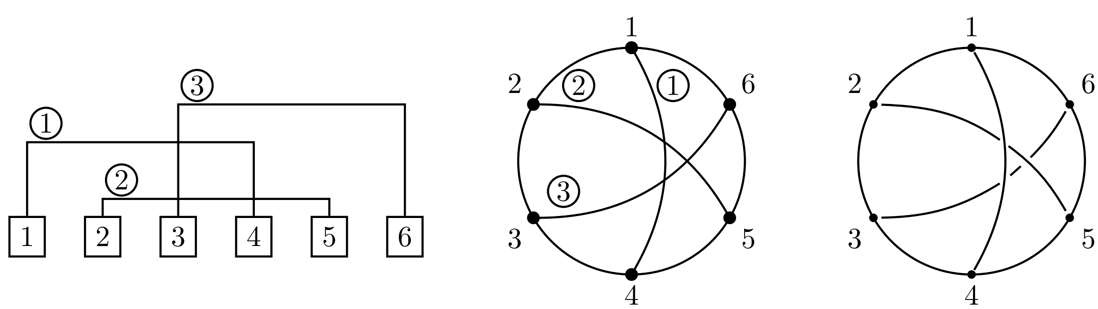



Figure 2.6: Standard ordered pair partition $\pi = (\{1,4\},\{2,5\},\{3,6\}) \in \mathcal{OP}_2(6)$ pictured in three different ways: (i) left, (ii) middle, and (iii) right.

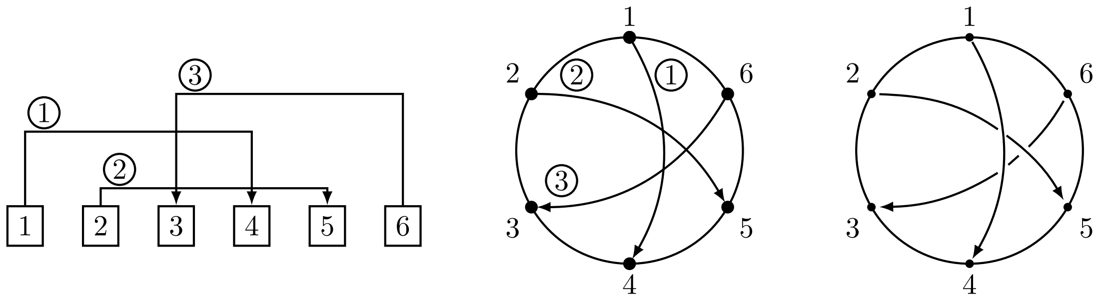



Figure 2.7: Three ways to picture a directed standard ordered pair partition $\big(\{1,4\},\{2,5\},\{3,6\}\big) \in \mathcal{OP}_2(6)$.

**Definition 2.1.33.** (i) The map $\boldsymbol{\varepsilon}\colon [n] \to \{*,1\}$ is called a *direction map* (*associated to the set of pair partitions* $\mathcal{P}_2(n)$) if the pre-images of $\boldsymbol{\varepsilon}$ satisfy $\operatorname{card}\boldsymbol{\varepsilon}^{-1}(\{*\}) = \operatorname{card}\boldsymbol{\varepsilon}^{-1}(\{1\})$. Such a map $\boldsymbol{\varepsilon}$ is also said to be *balanced.*
(ii) Let $\pi \in \mathcal{P}_2(n)$ be given. The direction map $\boldsymbol{\varepsilon}\colon [n] \to \{*,1\}$ is said to be $\pi$*-balanced* if $\boldsymbol{\varepsilon}(V) = \{*,1\}$ for every $V \in \pi$.
(iii) Suppose the direction map $\boldsymbol{\varepsilon}\colon [n] \to \{*,1\}$ is given for $n = 2k$ with $k \in \mathbb{N}$. Then

$$\mathcal{P}_2(n,\boldsymbol{\varepsilon}) := \left\{ \pi \in \mathcal{P}_2(n) \,\middle|\, \begin{array}{l} \boldsymbol{\varepsilon}(\min V) \neq \boldsymbol{\varepsilon}(\max V) \text{ for all } V \in \pi \\ \text{and } \min V_1 < \min V_2 < \cdots < \min V_k \end{array} \right\}$$

is called the $\boldsymbol{\varepsilon}$*-restricted set of (standard ordered) pair partitions.* Furthermore,

$$\mathcal{OP}_2(n,\boldsymbol{\varepsilon}) := \big\{ \pi \in \mathcal{OP}_2(n) \big| \boldsymbol{\varepsilon}(\min V) \neq \boldsymbol{\varepsilon}(\max V) \text{ for all } V \in \pi \big\}$$

is called the $\boldsymbol{\varepsilon}$*-restricted set of ordered pair partitions.*

An $\boldsymbol{\varepsilon}$-restricted set of ordered pair partitions is illustrated in Figure 2.8. Actually Figure 2.8 is an example of standard ordered pair partition. Figure 2.9 shows the more general case of a possibly not-standard ordered pair partition, where a geometric representation through underpass or overpass strands is specified by the permutation $\sigma$.

We recall from Corollary 2.1.27 that there exists a bijective correspondence between ordered pair partitions of $\mathcal{OP}_2(2k)$ and pairs of (standard ordered) pair partitions of $\mathcal{P}_2(2k)$ and permutations in $S_k$. We address next that this correspondence is also valid for $\boldsymbol{\varepsilon}$-restricted sets of ordered pair partitions.

**Corollary 2.1.34.** *Let the direction map* $\boldsymbol{\varepsilon}\colon [n] \to \{*,1\}$ *be given for* $n = 2k$ *with* $k \in \mathbb{N}$*. The map*

$$\mathcal{P}_2(n,\boldsymbol{\varepsilon}) \times S_k \ni (\{V_1,\ldots,V_k\},\sigma) \mapsto (V_{\sigma(1)},\ldots,V_{\sigma(k)}) \in \mathcal{OP}_2(A,\boldsymbol{\varepsilon})$$

*is bijective.*

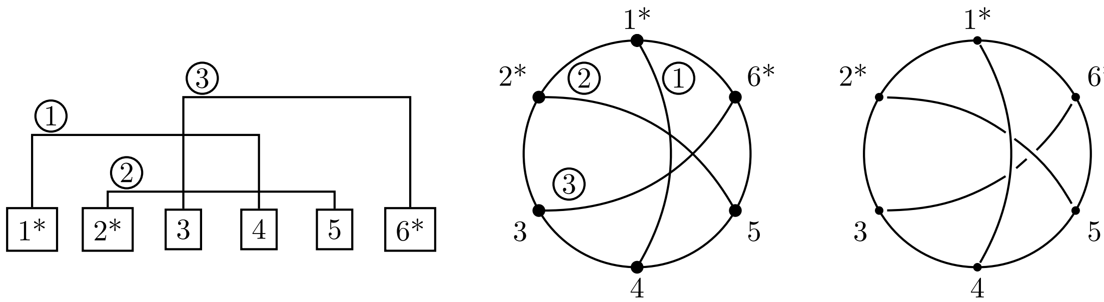



Figure 2.8: Three alternative ways to picture the directed standard ordered pair partition $\big(\{1,4\},\{2,5\},\{3,6\}\big) \in \mathcal{OP}_2(6,\boldsymbol{\varepsilon})$ with $\boldsymbol{\varepsilon}(1)=\boldsymbol{\varepsilon}(2)=\boldsymbol{\varepsilon}(6)=*$ and $\boldsymbol{\varepsilon}(3)=\boldsymbol{\varepsilon}(4)=\boldsymbol{\varepsilon}(5)=1$.

*Proof.* We have already established in Corollary 2.1.27 that the bijective map

$$\mathcal{P}([n],k)\times S_k \ni (\{V_1,\dots,V_k\},\sigma)\mapsto (V_{\sigma(1)},\dots,V_{\sigma(k)})\in \mathcal{OP}([n],k)$$

restricts to a bijection from $\mathcal{P}_2(A)\times S_k$ to $\mathcal{OP}_2(A)$. This bijection maps the block $V_i$ to the block $V_{\sigma(i)}$. Clearly, the additional property $\boldsymbol{\varepsilon}(\min V)\neq\boldsymbol{\varepsilon}(\max V)$ is preserved under this bijection. Consequently, this bijection restricts further to a bijection from $\mathcal{P}_2(n,\boldsymbol{\varepsilon})\times S_k$ onto $\mathcal{OP}_2(n,\boldsymbol{\varepsilon})$. $\square$

Building on the geometric idea of an oriented crossing, the orientation of crossings is determined by the data of an $\boldsymbol{\varepsilon}$-directed ordered pair partition in $\pi\in\mathcal{OP}_2(2k,\boldsymbol{\varepsilon})$ or, equivalently, of the triple

$$(\overline{\pi},\boldsymbol{\varepsilon},\sigma)\in\mathcal{P}_2(2k)\times\big\{f\colon[2k]\to\{*,1\}\big\}\times S_k.$$

For such a triple $(\pi,\boldsymbol{\varepsilon},\sigma)$, the orientation of each crossing of the pair partition $\pi$ is uniquely determined by the geometric representation of the triple, using the right-hand rule and as illustrated in Example 2.1.37 below.

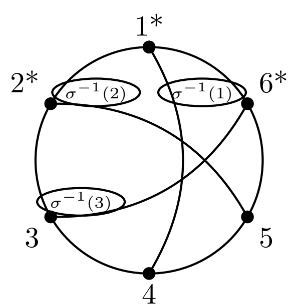



Figure 2.9: $\boldsymbol{\varepsilon}$-Restricted ordered pair partition $(V_{\sigma(1)},V_{\sigma(2)},V_{\sigma(3)})\in\mathcal{OP}_2(6,\boldsymbol{\varepsilon})$ with $\boldsymbol{\varepsilon}(1)=\boldsymbol{\varepsilon}(2)=\boldsymbol{\varepsilon}(6)=*$ and $\boldsymbol{\varepsilon}(3)=\boldsymbol{\varepsilon}(4)=\boldsymbol{\varepsilon}(5)=1$, blocks $V_1=\{1,4\}$, $V_2=\{2,5\}$, $V_3=\{3,6\}$ and permutation $\sigma\in S_3$ (left), and $(V_2,V_3,V_1)\in\mathcal{OP}_2(6,\boldsymbol{\varepsilon})$ with $\sigma(1)=2$, $\sigma(2)=3$ and $\sigma(3)=1$ (right).

**Definition 2.1.35.** Let an $\boldsymbol{\varepsilon}$-directed ordered pair partition $\pi \in \mathcal{OP}_2(2k, \boldsymbol{\varepsilon})$ be given by the triple

$$(\overline{\pi}, \boldsymbol{\varepsilon}, \sigma) \in \mathcal{P}_2(2k) \times \big\{f\colon [2k] \to \{*, 1\}\big\} \times S_k.$$

Then the number of crossings of $\pi$ which have positive orientation and negative orientation is denoted by $\mathrm{cr}_+(\overline{\pi}, \boldsymbol{\varepsilon}, \sigma)$ and $\mathrm{cr}_-(\overline{\pi}, \boldsymbol{\varepsilon}, \sigma)$, respectively.

The number of crossings with positive or negative orientation sums up of course to the number of all crossings of $\overline{\pi}$:

$$\mathrm{cr}_+(\overline{\pi}, \boldsymbol{\varepsilon}, \sigma) + \mathrm{cr}_-(\overline{\pi}, \boldsymbol{\varepsilon}, \sigma) = \mathrm{cr}(\overline{\pi}).$$

Explicit formulas are available for the number of oriented crossings. Let sgn denote the signum function such that $x = |x|\mathrm{sgn}(x)$ for all $x \in \mathbb{R}$.

**Lemma 2.1.36** ([MN01, 1.4.2])**.** *Let the $\boldsymbol{\varepsilon}$-directed ordered pair partition $\pi = (V_{\sigma(1)}, \ldots, V_{\sigma(k)}) \in \mathcal{OP}_2(2k, \boldsymbol{\varepsilon})$ be given by the triple $(\overline{\pi}, \boldsymbol{\varepsilon}, \sigma)$ where $\overline{\pi} = \{V_1, \ldots, V_k\}$ is in standard order. Then one has:*

$$\mathrm{cr}_+(\overline{\pi}, \boldsymbol{\varepsilon}, \sigma) = \mathrm{card}\left\{(i,j) \,\middle|\, \begin{array}{l} 1 \le i < j \le k,\ V_i \text{ and } V_j \text{ cross}, \\ \boldsymbol{\varepsilon}(\min V_i) \cdot \boldsymbol{\varepsilon}(\min V_j) = \mathrm{sgn}\big(\sigma(j) - \sigma(i)\big) \end{array}\right\},$$
$$\mathrm{cr}_-(\overline{\pi}, \boldsymbol{\varepsilon}, \sigma) = \mathrm{card}\left\{(i,j) \,\middle|\, \begin{array}{l} 1 \le i < j \le k,\ V_i \text{ and } V_j \text{ cross}, \\ \boldsymbol{\varepsilon}(\min V_i) \cdot \boldsymbol{\varepsilon}(\min V_j) = -\mathrm{sgn}\big(\sigma(j) - \sigma(i)\big) \end{array}\right\}.$$

*Here the product '$\boldsymbol{\varepsilon}(\min V_i) \cdot \boldsymbol{\varepsilon}(\min V_j)$' is evaluated according to the following convention: '$* \cdot 1$' and '$1 \cdot *$' equal $-1$, the two other products '$* \cdot *$' and '$1 \cdot 1$' equal* $1$.

**Example 2.1.37.** For $n = 8$, consider the standard ordered pair partition

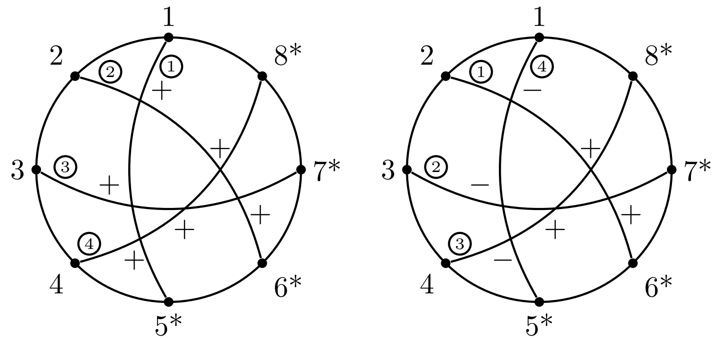
$$\pi = \{\{1,5\}, \{2,6\}, \{3,7\}, \{4,8\}\} \in \mathcal{P}_2(8).$$

Let the direction map $\boldsymbol{\varepsilon} : [8] \to \{1, *\}$ be given by $\boldsymbol{\varepsilon}(1) = \boldsymbol{\varepsilon}(2) = \boldsymbol{\varepsilon}(3) = \boldsymbol{\varepsilon}(4) = 1$ and $\boldsymbol{\varepsilon}(5) = \boldsymbol{\varepsilon}(6) = \boldsymbol{\varepsilon}(7) = \boldsymbol{\varepsilon}(8) = *$, and the permutation $\sigma \in S_4$ be given by $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 4$, and $\sigma(4) = 1$. Then the $\boldsymbol{\varepsilon}$-restricted standard ordered pair partition $\big(\{1,5\}, \{2,6\}, \{3,7\}, \{4,8\}\big) \in \mathcal{OP}_2(8, \boldsymbol{\varepsilon})$ has 6 positive crossings and 0 negative crossings:

$$\mathrm{cr}_+(\pi, \boldsymbol{\varepsilon}, \sigma_0) = 6, \qquad \mathrm{cr}_-(\pi, \boldsymbol{\varepsilon}, \sigma_0) = 0.$$

Here $\sigma_0 \in S_8$ denotes the neutral element of the group. The $\boldsymbol{\varepsilon}$-restricted ordered partition $\big(\{2,6\}, \{3,7\}, \{4,8\}, \{1,5\}\big) \in \mathcal{OP}_2(8, \boldsymbol{\varepsilon})$ has 3 positive crossings and 3 negative crossings:

$$\mathrm{cr}_+(\pi, \boldsymbol{\varepsilon}, \sigma) = 3, \qquad \mathrm{cr}_-(\pi, \boldsymbol{\varepsilon}, \sigma) = 3.$$




## 2.2 Basics of *-Algebraic Probability Theory

In this section we collect basic definitions that are related to noncommutative probability spaces, which can be found in various books such as [NS06, Sp19].

**Definition 2.2.1.** A **-algebraic probability space* $(\mathcal{A}, \varphi)$ consists of a unital *-algebra $\mathcal{A}$ over $\mathbb{C}$ and a $\mathbb{C}$-linear functional $\varphi\colon \mathcal{A} \to \mathbb{C}$ such that $\varphi(1_{\mathcal{A}}) = 1$ and $\varphi(x^*x) \geq 0$ for all $x \in \mathcal{A}$. Here $1_{\mathcal{A}}$ denotes the identity of $\mathcal{A}$. A *-algebraic probability space $(\mathcal{A}, \varphi)$ is said to be *tracial* if $\varphi$ is a trace, i.e. one has $\varphi(ab) = \varphi(ba)$ for all $a, b \in \mathcal{A}$. A *-algebraic probability space $(\mathcal{A}, \varphi)$ is said to be *classical* if $\mathcal{A}$ is commutative, i.e. one has $ab = ba$ for all $a, b \in \mathcal{A}$.

Given the *-algebraic probability space $(\mathcal{A}, \varphi)$, an element $a \in \mathcal{A}$ is also called a *random variable*. The random variable $a \in \mathcal{A}$ is said to be *centered* if $\varphi(a) = 0$. Furthermore, a random variable $a \in \mathcal{A}$ is said to be *self-adjoint* if $a = a^*$ and, more generally, to be *normal* if $a^*a = aa^*$.

Frequently, we will make use of the fact that $\varphi(a^*) = \overline{\varphi(a)}$ for all $a \in \mathcal{A}$. In particular, one has $\varphi(a) \in \mathbb{R}$ for any self-adjoint random variable $a \in \mathcal{A}$. Moreover, one has the following Cauchy-Schwarz inequality for a *-algebraic probability space $(\mathcal{A}, \varphi)$:

$$|\varphi(x^*y)| \leq \sqrt{\varphi(x^*x)}\ \sqrt{\varphi(y^*y)} \qquad \text{for } x, y \in \mathcal{A}.$$

Note that the map $\mathcal{A} \times \mathcal{A} \ni (x, y) \mapsto \varphi(x^*y) \in \mathbb{C}$ is sesquilinear and thus defines a semi-inner product on the vector space $\mathcal{A}$.

**Definition 2.2.2.** Let $(\mathcal{A}, \varphi)$ be an algebraic probability space and let $a \in \mathcal{A}$. Then $\varphi(a^n)$ is called the *$n$-th moment* of the random variable $a$. More generally, for $n, d \in \mathbb{N}$ and $i(1), \ldots, i(n) \in [d]$, we call $\varphi(a_{i(1)} \cdots a_{i(n)})$ a *joint moment (of order $n$)* of the family of random variables $a_1, a_2, \ldots, a_d \in \mathcal{A}$.

Occasionally, joint moments of random variables are also addressed as *mixed moments*. For example, $\varphi(a^m b^n)$ is a mixed moment of the two distinct random variables $a, b \in \mathcal{A}$, for any $m, n \in \mathbb{N}$.

**Definition 2.2.3.** Let $(\mathcal{A},\varphi)$ be a *-algebraic probability space. Two random variables $a,b\in\mathcal{A}$ are said to be *identically distributed* if

$$\varphi\big(a^{\varepsilon(1)}a^{\varepsilon(2)}\cdots a^{\varepsilon(n)}\big)=\varphi\big(b^{\varepsilon(1)}b^{\varepsilon(2)}\cdots b^{\varepsilon(n)}\big)$$

for all $\boldsymbol{\varepsilon}\colon[n]\to\{1,*\}$ and $n\in\mathbb{N}$. More generally, for some index set $I$, two families of random variables $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$ in $\mathcal{A}$ are said to be *identically distributed* if, for any $n\in\mathbb{N}$,

$$\varphi\big(a_{\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)}\cdots a_{\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\big)=\varphi\big(b_{\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)}\cdots b_{\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\big)$$

for all $\mathbf{i}\colon[n]\to I$ and $\boldsymbol{\varepsilon}\colon[n]\to\{1,*\}$.

If $a,b\in\mathcal{A}$ are normal, then $a$ and $b$ are identically distributed if

$$\varphi\big((a^*)^k a^\ell\big)=\varphi\big((b^*)^k b^\ell\big)$$

for all $k,\ell\geq 0$. Note that a classical *-algebraic probability space contains only normal random variables. If $a,b$ are self-adjoint, then $a$ and $b$ are identically distributed if

$$\varphi\big(a^n\big)=\varphi\big(b^n\big)$$

for all $n\in\mathbb{N}$.

**Definition 2.2.4.** Let $(\mathcal{A},\varphi)$ be a classical *-algebraic probability space. The family of random variables $\{a_i\}_{i\in I}\subset\mathcal{A}$, for some index set $I$, is said to be *(classically) independent* if, for any $n\in\mathbb{N}$,

$$\varphi\big(a_{i_1}^{k_1}(a_{i_1}^*)^{\ell_1}a_{i_2}^{k_2}(a_{i_2}^*)^{\ell_2}\cdots a_{i_n}^{k_n}(a_{i_n}^*)^{\ell_n}\big)=\varphi\big(a_{i_1}^{k_1}(a_{i_1}^*)^{\ell_1}\big)\varphi\big(a_{i_2}^{k_2}(a_{i_2}^*)^{\ell_2}\big)\cdots\varphi\big(a_{i_n}^{k_n}(a_{i_n}^*)^{\ell_n}\big)$$

for any distinct $i_1,\dots,i_n\in I$ and any $k_1,\dots,k_n,\ell_1,\dots\ell_n\in\mathbb{N}$.

**Definition 2.2.5.** Let $(\mathcal{A}_N,\varphi_N)$, with $N\in\mathbb{N}$, and $(\mathcal{A},\varphi)$ be *-algebraic probability spaces. Consider the random variables $a_N\in\mathcal{A}_N$ for each $N\in\mathbb{N}$ and $a\in\mathcal{A}$. The sequence $(a_N)_{N\in\mathbb{N}}$ is said to *converge in distribution* towards $a$, in symbols:

$$a_N\xrightarrow{\text{distr}}a,$$

if we have

$$\lim_{N\to\infty}\varphi_N\big(a_N^{\boldsymbol{\varepsilon}(1)}\cdots a_N^{\boldsymbol{\varepsilon}(n)}\big)=\varphi\big(a^{\boldsymbol{\varepsilon}(1)}\cdots a^{\boldsymbol{\varepsilon}(n)}\big)$$

for all $n\in\mathbb{N}$ and $\boldsymbol{\varepsilon}\colon[n]\to\{1,*\}$.

**Definition 2.2.6.** Let $(\mathcal{A}_N,\varphi_N)$, with $N\in\mathbb{N}$, and $(\mathcal{A},\varphi)$ be *-algebraic probability spaces, and let $C$ be a set. Consider for each $c\in C$ the random variables $a_{c,N}\in\mathcal{A}_N$ and $a_c\in\mathcal{A}$. The tuple $(a_{c,N})_{c\in C}$ is said to *converge in distribution* towards the tuple $(a_c)_{c\in C}$, in symbols:

$$(a_{c,N})_{c\in C}\xrightarrow{\text{distr}}(a_c)_{c\in C},$$

if all joint moments converge towards the corresponding joint moments i.e.

$$\lim_{N\to\infty} \varphi_N\big(a^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),N} \cdots a^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),N}\big) = \varphi\big(a^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1)} \cdots a^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n)}\big)$$

for all $n \in \mathbb{N}$, $\mathbf{t}\colon [n] \to C$, and $\boldsymbol{\varepsilon}\colon [n] \to \{1, *\}$.

**Definition 2.2.7.** Let $(\mathcal{A}, \varphi)$ be a *-algebraic probability space.

(i) An *endomorphism* $\alpha$ of $(\mathcal{A}, \varphi)$ is a unital *-homomorphism of $\mathcal{A}$ such that $\varphi \circ \alpha = \varphi$. In particular, $\alpha$ is said to be an *automorphism* of $(\mathcal{A}, \varphi)$ if $\alpha$ is a (unital) *-automorphism of $\mathcal{A}$. The set of endomorphisms of $(\mathcal{A}, \varphi)$ is denoted by $\mathrm{End}(\mathcal{A}, \varphi)$, and the set of automorphisms of $(\mathcal{A}, \varphi)$ is denoted by $\mathrm{Aut}(\mathcal{A}, \varphi)$.

(ii) $\mathcal{A}^\alpha$ denotes the fixed point algebra of $\alpha \in \mathrm{End}(\mathcal{A}, \varphi)$, i.e. $\mathcal{A}^\alpha = \{a \in \mathcal{A} \mid \alpha(a) = a\}$.

## 2.3 Distributional Symmetries

We start with introducing the most common distributional symmetries and invariance principles. Our approach adapts that of [GK09, Kö10] to the framework of *-algebraic probability spaces.

We will introduce presentations of the symmetric groups $S_n$ and $S_\infty$ via relations for generators in Definition 2.3.4. Here we just remind that the *symmetric group* $S_n$ is the group of all bijections on the set $[n]$ for $n \in \mathbb{N}$. Furthermore, the *infinite symmetric group* $S_\infty$ is the inductive limit of the groups $S_n$ for $n \to \infty$.

**Definition 2.3.1.** Let $(\mathcal{A}, \varphi)$ be a *-algebraic probability space. The sequence of random variables $\mathbf{x} \equiv (x_n)_{n=1}^\infty \subset \mathcal{A}$ is said to be

(i) *exchangeable* if, for all $n \in \mathbb{N}$ and $\boldsymbol{\varepsilon}\colon [n] \to \{1, *\}$,

$$\varphi(x^{\boldsymbol{\varepsilon}(1)}_{\mathbf{i}(1)} \cdots x^{\boldsymbol{\varepsilon}(n)}_{\mathbf{i}(n)}) = \varphi(x^{\boldsymbol{\varepsilon}(1)}_{\sigma(\mathbf{i}(1))} \cdots x^{\boldsymbol{\varepsilon}(n)}_{\sigma(\mathbf{i}(n))})$$

for all $\mathbf{i}\colon [n] \to \mathbb{N}$ and $\sigma \in S_\infty$;

(ii) *spreadable* if, for all $n \in \mathbb{N}$ and $\boldsymbol{\varepsilon}\colon [n] \to \{1, *\}$,

$$\varphi(x^{\boldsymbol{\varepsilon}(1)}_{\mathbf{i}(1)} \cdots x^{\boldsymbol{\varepsilon}(n)}_{\mathbf{i}(n)}) = \varphi(x^{\boldsymbol{\varepsilon}(1)}_{\mathbf{j}(1)} \cdots x^{\boldsymbol{\varepsilon}(n)}_{\mathbf{j}(n)})$$

whenever $\mathbf{i}, \mathbf{j}\colon [n] \to \mathbb{N}$ are order equivalent;

(iii) *stationary* if, for all $k, n \in \mathbb{N}$ and $\boldsymbol{\varepsilon}\colon [n] \to \{1, *\}$,

$$\varphi(x^{\boldsymbol{\varepsilon}(1)}_{\mathbf{i}(1)} x^{\boldsymbol{\varepsilon}(2)}_{\mathbf{i}(2)} \cdots x^{\boldsymbol{\varepsilon}(n)}_{\mathbf{i}(n)}) = \varphi(x^{\boldsymbol{\varepsilon}(1)}_{\mathbf{i}(1)+k} x^{\boldsymbol{\varepsilon}(2)}_{\mathbf{i}(2)+k} \cdots x^{\boldsymbol{\varepsilon}(n)}_{\mathbf{i}(n)+k})$$

for all $\mathbf{i}\colon [n] \to \mathbb{N}$;

(iv) *identically distributed* if, for all $k, n \in \mathbb{N}$ and $\boldsymbol{\varepsilon}\colon [n] \to \{1, *\}$,

$$\varphi(x_1^{\boldsymbol{\varepsilon}(1)} x_1^{\boldsymbol{\varepsilon}(2)} \cdots x_1^{\boldsymbol{\varepsilon}(n)}) = \varphi(x_k^{\boldsymbol{\varepsilon}(1)} x_k^{\boldsymbol{\varepsilon}(2)} \cdots x_k^{\boldsymbol{\varepsilon}(n)}).$$

**Lemma 2.3.2.** *One has the following hierarchy of distributional symmetries:*

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv).$$

*Proof.* $(i) \Rightarrow (ii)$ follows from the fact that $\mathbf{i} \sim_{\mathcal{O}} \mathbf{j}$ if and only if there exists some permutation $\sigma \in S_\infty$ such that $\mathbf{j} = \sigma \circ \mathbf{i}$ and $\sigma|_{\{\mathbf{i}(1),\ldots,\mathbf{i}(n)\}}$ is order preserving. All other implications are clear. □

**Remark 2.3.3.** (i) The role of the infinite symmetric group $S_\infty$ in the above definition of exchangeability could be taken by the larger group $S_{\mathbb{N}}$, the group of all bijections on $\mathbb{N}$. As exchangeability is about the invariance of joint distribution, replacing $S_\infty$ by $S_{\mathbb{N}}$ would be without effect since any joint distribution involves only finitely many random variables. More precisely, let $\mathbf{i}(1), \ldots, \mathbf{i}(n) \in \mathbb{N}$ for some $n \in \mathbb{N}$ and consider some bijection $\widehat{\sigma} \in S_{\mathbb{N}}$. Then there exist a permutation $\sigma \in S_\infty$ such that $\widehat{\sigma}(\mathbf{i}(k)) = \sigma(\mathbf{i}(k))$ for $k = 1, 2, \ldots, n$. As an immediate consequence, a sequence $\mathbf{x} \equiv (x_n)_{n=1}^\infty \subset \mathcal{A}$ is exchangeable if and only if, for all $n \in \mathbb{N}$,

$$\varphi(x_{\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}) = \varphi(x_{\beta(\mathbf{i}(1))}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\beta(\mathbf{i}(n))}^{\boldsymbol{\varepsilon}(n)})$$

for all $\boldsymbol{\varepsilon}\colon [n] \to \{1, *\}$, $\mathbf{i}\colon [n] \to \mathbb{N}$, and all bijections $\beta\colon \mathbb{N} \to \mathbb{N}$.
(ii) The strictly increasing (or order preserving) maps on $\mathbb{N}$ form a monoid which contains the so-called partial shift monoid

$$\mathcal{S} = \langle (\theta_n)_{n=1}^\infty \mid \theta_k \theta_\ell = \theta_{\ell+1}\theta_k, 1 \leq k \leq \ell < \infty \rangle^+$$

as a submonoid. More concretely, this partial shift monoid $\mathcal{S}$ can be seen to be isomorphic to the monoid generated by the partial shifts, denoted for notational simplicity by the same symbols, $(\theta_n)_{n=1}^\infty\colon \mathbb{N} \to \mathbb{N}$, where

$$\theta_n(i) = \begin{cases} i & \text{if } n > i, \\ i+1 & \text{if } n \leq i \end{cases}.$$

Roughly phrasing, the partial shifts monoid $\mathcal{S}$ relates to the monoid of strictly increasing maps on $\mathbb{N}$ as the infinite symmetric group $S_\infty$ relates to the group of all bijections on $\mathbb{N}$.
(iii) Instead of considering order equivalent sequences for the definition of spreadability, one could alternatively demand that the joint distributions are invariant when replacing the index tuple $\big(\mathbf{i}(1), \ldots, \mathbf{i}(n)\big)$ of the considered random variables by the index tuple $\big(\theta(\mathbf{i}(1)), \ldots, \theta(\mathbf{i}(n))\big)$ for any strictly increasing (or order preserving) map $\theta\colon \mathbb{N} \to \mathbb{N}$. Consequently, a sequence of random variables $\mathbf{x} \equiv (x_n)_{n=1}^\infty \subset \mathcal{A}$ is spreadable if and only if, for all $n \in \mathbb{N}$,

$$\varphi(x_{\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}) = \varphi(x_{\theta(\mathbf{i}(1))}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\theta(\mathbf{i}(n))}^{\boldsymbol{\varepsilon}(n)})$$

for all $\boldsymbol{\varepsilon}\colon [n] \to \{1, *\}$, $\mathbf{i}\colon [n] \to \mathbb{N}$, and all order preserving maps $\theta\colon \mathbb{N} \to \mathbb{N}$. This equivalence is also addressed in Proposition 2.3.12.

We continue with further studying the notions of exchangeability and spreadability, in particular to provide alternative characterizations for them. These alternative characterizations provide us with a constructive procedure for creating exchangeable and spreadable sequences (see also [GK09, EGK17]). Furthermore, this alternative characterization of exchangeability paves the way to a definition of braidability, which was found as a new distributional symmetry intermediate to exchangeability and spreadability by Gohm and Köstler (see [GK09, GK12]).

### 2.3.1 Exchangeability

We provide a constructive procedure for an exchangeable sequence, given a certain representation of the infinite symmetric group $S_\infty$ in the context of *-algebraic probability spaces. For this purpose, we will make use of the fact that symmetric groups can be presented in terms of generators and relations. Our presentation follows [GK09, EGK17] and adapts therein arguments to the setting of *-algebraic probability spaces.

**Definition 2.3.4.** The symmetric group $S_n$, with $1 \le n \le \infty$, is presented by the Coxeter generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ satisfying the relations

$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \qquad \text{if } |i-j| = 1, \tag{2.2}$$
$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{if } |i-j| > 1, \tag{2.3}$$
$$\sigma_i^2 = e. \tag{2.4}$$

Here $e =: \sigma_0$ denotes the neutral element of $S_n$.

We will refer to (2.2) also as braid relations and to (2.3) as commutation relations. The symmetric group $S_n$ can be seen to be isomorphic to the group of all bijections on the set $[n]$, such that the Coxeter generator $\sigma_i$ corresponds to the bijection on $[n]$, which transposes $i$ and $i+1$. Throughout we will identify $S_n$ with the group of all bijections on $[n]$ such that

$$\sigma_i(k) = \begin{cases} k+1 & \text{if } k = i, \\ k-1 & \text{if } k = i+1, \\ k & \text{otherwise.} \end{cases}$$

**Proposition 2.3.5.** *Let $(\mathcal{A}, \varphi)$ be a *-algebraic probability space which is equipped with a representation $\rho\colon S_\infty \to \operatorname{Aut}(\mathcal{A}, \varphi)$. Suppose further that the sequence $(x_n)_{n=1}^\infty \subset \mathcal{A}$ satisfies*

$$x_1 = \rho(\sigma_k)x_1 \qquad (k \ge 2), \tag{2.5}$$
$$x_{n+1} = \rho(\sigma_n\sigma_{n-1}\cdots\sigma_1)x_1 \quad (n \ge 1). \tag{2.6}$$

*Then* $(x_n)_{n=1}^{\infty}$ *is exchangeable.*

*Proof.* Suppose the sequence $(x_n)_{n=1}^{\infty}$ is given as stated in the proposition. Assume for now that, for any $\boldsymbol{\varepsilon}\colon [n] \to \{1,*\}$ and $\boldsymbol{\ell}\colon [n] \to \mathbb{N}$,

$$\varphi(x_{\boldsymbol{\ell}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\boldsymbol{\ell}(n)}^{\boldsymbol{\varepsilon}(n)}) = \varphi(x_{\sigma_i(\boldsymbol{\ell}(1))}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\sigma_i(\boldsymbol{\ell}(n))}^{\boldsymbol{\varepsilon}(n)}) \qquad (i \in \mathbb{N}). \tag{2.7}$$

Then this immediately implies the exchangeability of the sequence $(x_n)_{n=1}^{\infty}$, since any permutation $\sigma \in S_\infty$ can be written as a monomial in the Coxeter generators $\sigma_i$, i.e. one obtains

$$\varphi(x_{\boldsymbol{\ell}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\boldsymbol{\ell}(n)}^{\boldsymbol{\varepsilon}(n)}) = \varphi(x_{\sigma(\boldsymbol{\ell}(1))}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\sigma(\boldsymbol{\ell}(n))}^{\boldsymbol{\varepsilon}(n)}) \qquad (\sigma \in S_\infty).$$

Thus we are left to verify the invariance property claimed in (2.7). Indeed, we obtain from the invariance of the state $\varphi$ under the *-homomorphism $\rho(\sigma_i)$ that

$$\begin{aligned}\varphi(x_{\boldsymbol{\ell}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\boldsymbol{\ell}(n)}^{\boldsymbol{\varepsilon}(n)}) &= \varphi \circ \rho(\sigma_i)\big(x_{\boldsymbol{\ell}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\boldsymbol{\ell}(n)}^{\boldsymbol{\varepsilon}(n)}\big) \\ &= \varphi\big(\rho(\sigma_i)(x_{\boldsymbol{\ell}(1)}^{\boldsymbol{\varepsilon}(1)}) \cdots \rho(\sigma_i)(x_{\boldsymbol{\ell}(n)}^{\boldsymbol{\varepsilon}(n)})\big).\end{aligned}$$

Consequently, it suffices to verify

$$\rho(\sigma_i)(x_n) = x_{\sigma_i(n)}$$

for all $i, n \in \mathbb{N}$. We consider separately the four cases $n = i$, $n = i+1$, $n > i+1$, and $n < i$.

**Case** $n = i$**:** We verify from the given properties of the sequence and the action of Coxeter generators on natural numbers that

$$\begin{aligned}\rho(\sigma_i)x_n &= \rho(\sigma_n)x_n \\ &= \rho(\sigma_n)\rho(\sigma_{n-1}\sigma_{n-2}\ldots\sigma_1)x_1 \\ &= \rho(\sigma_n\sigma_{n-1}\ldots\sigma_1)x_1 \\ &= x_{n+1} \\ &= x_{\sigma_i(n)}.\end{aligned}$$

**Case** $n = i+1 \Leftrightarrow i = n-1$**:** By making use of (2.4), we compute that

$$\begin{aligned}\rho(\sigma_i)x_n &= \rho(\sigma_{n-1})x_n \\ &= \rho(\sigma_{n-1})\rho(\sigma_{n-1}\cdots\sigma_1)x_1 \\ &= \rho(\sigma_{n-1}\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)x_1 \\ &= \rho(\sigma_{n-2}\sigma_{n-3}\cdots\sigma_1)x_1 \\ &= x_{n-1} \\ &= x_{\sigma_i(n)}.\end{aligned}$$

**Case $n > i+1 \Leftrightarrow i < n-1$:** Using repeatedly the commutation relations (2.3), once the braid relation (2.2) and finally the invariance property (2.5), we obtain

$$\begin{aligned}
\rho(\sigma_i)x_n &= \rho(\sigma_i)\rho(\sigma_{n-1}\cdots\sigma_1)x_1 \\
&= \rho(\sigma_i\sigma_{n-1}\cdots\sigma_{i+2}\sigma_{i+1}\sigma_i\sigma_{i-1}\cdots\sigma_1)x_1 \\
&= \rho(\sigma_{n-1}\cdots\sigma_{i+2}\sigma_i\sigma_{i+1}\sigma_i\sigma_{i-1}\cdots\sigma_1)x_1 \\
&= \rho(\sigma_{n-1}\cdots\sigma_{i+2}\sigma_{i+1}\sigma_i\sigma_{i+1}\sigma_{i-1}\cdots\sigma_1)x_1 \\
&= \rho(\sigma_{n-1}\cdots\sigma_{i+2}\sigma_{i+1}\sigma_i\sigma_{i-1}\cdots\sigma_1\sigma_{i+1})x_1 \\
&= \rho(\sigma_{n-1}\cdots\sigma_1)\rho(\sigma_{i+1})x_1 \\
&= \rho(\sigma_{n-1}\cdots\sigma_1)x_1 \\
&= x_n \\
&= x_{\sigma_i(n)}.
\end{aligned}$$

**Case $n < i$:** Here we use the commutation relations (2.3) and, as $i > 1$ in this case, the invariance property (2.5) to compute

$$\begin{aligned}
\rho(\sigma_i)x_n &= \rho(\sigma_i)\rho(\sigma_{n-1}\cdots\sigma_1)x_1 \\
&= \rho(\sigma_{n-1}\cdots\sigma_1)\rho(\sigma_i)x_1 \\
&= x_n \\
&= x_{\sigma_i(n)}.
\end{aligned}$$

Altogether, we have verified that $\rho(\sigma_i)x_n = x_{\sigma_i(n)}$ for all $i, n \in \mathbb{N}$. □

**Remark 2.3.6.** Actually, the converse of Proposition 2.3.5 is also true if the *-algebraic probability space $(\mathcal{A}, \varphi)$ is equipped with a state $\varphi$, which restricts to a faithful state on the unital *-algebra generated by the sequence $(x_n)_{n=0}^{\infty}$. As we will not make further use of this converse, we omit its proof which can be transferred in a straightforward manner from the arguments provided in the proof of [GK09, Theorem 1.9].

### 2.3.2 Spreadability

We present some basics on spreadability as a distributional invariance principle in the context of *-algebraic probability spaces and refer the reader to [Kö10, GK09, EGK17] for further information on it. Similar to exchangeability, we present a procedure on how to construct a spreadable sequence.

**Definition 2.3.7.** Let $\mathcal{S}$ be the monoid of strictly increasing maps on $\mathbb{N}$ which is generated by the *partial shifts* $(\theta_n)_{n=1}^{\infty}$, where

$$\theta_n(\ell) = \begin{cases} \ell & \text{if } n > \ell; \\ \ell+1 & \text{if } n \leq \ell. \end{cases}$$

We will refer to $\mathcal{S}$ also as the *partial shifts monoid*.

The following relations are easily verified.

**Proposition 2.3.8.** *One has*

$$\theta_k\theta_n = \theta_{n+1}\theta_k \tag{2.8}$$

*for any* $1 \leq k \leq n < \infty$.

**Remark 2.3.9.** Note that our labelling of the partial shifts $\theta_n$ differs from that in [GK09, GK12, EGK17], as we use $\mathbb{N}$ instead of $\mathbb{N}_0$ for labelling. Nevertheless, our labelling of partial shifts maintains all relations, in particular the convention that "$\theta_n$ starts shifting at the point $n$" or, equivalently, "the strictly increasing map $\theta_n$ omits the point $n$".

Frequently, we will make use of that spreadability can be equivalently formulated in terms of invariance properties with respect to actions of the monoid $\mathcal{S}$ on the index set of a sequence. Related considerations are elementary, but require some technical preparation for the sake of clarity of the arguments. (This corrects an erroneous statement in [Kö10, Remark 1.9].)

**Lemma 2.3.10.** *The following are equivalent for two functions* $\mathbf{i}, \mathbf{j} \colon [n] \to \mathbb{N}$*:*

*(a) there exists* $\theta, \widetilde{\theta} \in \mathcal{S}$ *and* $\mathbf{h} \colon [n] \to \mathbb{N}$ *such that* $\mathbf{i} = \theta \circ \mathbf{h}$ *and* $\mathbf{j} = \widetilde{\theta} \circ \mathbf{h}$*;*

*(b)* $\mathbf{i}$ *and* $\mathbf{j}$ *are order equivalent, in symbols:* $\mathbf{i} \sim_{\mathcal{O}} \mathbf{j}$.

*Proof.* '(a) $\Rightarrow$ (b)': The monoid $\mathcal{S}$ is generated by partial shifts $\theta_i$, which are order preserving. Thus any $\theta, \widetilde{\theta} \in \mathcal{S}$ are order preserving. Consequently $\mathbf{i} = \theta \circ \mathbf{h}$ and $\mathbf{j} = \widetilde{\theta} \circ \mathbf{h}$ implies $\mathbf{i} \sim_{\mathcal{O}} \mathbf{h}$ and $\mathbf{j} \sim_{\mathcal{O}} \mathbf{h}$, respectively. Now $\mathbf{i} \sim_{\mathcal{O}} \mathbf{j}$ follows from the transitivity of equivalence relations.
'(b) $\Rightarrow$ (a)': Suppose $\mathbf{i}$ and $\mathbf{j}$ are order equivalent. Then the range of $\mathbf{i}$ and $\mathbf{j}$ have the same cardinality $m \in \mathbb{N}$ with $1 \leq m \leq n$. Furthermore, there exists a unique $\mathbf{h} \colon [n] \to \mathbb{N}$ with range $\{1, \ldots, m\}$ such that $\mathbf{h} \sim_{\mathcal{O}} \mathbf{i}$ and $\mathbf{h} \sim_{\mathcal{O}} \mathbf{j}$. Since $\mathbf{h}(k) \leq \mathbf{i}(k)$ for all $k = 1, \ldots, n$, there exist some $\theta \in \mathcal{S}$ such that $\mathbf{i} = \theta \circ \mathbf{h}$. Similarly, we conclude that there exists some $\widetilde{\theta} \in \mathcal{S}$ such that $\mathbf{j} = \widetilde{\theta} \circ \mathbf{h}$. □

Let us illustrate '(b) $\Rightarrow$ (a)' of Lemma 2.3.10.

**Example 2.3.11.** Consider the two order equivalent 7-tuples

$$\begin{aligned}
\big(\mathbf{i}(1), \ldots, \mathbf{i}(7)\big) &= (3, 5, 3, 6, 5, 6, 3), \\
\big(\mathbf{j}(1), \ldots, \mathbf{j}(7)\big) &= (1, 4, 1, 7, 4, 7, 1),
\end{aligned}$$

which have $\operatorname{Ran}(\mathbf{i}) = \{3, 5, 6\}$ and $\operatorname{Ran}(\mathbf{j}) = \{1, 4, 7\}$. Now consider the tuple

$$\big(\mathbf{h}(1), \ldots, \mathbf{h}(7)\big) = (1, 2, 1, 3, 2, 3, 1)$$

which has the range $\{1, 2, 3\}$ and which is order equivalent to each of $\mathbf{i}$ and $\mathbf{j}$. An elementary computation shows that one has $\mathbf{i} = (\theta_4\theta_1^2) \circ \mathbf{h}$ and $\mathbf{j} = (\theta_5^2\theta_2^2) \circ \mathbf{h}$.

**Proposition 2.3.12.** *Let $(\mathcal{A}, \varphi)$ be a *-algebraic probability space. The sequence $\mathbf{x} \equiv (x_n)_{n=1}^{\infty} \subset \mathcal{A}$ is spreadable if and only if one of the following equivalent (and thus both) conditions are satisfied for all fixed $n \in \mathbb{N}$:*

*(a) The sequence $\mathbf{x}$ satisfies*

$$\varphi(x_{\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}) = \varphi(x_{\theta(\mathbf{i}(1))}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\theta(\mathbf{i}(n))}^{\boldsymbol{\varepsilon}(n)})$$

*for all $\boldsymbol{\varepsilon}\colon [n] \to \{1, *\}$, $\mathbf{i}\colon [n] \to \mathbb{N}$, and $\theta \in \mathcal{S}$.*

*(b) The sequence $\mathbf{x}$ satisfies*

$$\varphi(x_{\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}) = \varphi(x_{\mathbf{j}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{j}(n)}^{\boldsymbol{\varepsilon}(n)})$$

*for all $\boldsymbol{\varepsilon}\colon [n] \to \{1, *\}$ and $\mathbf{i}, \mathbf{j}\colon [n] \to \mathbb{N}$ with $\mathbf{i} \sim_{\mathcal{O}} \mathbf{j}$.*

*Proof.* We note that the validity of property '(b)' for all $n \in \mathbb{N}$ defines spreadability, see Definition 2.3.1 (ii).
'(a) $\Rightarrow$ (b)': Suppose $\mathbf{i} \sim_{\mathcal{O}} \mathbf{j}$. By Lemma 2.3.10, there exists $\mathbf{h}\colon [n] \to \mathbb{N}$ and $\theta, \widetilde{\theta} \in \mathcal{S}$ such that $\mathbf{i} = \theta \circ \mathbf{h}$ and $\mathbf{j} = \widetilde{\theta} \circ \mathbf{h}$. Now (a) implies

$$\begin{aligned} \varphi(x_{\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}) &= \varphi(x_{\theta(\mathbf{h}(1))}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\theta(\mathbf{h}(n))}^{\boldsymbol{\varepsilon}(n)}) = \varphi(x_{\mathbf{h}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{h}(n)}^{\boldsymbol{\varepsilon}(n)}) \\ &= \varphi(x_{\widetilde{\theta}(\mathbf{h}(1))}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\widetilde{\theta}(\mathbf{h}(n))}^{\boldsymbol{\varepsilon}(n)}) = \varphi(x_{\mathbf{j}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{j}(n)}^{\boldsymbol{\varepsilon}(n)}). \end{aligned}$$

'(b) $\Rightarrow$ (a)': Since any $\theta \in \mathcal{S}$ is order preserving, one has $\mathbf{i} \sim_{\mathcal{O}} \theta(\mathbf{i})$. This ensures the validity of the claimed implication. □

Next we turn our attention to a result, which allows us to construct a spreadable sequence from knowing the representation of the partial shifts monoid $\mathcal{S}$.

**Proposition 2.3.13.** *Let $(\mathcal{A}, \varphi)$ be a *-algebraic probability space which is equipped with a representation $\varrho\colon \mathcal{S} \to \mathrm{End}(\mathcal{A}, \varphi)$. Suppose further that the sequence $(x_n)_{n=1}^{\infty} \subset \mathcal{A}$ satisfies*

$$x_1 = \varrho(\theta_k) x_1 \quad (k \geq 2), \tag{2.9}$$

$$x_{n+1} = \varrho(\theta_1^n) x_1 \quad (n \geq 1). \tag{2.10}$$

*Then $(x_n)_{n=1}^{\infty}$ is spreadable.*

Note that (2.10) implies the stationarity of a spreadable sequence, since $\varphi \circ \varrho(\theta_1) = \varphi$.

*Proof.* Suppose the sequence $(x_n)_{n=1}^{\infty}$ is given as stated in the proposition. Let $n \in \mathbb{N}$. Assume for now that, for any $\boldsymbol{\varepsilon}\colon [n] \to \{1, *\}$ and $\boldsymbol{\ell}\colon [n] \to \mathbb{N}$,

$$\varphi(x_{\boldsymbol{\ell}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\boldsymbol{\ell}(n)}^{\boldsymbol{\varepsilon}(n)}) = \varphi(x_{\theta_i(\boldsymbol{\ell}(1))}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\theta_i(\boldsymbol{\ell}(n))}^{\boldsymbol{\varepsilon}(n)}) \qquad (i \in \mathbb{N}). \tag{2.11}$$

Then this immediately implies the spreadability of the sequence $(x_n)_{n=1}^{\infty}$, since any $\theta \in \mathcal{S}$ can be written as a monomial in the generators $\theta_i$, i.e. one obtains

$$\varphi(x_{\boldsymbol{\ell}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\boldsymbol{\ell}(n)}^{\boldsymbol{\varepsilon}(n)}) = \varphi(x_{\theta(\boldsymbol{\ell}(1))}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\theta(\boldsymbol{\ell}(n))}^{\boldsymbol{\varepsilon}(n)}) \qquad (\theta \in \mathcal{S}),$$

which is equivalent to spreadability by Proposition 2.3.12. Thus we are left to verify the invariance property claimed in (2.11). Indeed, we obtain from the invariance of the state $\varphi$ under the *-homomorphism $\varrho(\theta_i)$ that

$$\begin{aligned} \varphi(x_{\boldsymbol{\ell}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\boldsymbol{\ell}(n)}^{\boldsymbol{\varepsilon}(n)}) &= \varphi \circ \varrho(\theta_i)\big(x_{\boldsymbol{\ell}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\boldsymbol{\ell}(n)}^{\boldsymbol{\varepsilon}(n)}\big) \\ &= \varphi\big(\varrho(\theta_i)(x_{\boldsymbol{\ell}(1)}^{\boldsymbol{\varepsilon}(1)}) \cdots \varrho(\theta_i)(x_{\boldsymbol{\ell}(n)}^{\boldsymbol{\varepsilon}(n)})\big). \end{aligned}$$

Consequently, it suffices to verify

$$\varrho(\theta_i)(x_n) = x_{\theta_i(n)}$$

for all $i, n \in \mathbb{N}$. We consider separately the two cases $i > n$ and $i \leq n$.

**Case** $i > n$**:** We verify from the given properties of the sequence and the action of the partial shifts $\theta_k$ on natural numbers that

$$\begin{aligned} \varrho(\theta_i)x_n &= \rho(\theta_i\theta_1^{n-1})x_1 \\ &= \rho(\theta_1^{n-1}\theta_{i-n+1})x_1 \\ &= \rho(\theta_1^{n-1})x_1 \\ &= x_n = x_{\theta_i(n)}. \end{aligned}$$

Here we have used first a multiple times the relations (2.8) and then that $i > n \Leftrightarrow i - n + 1 > 1$ which entails $\varrho(\theta_{i-n+1})x_1 = x_1$ by the localization property (2.9).

**Case** $i \leq n$**:** Using again repeatedly the relations (2.8), we compute that

$$\begin{aligned} \varrho(\theta_i)x_n &= \rho(\theta_i\theta_1^{n-1})x_1 \\ &= \rho(\theta_i\theta_1^{i-1}\theta_1^{n-i})x_1 \\ &= \rho(\theta_1^{i}\theta_1^{n-i})x_1 \\ &= \rho(\theta_1^{n})x_1 \\ &= x_{n+1} = x_{\theta_i(n)}. \end{aligned}$$

Altogether, we have shown that $\varrho(\theta_i)x_n = x_{\theta_i(n)}$ for all $i, n \in \mathbb{N}$. □

### 2.3.3 Braidability

We discuss the distributional symmetry of braidability in the framework of *-algebraic probability spaces (see also [EGK17]) and show that braidability implies spreadability. Our presented notion of braidability in Definition 2.3.14 generalizes exchangeability (in the sense of Definition 2.3.1), whenever the latter realized as done in Proposition 2.3.5 (see also Remark 2.3.6).

Let $n \geq 2$. The *braid group* $B_n$ is presented by the so-called *Artin generators* $\widetilde{\sigma}_1, \widetilde{\sigma}_2, \ldots, \widetilde{\sigma}_{n-1}$ satisfying the following relations:

$$\widetilde{\sigma}_i\widetilde{\sigma}_j\widetilde{\sigma}_i = \widetilde{\sigma}_j\widetilde{\sigma}_i\widetilde{\sigma}_j \qquad \text{if } |i-j| = 1, \tag{2.12}$$

$$\widetilde{\sigma}_i\widetilde{\sigma}_j = \widetilde{\sigma}_j\widetilde{\sigma}_i \qquad \text{if } |i-j| > 1. \tag{2.13}$$

The inductive limit of the braid groups $B_n$ for $n \to \infty$ is denoted by $B_\infty$ and called the *infinite braid group*.

**Definition 2.3.14.** Let $(\mathcal{A}, \varphi)$ be a *-algebraic probability space. The sequence of random variables $(x_n)_{n\geqslant 1} \subset \mathcal{A}$ is said to be *braidable* if there exists a representation $\rho\colon B_\infty \to \mathrm{Aut}(\mathcal{A}, \varphi)$ such that:

$$x_1 = \rho(\widetilde{\sigma}_k)x_1 \qquad \text{if } k \geq 2, \tag{2.14}$$

$$x_{n+1} = \rho(\widetilde{\sigma}_n \cdots \widetilde{\sigma}_1)x_1 \qquad \text{if } n \geq 1. \tag{2.15}$$

**Proposition 2.3.15.** *The exchangeable sequence $(x_n)_{n=1}^\infty$ from Proposition 2.3.5 is braidable.*

*Proof.* Suppose $\rho\colon S_\infty \to \mathrm{Aut}(\mathcal{A}, \varphi)$ is a representation of the infinite group $S_\infty$. Let $\epsilon\colon B_\infty \to S_\infty$ denote the epimorphism, which maps Artin generators $\widetilde{\sigma}_n$ to Coxeter generators $\sigma_n$ for all $n \in \mathbb{N}$. Then $\varrho := \rho \circ \epsilon\colon B_\infty \to \mathrm{Aut}(\mathcal{A}, \varphi)$ defines a representation of such that an exchangeable sequence is seen to be braidable. □

**Theorem 2.3.16.** *Let $(\mathcal{A}, \varphi)$ be a *-algebraic probability space. Then a braidable sequence $(x_n)_{n=1}^\infty \subset \mathcal{A}$ is spreadable.*

We prepare the proof of this theorem with an elementary result on order equivalent functions.

**Lemma 2.3.17.** *Suppose $\mathbf{i}, \mathbf{j}\colon [n] \to \mathbb{N}$ are order equivalent. Then there exists a finite sequence of functions $(\mathbf{h}_a)_{a=0}^b\colon [n] \to \mathbb{N}$ such that $\mathbf{h}_0 = \mathbf{i}$, $\mathbf{h}_b = \mathbf{j}$ and , for all $0 \leq a < b$,*

*(i)* $\mathbf{h}_a \sim_\mathcal{O} \mathbf{h}_{a+1}$*;*

*(ii)* $\mathrm{Ran}(\mathbf{h}_a)\backslash\{r_a, r_a + 1\} = \mathrm{Ran}(\mathbf{h}_{a+1})\backslash\{r_a, r_a + 1\}$ *for some* $r_a \in \mathbb{N}$.

We omit the elementary proof of this lemma and instead illustrate it by an example. Note that Lemma 2.3.17 permits the trivial case $\mathbf{i} = \mathbf{j}$, but we can of course assume $\mathbf{i} \neq \mathbf{j}$ without loss of generality in the following.

**Example 2.3.18.** Consider the two order equivalent functions $i, j\colon [7] \to \mathbb{N}$ given by the two 7-tuples

$$\big(\mathbf{i}(1), \ldots, \mathbf{i}(7)\big) = (4, 2, 7, 4, 6, 7, 4),$$
$$\big(\mathbf{j}(1), \ldots, \mathbf{j}(7)\big) = (3, 1, 9, 3, 8, 9, 3).$$

One has $\mathrm{Ran}(\mathbf{i}) = \{2, 4, 6, 7\}$ and $\mathrm{Ran}(\mathbf{j}) = \{1, 3, 8, 9\}$. We construct first order equivalent functions to match the maximal element in the range of $\mathbf{i}$ and with that of $\mathbf{j}$. Next we construct order equivalent functions to match the second largest element in the range of $\mathbf{i}$ and with that of $\mathbf{j}$, and so on until we have matched the two smallest elements. Altogether, it is elementary to find order equivalent functions $\mathbf{h}_0, \ldots, \mathbf{h}_6$ such that $\mathbf{h}_0 = \mathbf{i}$, $\mathbf{h}_6 = \mathbf{j}$ and

$$\begin{aligned}
\mathrm{Ran}\,\mathbf{h}_0 &= \{2, 4, 6, 7\} = \mathrm{Ran}\,\mathbf{i},\\
\mathrm{Ran}\,\mathbf{h}_1 &= \{2, 4, 6, 8\},\\
\mathrm{Ran}\,\mathbf{h}_2 &= \{2, 4, 6, 9\},\\
\mathrm{Ran}\,\mathbf{h}_3 &= \{2, 4, 7, 9\},\\
\mathrm{Ran}\,\mathbf{h}_4 &= \{2, 4, 8, 9\},\\
\mathrm{Ran}\,\mathbf{h}_5 &= \{2, 3, 8, 9\},\\
\mathrm{Ran}\,\mathbf{h}_6 &= \{1, 3, 8, 9\} = \mathrm{Ran}\,\mathbf{j}.
\end{aligned}$$

Consequently, two consecutive functions $\mathbf{h}_a$ have all but one point in common in their range.

*Proof of Theorem 2.3.16.* Due to Lemma 2.3.17, it suffices to provide a proof for the situation where the order equivalent functions $\mathbf{i}$ and $\mathbf{j}$ have a range which differs only in a single point. To be more precise, consider $\mathbf{i}\colon [n] \to \mathbb{N}$ and suppose that $r \in \mathrm{Ran}(\mathbf{i})$ and $r + 1 \notin \mathrm{Ran}(\mathbf{i})$. Let $\mathbf{j}\colon [n] \to \mathbb{N}$ be such that $\mathbf{j}(k) = \mathbf{i}(k)$ whenever $\mathbf{i}(k) \neq r$ and $\mathbf{j}(k) = r + 1$ whenever $\mathbf{i}(k) = r$. In other words,

$$\mathbf{j}(k) = \begin{cases} \mathbf{i}(k) & \text{for } k \in \{1, n\} \text{ and } \mathbf{i}(k) \neq r,\\ \mathbf{i}(k) + 1 & \text{for } k \in \{1, n\} \text{ and } \mathbf{i}(k) = r. \end{cases}$$

Clearly $\mathbf{i}$ and $\mathbf{j}$ are order equivalent. We show next that, for a braidable sequence $(x_n)_{n=1}^{\infty}$,

$$\varphi(x_{\mathbf{i}(1)} \cdots x_{\mathbf{i}(n)}) = \varphi(x_{\mathbf{j}(1)} \cdots x_{\mathbf{j}(n)})$$

where $\mathbf{i}$ and $\mathbf{j}$ are as above.

$$\varphi(x_{\mathbf{i}(1)} \cdots x_{\mathbf{i}(n)}) = \varphi \circ \rho(\widetilde{\sigma}_r)(x_{\mathbf{i}(1)} \cdots x_{\mathbf{i}(n)}) = \varphi\big(\rho(\widetilde{\sigma}_r)(x_{\mathbf{i}(1)}) \cdots \rho(\widetilde{\sigma}_r)(x_{\mathbf{i}(n)})\big).$$

We are left to prove that

$$\rho(\widetilde{\sigma}_r)(x_k) = \begin{cases} x_k & \text{if } r \neq k,\\ x_{k+1} & \text{if } r = k. \end{cases}$$

Thus we will look separately at three cases $r = k$, $r < k$, and $r > k$.

**Case $r = k$:** Using the definition of the sequence $(x_n)_{n=1}^{\infty}$ from (2.15), we obtain

$$\begin{aligned}
\rho(\widetilde{\sigma}_r)(x_k) &= \rho(\widetilde{\sigma}_k)(x_k) \\
&= \rho(\widetilde{\sigma}_k)\rho(\widetilde{\sigma}_{k-1}\cdots\widetilde{\sigma}_1)x_1 \\
&= \rho(\widetilde{\sigma}_k\widetilde{\sigma}_{k-1}\cdots\widetilde{\sigma}_1)x_1 \\
&= x_{k+1}.
\end{aligned}$$

**Case $r < k$:** We claim that $r = k-1 \Leftrightarrow r+1 = k$ cannot occur. Indeed, by our assumptions on $\mathbf{i}$, we know that $r \in \operatorname{Ran}(\mathbf{i})$ and $r+1 \notin \operatorname{Ran}(\mathbf{i}) \ni k$. Consequently, $r < k$ implies $r < k-1$. Thus we compute, by using repeatedly the commutation relations (2.13), once the braid relation (2.12), and finally Definition 2.3.14,

$$\begin{aligned}
\rho(\widetilde{\sigma}_r)(x_k) &= \rho(\widetilde{\sigma}_r)\rho(\widetilde{\sigma}_{k-1}\cdots\widetilde{\sigma}_1)x_1 \\
&= \rho(\widetilde{\sigma}_r\widetilde{\sigma}_{k-1}\cdots\widetilde{\sigma}_1)x_1 \\
&= \rho(\widetilde{\sigma}_r\widetilde{\sigma}_{k-1}\cdots\widetilde{\sigma}_{r+2}\widetilde{\sigma}_{r+1}\widetilde{\sigma}_r\widetilde{\sigma}_{r-1}\cdots\widetilde{\sigma}_1)x_1 \\
&= \rho(\widetilde{\sigma}_{k-1}\cdots\widetilde{\sigma}_{r+2}(\widetilde{\sigma}_r\widetilde{\sigma}_{r+1}\widetilde{\sigma}_r)\widetilde{\sigma}_{r-1}\cdots\widetilde{\sigma}_1)x_1 \\
&= \rho(\widetilde{\sigma}_{k-1}\cdots\widetilde{\sigma}_{r+2}(\widetilde{\sigma}_{r+1}\widetilde{\sigma}_r\widetilde{\sigma}_{r+1})\widetilde{\sigma}_{r-1}\cdots\widetilde{\sigma}_1)x_1 \\
&= \rho(\widetilde{\sigma}_{k-1}\cdots\widetilde{\sigma}_{r+2}\widetilde{\sigma}_{r+1}\widetilde{\sigma}_r\widetilde{\sigma}_{r-1}\cdots\widetilde{\sigma}_1\widetilde{\sigma}_{r+1})x_1 \\
&= \rho(\widetilde{\sigma}_{k-1}\cdots\widetilde{\sigma}_{r+2}\widetilde{\sigma}_{r+1}\widetilde{\sigma}_r\widetilde{\sigma}_{r-1}\cdots\widetilde{\sigma}_1)x_1 \\
&= x_k.
\end{aligned}$$

**Case $r > k$:** We use the commutation relations (2.13) and Definition 2.3.14 to obtain

$$\begin{aligned}
\rho(\widetilde{\sigma}_r)(x_k) &= \rho(\widetilde{\sigma}_r)\rho(\widetilde{\sigma}_{k-1}\cdots\widetilde{\sigma}_1)x_1 \\
&= \rho(\widetilde{\sigma}_r\widetilde{\sigma}_{k-1}\cdots\widetilde{\sigma}_1)x_1 \\
&= \rho(\widetilde{\sigma}_{k-1}\cdots\widetilde{\sigma}_1\widetilde{\sigma}_r)x_1 \\
&= \rho(\widetilde{\sigma}_{k-1}\cdots\widetilde{\sigma}_1)x_1 \\
&= x_k.
\end{aligned}$$

□

So far we have verified that braidability implies spreadability on the level of a distributional symmetry, where the latter may not invoke the existence of a representation of the partial shifts monoid $\mathcal{S}$ in $\operatorname{End}(\mathcal{A},\varphi)$. Next we strengthen Theorem 2.3.16 by constructing a representation of $\mathcal{S}$ from the representation of the braid group $B_\infty$ in $\operatorname{Aut}(\mathcal{A},\varphi)$.

Suppose the *-algebraic probability space $(\mathcal{A},\varphi)$ is equipped with the representation $\rho\colon B_\infty \to \operatorname{Aut}(\mathcal{A},\varphi)$. Recall that $\mathcal{A}^{\rho(\widetilde{\sigma}_k)}$ denotes the fixed point *-subalgebra of $\rho(\widetilde{\sigma}_k)$ in $\mathcal{A}$. Putting

$$\mathcal{A}_n := \bigcap_{k\geq n} \mathcal{A}^{\rho(\widetilde{\sigma}_k)} \qquad (n \in \mathbb{N}),$$

we obtain a tower of fixed point *-subalgebras such that

$$\mathcal{A}_1 \subset \mathcal{A}_2 \subset \ldots \subset \bigcup_{n=1}^{\infty} \mathcal{A}_n =: \mathcal{A}_\infty \subset \mathcal{A}.$$

**Definition 2.3.19.** Let $(\mathcal{A}, \varphi)$ be a *-algebraic probability space which is equipped with a representation $\rho\colon B_\infty \to \mathrm{Aut}(\mathcal{A}, \varphi)$. The representation $\rho$ is said to have the *generating property* if

$$\mathcal{A} = \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} \mathcal{A}^{\rho(\widetilde{\sigma}_k)} = \mathcal{A}_\infty.$$

In general a representation $\rho$ of $B_\infty$ may fail to enjoy the generating property (see [GK09, Proposition 3.3]).

**Proposition 2.3.20.** *A representation $\rho\colon B_\infty \to \mathrm{Aut}(\mathcal{A}, \varphi)$ restricts to a representation $\rho_\infty\colon B_\infty \to \mathrm{Aut}(\mathcal{A}_\infty, \varphi_\infty)$ which has the generating property. Here $\varphi_\infty$ denotes the restriction of $\varphi$ to $\mathcal{A}_\infty$.*

*Proof.* To ensure $\rho(B_\infty)(\mathcal{A}_\infty) \subset \mathcal{A}_\infty$, it suffices to show that $\rho(\widetilde{\sigma}_m)(\mathcal{A}_\infty) \subset \mathcal{A}_\infty$ for any $m \in \mathbb{N}$. Note that $x \in \mathcal{A}_\infty$ if and only if $x \in \mathcal{A}_N$ for some $N \in \mathbb{N}$. We will show that there exists some $n \in \mathbb{N}$ such that

$$\rho(\widetilde{\sigma}_k)\rho(\widetilde{\sigma}_m)x = \rho(\widetilde{\sigma}_m)x \qquad (k \geq n). \tag{2.16}$$

This implies $\rho(\widetilde{\sigma}_m)x \in \mathcal{A}_n \subset \mathcal{A}_\infty$. So we are left to verify (2.16). We claim that $\rho(\widetilde{\sigma}_k)(\rho(\widetilde{\sigma}_m)x) = \rho(\widetilde{\sigma}_m)x$ for all $k \geq n := N + m + 1$. (Note that $m, N \geq 1$.) Indeed, this is the case, since $\widetilde{\sigma}_k$ and $\widetilde{\sigma}_m$ commute and thus

$$\rho(\widetilde{\sigma}_k\widetilde{\sigma}_m)x = \rho(\widetilde{\sigma}_m\widetilde{\sigma}_k)x = \rho(\widetilde{\sigma}_m)x.$$

Here we have used for the last equality that $x$ is localized in $\mathcal{A}_N$ and thus $\mathcal{A}_N \subset \mathcal{A}^{\rho(\widetilde{\sigma}_k)}$ since $k > N$. $\square$

We will make use of the following well-known relations between Artin generators.

**Lemma 2.3.21.** *Let $\widetilde{\sigma}_i$, for $i \in \mathbb{N}$, denote an Artin generator of $B_\infty$. Then one has*

$$\widetilde{\sigma}_{n+2}(\widetilde{\sigma}_{k+1}\widetilde{\sigma}_{k+2}\cdots\widetilde{\sigma}_N) = (\widetilde{\sigma}_{k+1}\widetilde{\sigma}_{k+2}\cdots\widetilde{\sigma}_N)\widetilde{\sigma}_{n+1} \qquad \textit{for } k \leq n \leq N-2.$$

*Proof.* Let's first consider the case $k = n$. Using the relations for Artin generators, one has

$$\begin{aligned}
\widetilde{\sigma}_{k+2}(\widetilde{\sigma}_{k+1}\widetilde{\sigma}_{k+2}\widetilde{\sigma}_{k+3}\cdots\widetilde{\sigma}_N) &= (\widetilde{\sigma}_{k+2}\widetilde{\sigma}_{k+1}\widetilde{\sigma}_{k+2})\widetilde{\sigma}_{k+3}\cdots\widetilde{\sigma}_N \\
&= (\widetilde{\sigma}_{k+1}\widetilde{\sigma}_{k+2}\widetilde{\sigma}_{k+1})\widetilde{\sigma}_{k+3}\cdots\widetilde{\sigma}_N \qquad \text{(by (2.12))} \\
&= (\widetilde{\sigma}_{k+1}\widetilde{\sigma}_{k+2}\widetilde{\sigma}_{k+3}\cdots\widetilde{\sigma}_N)\widetilde{\sigma}_{k+1} \qquad \text{(by (2.13))}.
\end{aligned}$$

The calculation for the more general case $k < n$ proceeds along similar lines, after initially using

$$\begin{aligned}\widetilde{\sigma}_{n+2}(\widetilde{\sigma}_{k+1}\cdots\widetilde{\sigma}_N) &= \widetilde{\sigma}_{n+2}(\widetilde{\sigma}_{k+1}\cdots\widetilde{\sigma}_n\widetilde{\sigma}_{n+1}\widetilde{\sigma}_{n+2}\cdots\widetilde{\sigma}_N)\\ &= (\widetilde{\sigma}_{k+1}\cdots\widetilde{\sigma}_n)\widetilde{\sigma}_{n+2}(\widetilde{\sigma}_{n+1}\widetilde{\sigma}_{n+2}\cdots\widetilde{\sigma}_N) \qquad \text{(by (2.13))}.\end{aligned}$$

□

**Theorem 2.3.22.** *Suppose $\rho\colon B_\infty \to \mathrm{Aut}(\mathcal{A},\varphi)$ has the generating property. Then the limits*

$$\begin{aligned}\alpha_1(x) &= \lim_{N\longrightarrow\infty}\rho(\widetilde{\sigma}_1\widetilde{\sigma}_2\cdots\widetilde{\sigma}_N)(x),\\ \alpha_2(x) &= \lim_{N\longrightarrow\infty}\rho(\widetilde{\sigma}_2\widetilde{\sigma}_3\cdots\widetilde{\sigma}_N)(x),\\ &\;\;\vdots\\ \alpha_n(x) &= \lim_{N\longrightarrow\infty}\rho(\widetilde{\sigma}_n\widetilde{\sigma}_{n+1}\cdots\widetilde{\sigma}_N)(x),\\ &\;\;\vdots\end{aligned}$$

*exist for any $x \in \mathcal{A}$. Furthermore, the maps $\mathcal{S} \ni \theta_n \mapsto \varrho(\theta_n) := \alpha_n \in \mathrm{End}(\mathcal{A},\varphi)$ multiplicatively extend to a representation $\varrho\colon \mathcal{S} \to \mathrm{End}(\mathcal{A},\varphi)$.*

*Proof.* We show first the existence of the claimed limits for some $x \in \mathcal{A}$. The generating property of $\rho$ ensures $\mathcal{A} = \mathcal{A}_\infty$ and thus there exists some $M \in \mathbb{N}$ such that $x \in \mathcal{A}_M$. Thus $\rho(\widetilde{\sigma}_m)(x) = x$ for all $m \geq M$. Consequently,

$$\lim_{N\longrightarrow\infty}\rho(\widetilde{\sigma}_n\widetilde{\sigma}_{n+1}\cdots\widetilde{\sigma}_N)(x) = \rho(\widetilde{\sigma}_n\widetilde{\sigma}_{n+1}\cdots\widetilde{\sigma}_M)(x) \tag{2.17}$$

for each $n$. Given $x \in \mathcal{A}_\infty$ and $n \in \mathbb{N}$, denote by $\alpha_n(x)$ the corresponding limit element in (2.17).

We show next that the map $\mathcal{A} \ni x \mapsto \alpha_n(x) \in \mathcal{A}_\infty$ defines an endomorphism of the *-algebraic probability space $(\mathcal{A}_\infty,\varphi)$. Clearly, $\alpha_n(1_\mathcal{A}) = 1_\mathcal{A}$. Furthermore, for $x, y \in \mathcal{A}_\infty$, there exists some $M$ such that $x, y \in \mathcal{A}_M$. Thus we can compute

$$\begin{aligned}\alpha_n(xy) &= \lim_{N\longrightarrow\infty}\rho(\widetilde{\sigma}_n\widetilde{\sigma}_{n+1}\cdots\widetilde{\sigma}_N)(xy)\\ &= \rho(\widetilde{\sigma}_n\widetilde{\sigma}_{n+1}\cdots\widetilde{\sigma}_M)(xy)\\ &= \rho(\widetilde{\sigma}_n\widetilde{\sigma}_{n+1}\cdots\widetilde{\sigma}_M)(x)\;\rho(\widetilde{\sigma}_n\widetilde{\sigma}_{n+1}\cdots\widetilde{\sigma}_M)(y)\\ &= \lim_{N\longrightarrow\infty}\rho(\widetilde{\sigma}_n\widetilde{\sigma}_{n+1}\cdots\widetilde{\sigma}_N)(x)\;\lim_{N\longrightarrow\infty}\rho(\widetilde{\sigma}_n\widetilde{\sigma}_{n+1}\cdots\widetilde{\sigma}_N)(y)\\ &= \alpha_n(x)\alpha_n(y),\end{aligned}$$

since $\rho(\sigma) \in \mathrm{Aut}(\mathcal{A}, \varphi)$ for any $\widetilde{\sigma} \in B_\infty$. A similar calculation ensures

$$\begin{aligned}
\big(\alpha_n(x)\big)^* &= \big(\lim_{N\longrightarrow\infty} \rho(\widetilde{\sigma}_n\widetilde{\sigma}_{n+1}\cdots\widetilde{\sigma}_N)(x)\big)^* \\
&= \big(\rho(\widetilde{\sigma}_n\widetilde{\sigma}_{n+1}\cdots\widetilde{\sigma}_M)(x)\big)^* \\
&= \rho(\widetilde{\sigma}_n\widetilde{\sigma}_{n+1}\cdots\widetilde{\sigma}_M)(x^*) \\
&= \lim_{N\longrightarrow\infty} \rho(\widetilde{\sigma}_n\widetilde{\sigma}_{n+1}\cdots\widetilde{\sigma}_N)(x^*) \\
&= \alpha_n(x^*).
\end{aligned}$$

Thus $\alpha_n$ is a unital *-homomorphism of $\mathcal{A}_\infty$ for which we conclude $\varphi \circ \alpha_n = \varphi$ from

$$\begin{aligned}
\varphi \circ \alpha_n(x) &= \varphi\big(\alpha_n(x)\big) \\
&= \varphi\big(\rho(\widetilde{\sigma}_n\widetilde{\sigma}_{n+1}\cdots\widetilde{\sigma}_M)(x)\big) \\
&= \varphi(x)
\end{aligned}$$

for any $x \in \mathcal{A}$. Altogether, this ensures $\alpha_n \in \mathrm{End}(\mathcal{A}, \varphi)$. We are left to verify that these endomorphisms $\alpha_n$ satisfy the relations of the generators $\theta_n$ of the partial shift monoid $\mathcal{S}$:

$$\alpha_k\alpha_\ell = \alpha_{\ell+1}\alpha_k \qquad (1 \le k \le \ell < \infty).$$

We know already that, given $x \in \mathcal{A}_\infty$, there exist some $n, m \in \mathbb{N}$ with $n \le m$ such that

$$\begin{aligned}
\alpha_{\ell+1}\alpha_k &= \lim_{N\longrightarrow\infty}\lim_{M\to\infty} \rho(\widetilde{\sigma}_{\ell+1}\cdots\widetilde{\sigma}_N)\rho(\widetilde{\sigma}_k\cdots\widetilde{\sigma}_M)(x) \\
&= \rho(\widetilde{\sigma}_{\ell+1}\cdots\widetilde{\sigma}_n)\rho(\widetilde{\sigma}_k\cdots\widetilde{\sigma}_m)(x) \\
&= \rho\big((\widetilde{\sigma}_{\ell+1}\cdots\widetilde{\sigma}_n)(\widetilde{\sigma}_k\cdots\widetilde{\sigma}_m)\big)(x) \\
&= \rho\big((\widetilde{\sigma}_k\cdots\widetilde{\sigma}_m)(\widetilde{\sigma}_\ell\cdots\widetilde{\sigma}_{n-1})\big)(x) \qquad (2.18) \\
&= \lim_{M\longrightarrow\infty}\lim_{N\longrightarrow\infty} \rho\big((\widetilde{\sigma}_k\cdots\widetilde{\sigma}_M)(\widetilde{\sigma}_\ell\cdots\widetilde{\sigma}_{N-1})\big)(x) \\
&= \alpha_k\alpha_\ell(x).
\end{aligned}$$

Here we repeatedly applied Lemma 2.3.21 to obtain (2.18). This establishes that the multiplicative extension of the maps $\mathcal{S} \ni \theta_n \mapsto \alpha_n =: \varrho(\theta_n)$ defines a representation $\varrho\colon \mathcal{S} \to \mathrm{End}(\mathcal{A}, \varphi)$. $\square$

# Chapter 3

# *-Algebraic Central Limit Theorems

We start this chapter with reviewing the classical central limit theorem, including a multivariate version of it. Also, we present singleton vanishing properties (SVPs), as they are known in the literature to play a role for multivariate versions of *-algebraic CLTs. Additionally, we refine the notion of exchangeability/spreadability of sequences of random variables to that of $C$-jointly and $C$-separately exchangeable/spreadable families of random variables, for some 'color set' $C$. We provide multivariate CLTs, which correspond to these refined notions of distributional symmetries or invariance principles. So far $C$-separately exchangeable/spreadable families of random variables have not been addressed explicitly in the published results on *-algebraic CLTs. Related results will be used later for CLTs associated to $\omega$-sequences of partial isometries (which we introduce in Chapter 4). Also, we discuss how one can construct $C$-jointly and $C$-separately exchangeable/spreadable sequences from a single exchangeable or spreadable sequence. Furthermore, we present factorization properties of mixed moments in the context of distributional invariance principles. We will make use of these factorization properties for SVPs when establishing concrete moment formulas for CLTs associated to $\omega$-sequences of partial isometries.

## 3.1 Classical Central Limit Theorems

In this section we review the combinatorial approaches to the classical CLT and its multivariate version. Our presentation follows [NS06, Sp19]. Recall the notion of a classical *-algebraic probability space from Definition 2.2.3 and the notion of independence from Definition 2.2.4.

**Theorem 3.1.1.** *Let $(\mathcal{A}, \varphi)$ be a classical *-algebraic probability space and let $\mathbf{x} \equiv (x_n)_{n=1}^{\infty} \subset \mathcal{A}$ be a sequence such that*

*(i) $x_i = x_i^*$ for $i \in \mathbb{N}$;*

*(ii)* $\mathbf{x}$ *is independent and identically distributed.*

*Then one has for*

$$S_N = \frac{1}{\sqrt{N}}\Big(x_1 + \ldots + x_N - N\varphi(x_1)\Big)$$

*that, for all* $n \in \mathbb{N}$,

$$\lim_{N\to\infty} \varphi(S_N^n) = \begin{cases} 0 & \text{if } n \text{ odd,} \\ (n-1)!!\Big(\varphi(x_1^2) - \varphi(x_1)^2\Big)^{\frac{n}{2}} & \text{if } n \text{ even.} \end{cases}$$

Note that $\varphi(x_1^2) = \varphi(x_1^* x_1) \geq 0$ as $x_1$ is self-adjoint, and that $\varphi(x_1^2) - \varphi(x_1)^2$ is the variance of the random variable $x_1$.

*Proof.* Since all $x_n$ are identically distributed,

$$\Big(x_1 + \ldots + x_N - N\varphi(x_1)\Big) = (x_1 - \varphi(x_1)) + \ldots + (x_N - \varphi(x_N)). \tag{3.1}$$

The sequence $\big(x_n - \varphi(x_n)\big)_{n=1}^{\infty}$ is independent, identically distributed, and centred. Thus it suffices to calculate the central limit for a centred sequence $(x_n)_{n=1}^{\infty}$. In other other words, we need to compute

$$\varphi(S_N^n) = \frac{1}{N^{\frac{n}{2}}} \sum_{\mathbf{i}(1),\ldots,\mathbf{i}(n)=1}^{N} \varphi(x_{\mathbf{i}(1)} \cdots x_{\mathbf{i}(n)}) \tag{3.2}$$

for $N \to \infty$. We will subdivide the proof of this theorem into several steps.

**Step 1.** The $n$-tuple of indices $(\mathbf{i}(1), \ldots, \mathbf{i}(n))$ uniquely corresponds to a function $\mathbf{i}\colon [n] \to \mathbb{N}$, which induces the kernel set partition $\pi := \ker(\mathbf{i}) = \{V_1, \ldots, V_k\} \in \mathcal{P}(n)$ (see Definition 2.1.8). Here $V_i$ denotes the blocks of $\pi$ for some $1 \leq k \leq n$ and $k$ is the size of the partition. We say that the two tuples $\mathbf{i}, \mathbf{j}\colon [n] \to \mathbb{N}$ are equivalent, in symbols: $\mathbf{i} \sim \mathbf{j}$, if

$$\mathbf{i}(r) = \mathbf{i}(s) \Leftrightarrow \mathbf{j}(r) = \mathbf{j}(s) \text{ for all } 1 \leq r, s \leq n.$$

We conclude from this that

$$\mathbf{i} \sim \mathbf{j} \quad \Longleftrightarrow \quad \ker(\mathbf{i}) = \ker(\mathbf{j}).$$

Since all the random variables are independent and identically distributed, we conclude that

$$\varphi(x_{\mathbf{i}(1)} \cdots x_{\mathbf{i}(n)}) = \varphi(x_{\mathbf{j}(1)} \cdots x_{\mathbf{j}(n)}) \tag{3.3}$$

whenever $\mathbf{i} \sim \mathbf{j}$. Thus

$$\varphi_\pi := \varphi(x_{\mathbf{i}(1)} \cdots x_{\mathbf{i}(n)}) \tag{3.4}$$

is well-defined for the kernel set partition $\pi = \ker(\mathbf{i})$.

**Step 2.** The cardinality of the set

$$\{\mathbf{i}\colon [n] \to [N] \mid \ker(\mathbf{i}) = \pi\}$$

for some partition $\pi \in \mathcal{P}(n)$ is given by

$$A_{|\pi|}^{(N)} = N \cdot (N-1) \cdots (N - |\pi| + 1) = \binom{N}{|\pi|} |\pi|!.$$

Note that $|\pi| = k$ if $\pi = \{V_1, \dots, V_k\}$ for some $1 \leq k \leq n$. Therefore, we can rewrite (3.2) as the finite sum

$$\varphi(S_N^n) \quad = \quad \frac{1}{N^{\frac{n}{2}}} \sum_{\pi \in \mathcal{P}(n)} A_{|\pi|}^{(N)} \varphi_\pi = \frac{1}{N^{\frac{n}{2}}} \sum_{\pi \in \mathcal{P}(n)} \binom{N}{|\pi|} |\pi|!\, \varphi_\pi. \tag{3.5}$$

As this finite sum is independent of the choice of $N$, the convergence of each summand can be discussed separately. This is done in Step 3, where one first shows that $\varphi_\pi$ vanishes whenever $\pi$ contains a singleton. Thus the finite sum over all partitions reduces further to a sum over partitions $\pi$ with $|\pi| \leq n/2$. Furthermore, one shows that $A_{|\pi|}^{(N)}/N^{n/2}$ vanishes for $N \to \infty$ whenever $|\pi| < n/2$. Thus we will arrive in Step 3 at the conclusion that the only contribution to the limit comes from pair partitions with $|\pi| = n/2$, where one has $A_{|\pi|}^{(N)}/N^{n/2} \to 1$ for $N \to \infty$.

**Step 3.** We need now to examine the contribution of different partitions. We will see that some of these partitions will vanish. First we assume that $\pi = \{V_1, \dots, V_k\}$ contains a singleton block. Thus there exists a block $V \in \pi$ such that $|V| = 1$, let us say $V = \{\ell\}$, for $\ell \in \ker(\mathbf{i})$. Since all $x_i$ are classically independent (see Definition 2.2.4) and centred, we conclude that

$$\begin{aligned} \varphi_\pi &= \varphi(x_{\mathbf{i}(1)} \cdots x_{\mathbf{i}(\ell-1)} x_{\mathbf{i}(\ell)} x_{\mathbf{i}(\ell+1)} \cdots x_{\mathbf{i}(n)}) \\ &= \varphi(x_{\mathbf{i}(\ell)}) \varphi(x_{\mathbf{i}(1)} \cdots x_{\mathbf{i}(\ell-1)} x_{\mathbf{i}(\ell+1)} \cdots x_{\mathbf{i}(n)}) = 0. \end{aligned}$$

Hence, if the partition $\pi = \ker(\mathbf{i})$ contains a singleton (i.e. a block $V$ with $|V| = 1$), this implies $\varphi_\pi = 0$. Therefore, no partitions with a singleton contributes to the sum. Thus only those partitions $\pi$ can contribute to the central limit for which each block of $\pi$ has at least two elements. This implies $k \leq n/2$ for the number of blocks of the partition $\pi$. We consider separately the two cases $k < n/2$ and $k = n/2$.

Let us first consider the case $k < \frac{n}{2}$. Since the monomial $A_k^{(N)} = N(N-1) \cdots (N - k + 1)$ contains strictly less than $n/2$ factors, one concludes

$$\lim_{N \to \infty} \frac{A_{|\pi|}^{(N)}}{N^{\frac{n}{2}}} = \frac{N(N-1) \cdots (N-k+1)}{N^{\frac{n}{2}}} = 0.$$

In particular, as $k < n/2$ for any odd number $n$, no moment $\varphi_\pi$ with an odd order $n$ can survive in the limit $N \to \infty$.
We are left to consider the case $k = \frac{n}{2}$, which can only occur for even $n$ and $\pi \in \mathcal{P}_2(n)$, i.e. when $\pi$ is a pair partition. Taking again the limit $N \to \infty$ we find that

$$\frac{A^{(N)}_{\frac{n}{2}}}{N^{\frac{n}{2}}} = \frac{N(N-1)\cdots(N-k+1)}{N \cdot N \cdots N} \longrightarrow 1$$

since numerator and denominator have the same number of factors.

Altogether, we can write (3.5) as

$$\lim_{N\to\infty} \varphi(S_N^n) = \sum_{\pi \in \mathcal{P}_2(n)} \varphi_\pi. \tag{3.6}$$

where we use the convention that summation over an empty set gives zero.

**Step 4.** Suppose $\pi \in \mathcal{P}_2(n)$ with $\pi = \{V_1, \dots, V_k\}$ and $n = 2k$. Note that there exists some $\mathbf{i}\colon [2k] \to [k]$ such that $\pi = \ker(\mathbf{i})$. It follows from commutativity and classical independence that

$$\varphi_\pi = \varphi(x_{\mathbf{i}(1)} \cdots x_{\mathbf{i}(2k)}) = \prod_{i=1}^{k} \varphi(x_i x_i).$$

Since all $x_i$ are identically distributed, we conclude further that

$$\varphi_\pi = \varphi(x_1 x_1)^k = \Big(\varphi(x_1 x_1)\Big)^{n/2}.$$

Thus (3.6) counts the number of all pair partitions of the set $[n]$:

$$|\mathcal{P}_2(\{1, \dots, 2k\})| = (2k-1) \cdot (2k-3) \cdots 1 = (2k-1)!!$$

Altogether, one arrives at

$$\lim_{N\to\infty} \varphi(S_N^n) = \sum_{\pi \in \mathcal{P}_2(n)} \varphi(x_1^* x_1)^{\frac{n}{2}} = (n-1)!!\varphi(x_1^2)^{\frac{n}{2}}$$

when the sequence $\mathbf{x}$ is assumed to be centred. The non-centred case is deduced from above equation by replacing $x_1$ (from the centred setting) by $x_1 - \varphi(x_1)$ (in the non-centred setting). This completes the proof of Theorem 3.1.1. $\square$

In the following, we review a multivariate version of the classical CLT and show that the limit distribution is described by the joint distribution of a *Gaussian family* as $N \to \infty$ (see [NS06, Remark 8.18]). To ease the notation, we limit our considerations to the case of centred random variables, as it is elementary to reduce the general case to such a setting.

**Notation 3.1.2.** We recall that $(n-1)!! = 0$ for an odd non-negative integer and $(-1)!! = 1$ by convention. Furthermore, we make use of the convention $0^0 = 1$.

**Theorem 3.1.3.** *Let $(\mathcal{A}, \varphi)$ be a classical *-algebraic probability space and suppose the d-tuple of sequences $\{(x_{t,n})_{n=1}^{\infty}\}_{t\in[d]} \subset \mathcal{A}$ is such that*

*(i) $x_{t,n} = (x_{t,n})^*$ for all $n \in \mathbb{N}$ and $t \in [d]$;*

*(ii) $\{x_{t,n} \mid n \in \mathbb{N}, t \in [d]\}$ is a set of mutually independent random variables;*

*(iii) $(x_{t,n})_{n\in\mathbb{N}}$ is a sequence of centred, identically distributed random variables (for each $t \in [d]$).*

*Then one has for*

$$S_{t,N} = \frac{x_{t,1} + \ldots + x_{t,N}}{\sqrt{N}}$$

*that*

$$\lim_{N\to\infty} \varphi(S_{\mathbf{t}(1),N} \cdots S_{\mathbf{t}(n),N}) = \prod_{c\in[d]} (n_c - 1)!!\varphi(x_{c,1}^2)^{\frac{n_c}{2}}$$

*where $\mathbf{t}\colon [n] \to [d]$ and $n_c = |\mathbf{t}^{-1}(\{c\})|$.*

*Proof.* We need to calculate, for some fixed $n \in \mathbb{N}$ and $\mathbf{t}(1), \ldots, \mathbf{t}(n) \in [d]$, the limit for $N \to \infty$ of

$$\varphi\big(S_{\mathbf{t}(1),N} \cdots S_{\mathbf{t}(n),N}\big) = \frac{1}{N^{\frac{n}{2}}} \sum_{\mathbf{i}(1),\ldots,\mathbf{i}(n)=1}^{N} \varphi\big(x_{\mathbf{t}(1),\mathbf{i}(1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}\big).$$

Since all random variables commute, we may assume without loss of generality that $1 \leq \mathbf{t}(1) \leq \mathbf{t}(2) \leq \cdots \leq \mathbf{t}(n) \leq d$. Let $\ker(\mathbf{t}) = \{W_1, \ldots, W_c\} \in \mathcal{P}(n)$ for some $1 \leq c \leq d$ such that $w_i < w_j$ for all $w_i \in W_i$ and $w_j \in W_j$ with $i < j$. Then

$$\sum_{\mathbf{i}\colon [n]\to[N]} \varphi\big(x_{\mathbf{t}(1),\mathbf{i}(1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}\big) = \sum_{\mathbf{i}\colon W_1\to[N]} \cdots \sum_{\mathbf{i}\colon W_c\to[N]} \varphi\big(x_{\mathbf{t}(1),\mathbf{i}(1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}\big).$$

For $1 \leq m \leq c$, let $n_m := |W_m|$ and let $\mathbf{i}_m\colon W_m \to [N]$ be the restriction of the index function $\mathbf{i}\colon [n] \to [N]$ . Furthermore, let $t_m := \mathbf{t}(w)$ for $w \in W_m$. Since all random variables are independent, and writing below $x_{t,n}$ as $x_n^{(t)}$), we can further factorize and regroup the right-hand side of above equation such that

$$\begin{aligned}
&\sum_{\mathbf{i}\colon [n]\to[N]} \varphi\big(x_{\mathbf{t}(1),\mathbf{i}(1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}\big) \\
&= \sum_{\mathbf{i}_1\colon W_1\to[N]} \cdots \sum_{\mathbf{i}_c\colon W_c\to[N]} \varphi\Big(\underbrace{x_{\mathbf{i}_1(1)}^{(\mathbf{t}(1))} \cdots x_{\mathbf{i}_1(n_1)}^{(\mathbf{t}(n_1))}}_{n_1 \text{ factors}}\Big) \cdots \varphi\Big(\underbrace{x_{\mathbf{i}_c(n-n_c+1)}^{(\mathbf{t}(n-n_c+1))} \cdots x_{\mathbf{i}_c(n)}^{(\mathbf{t}(n))}}_{n_c \text{ factors}}\Big) \\
&= \left(\sum_{\mathbf{i}_1\colon W_1\to[N]} \varphi\Big(\underbrace{x_{\mathbf{i}_1(1)}^{(t_1)} \cdots x_{\mathbf{i}_1(n_1)}^{(t_1)}}_{n_1 \text{ factors}}\Big)\right) \cdots \left(\sum_{\mathbf{i}_c\colon W_c\to[N]} \varphi\Big(\underbrace{x_{\mathbf{i}_c(n-n_c+1)}^{(t_c)} \cdots x_{\mathbf{i}_c(n)}^{(t_c)}}_{n_c \text{ factors}}\Big)\right).
\end{aligned}$$

Now each of these $c$ factors can be treated similar as it was done in Step 1 of the proof of the classical CLT, Theorem 3.1.1. For this purpose, let $W = \{w_1, w_2, \ldots, w_k\}$ be a finite subset of $\mathbb{N}$ such that $1 \le w_1 < w_2 < \cdots < w_k$ for some $k \in \mathbb{N}$. Then

$$\varphi_{\pi}^{(t)} := \left( \sum_{\mathbf{j}\colon W \to [N]} \varphi\Big( \underbrace{x_{\mathbf{j}(w_1)}^{(t)} \cdots x_{\mathbf{j}(w_k)}^{(t)}}_{k \text{ factors}} \Big) \right)$$

is well-defined for any $\mathbf{j}\colon W \to [N]$ with $\pi = \ker(\mathbf{j}) \in \mathcal{P}(W)$. The cardinality of the set

$$\{\mathbf{j}\colon W \to [N] \mid \ker(\mathbf{j}) = \pi\}$$

for some partition $\pi \in \mathcal{P}(W)$ is

$$A_{|\pi|}^{(N)} = \binom{N}{|\pi|} |\pi|! = N \cdot (N-1) \cdots (N - |\pi| + 1).$$

Thus we can write

$$\begin{aligned}
&\sum_{\mathbf{i}\colon [n] \to [N]} \varphi\big(x_{\mathbf{t}(1),\mathbf{i}(1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}\big) \\
&\qquad = \left( \sum_{\pi_1 \in \mathcal{P}(W_1)} A_{|\pi_1|}^{(N)} \varphi_{\pi_1}^{(t_1)} \right) \cdots \left( \sum_{\pi_c \in \mathcal{P}(W_c)} A_{|\pi_c|}^{(N)} \varphi_{\pi_c}^{(t_c)} \right).
\end{aligned}$$

Altogether, since $n = n_1 + n_2 + \ldots + n_c$, we arrive at

$$\begin{aligned}
&\varphi\big(S_{\mathbf{t}(1),N} \cdots S_{\mathbf{t}(n),N}\big) \\
&\quad = \frac{1}{N^{\frac{n}{2}}} \sum_{\mathbf{i}\colon [n] \to [N]} \varphi\big(x_{\mathbf{t}(1),\mathbf{i}(1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}\big) \\
&\quad = \left( N^{-\frac{n_1}{2}} \sum_{\pi_1 \in \mathcal{P}(W_1)} A_{|\pi_1|}^{(N)} \varphi_{\pi_1}^{(t_1)} \right) \cdots \left( N^{-\frac{n_c}{2}} \sum_{\pi_c \in \mathcal{P}(W_c)} A_{|\pi_c|}^{(N)} \varphi_{\pi_c}^{(t_c)} \right).
\end{aligned}$$

As explicitly shown in the proof of the univariate CLT, Theorem 3.1.1, only pair partitions survive as summands in the limit $N \to \infty$ in each of the $c$ factors such that

$$\begin{aligned}
&\lim_{N \to \infty} \varphi\big(S_{\mathbf{t}(1),N} \cdots S_{\mathbf{t}(n),N}\big) \\
&\qquad = \left( \sum_{\pi_1 \in \mathcal{P}_2(W_1)} \varphi_{\pi_1}^{(t_1)} \right) \cdots \left( \sum_{\pi_c \in \mathcal{P}_2(W_c)} \varphi_{\pi_c}^{(t_c)} \right) \\
&\qquad = \left( (n_1 - 1)!!\varphi(x_1^{(t_1)} x_1^{(t_1)})^{\frac{n_1}{2}} \right) \cdots \left( (n_c - 1)!!\varphi(x_1^{(t_c)} x_1^{(t_c)})^{\frac{n_c}{2}} \right).
\end{aligned}$$

Since $\{W_1, W_2, \ldots, W_c\}$ is a partition of the set $[n]$, we know that $n_i = |W_i| > 0$ for all $i = 1, 2, \ldots, c$. Permitting the case that a 'color' in $[d]$ may not appear, and making use of the convention $(0-1)!! = 1$, we can rewrite above equation as the $d$-fold product

$$\lim_{N\to\infty} \varphi\big(S_{\mathbf{t}(1),N} \cdots S_{\mathbf{t}(n),N}\big) \\ = \Big((n_1 - 1)!!\varphi(x_1^{(1)} x_1^{(1)})^{\frac{n_1}{2}}\Big) \cdots \Big((n_d - 1)!!\varphi(x_1^{(d)} x_1^{(d)})^{\frac{n_d}{2}}\Big),$$

where now $n_1 = |\mathbf{t}^{-1}(\{1\})|, \ldots, n_d = |\mathbf{t}^{-1}(\{d\})|$. □

It is well-known that a sequence of centred independent identically distributed random variables is exchangeable (see Definition 2.3.1) and satisfies a SFP (see Definition 3.2.1). This motivates us to take distributional invariance principles and certain factorization properties as an alternative starting point for the formulation of (univariate and) multivariate CLTs in the framework of *-algebraic probability spaces. Actually, we replace these factorization properties by so-called singleton vanishing properties, as investigated in the consecutive sections.

## 3.2 Singleton Vanishing Properties

The singleton factorization property (SFP) of a sequence of random variables is well-known to play an important role when proving algebraic CLTs. We first discuss some of its generalizations, as they are appropriate for multivariate CLTs. Afterwards, as a kind of further generalization, we present singleton vanishing properties (SVPs) for set-indexed families of random variables, as we will need them again for multivariate CLTs in *-algebraic probability theory.

**Definition 3.2.1.** Let $(\mathcal{A}, \varphi)$ be a *-algebraic probability space.

(i) The sequence $(x_n)_{n=1}^\infty \subset \mathcal{A}$ is said to have the *singleton factorization property* (SFP) if, for any $n \in \mathbb{N}$,

$$\varphi\big(x_{\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\big) = \varphi\big(x_{\mathbf{i}(\ell)}^{\boldsymbol{\varepsilon}(\ell)}\big) \cdot \varphi\big(x_{\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{i}(\ell-1)}^{\boldsymbol{\varepsilon}(\ell-1)} x_{\mathbf{i}(\ell+1)}^{\boldsymbol{\varepsilon}(\ell+1)} \cdots x_{\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\big)$$

for any $\boldsymbol{\varepsilon}\colon [n] \to \{1, *\}$ and $\mathbf{i}\colon [n] \to \mathbb{N}$ with $\{\ell\} \in \ker(\mathbf{i})$.

(ii) The family of sequences $\mathcal{X} \equiv \{(x_{c,n})_{n=1}^\infty \mid c \in C\} \subset \mathcal{A}$, for some index set $C$, is said to have the *$C$-joint singleton factorization property* (SFP) if, for any $n \in \mathbb{N}$,

$$\varphi\big(x_{\mathbf{t}(1),\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\big) \\ = \varphi\big(x_{\mathbf{t}(\ell),\mathbf{i}(\ell)}^{\boldsymbol{\varepsilon}(\ell)}\big) \cdot \varphi\big(x_{\mathbf{t}(1),\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{t}(\ell-1),\mathbf{i}(\ell-1)}^{\boldsymbol{\varepsilon}(\ell-1)} x_{\mathbf{t}(\ell+1),\mathbf{i}(\ell+1)}^{\boldsymbol{\varepsilon}(\ell+1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\big)$$

for any $\boldsymbol{\varepsilon}\colon [n] \to \{1, *\}$, any $\mathbf{i}\colon [n] \to \mathbb{N}$, and $\mathbf{t}\colon [n] \to C$ with $\{\ell\} \in \ker(\mathbf{i})$.

(iii) The family of sequences $\mathcal{X} \equiv \{(x_{c,n})_{n=1}^{\infty} \mid c \in C\} \subset \mathcal{A}$, for some index set $C$, is said to have the *$C$-separate singleton factorization property* (SFP) if, for any $n \in \mathbb{N}$,

$$\begin{aligned}&\varphi\big(x_{\mathbf{t}(1),\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\big) \\ &\qquad = \varphi\big(x_{\mathbf{t}(\ell),\mathbf{i}(\ell)}^{\boldsymbol{\varepsilon}(\ell)}\big) \cdot \varphi\big(x_{\mathbf{t}(1),\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{t}(\ell-1),\mathbf{i}(\ell-1)}^{\boldsymbol{\varepsilon}(\ell-1)} x_{\mathbf{t}(\ell+1),\mathbf{i}(\ell+1)}^{\boldsymbol{\varepsilon}(\ell+1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\big)\end{aligned}$$

for any $\boldsymbol{\varepsilon}\colon [n] \to \{1, *\}$, any $\mathbf{i}\colon [n] \to \mathbb{N}$, and $\mathbf{t}\colon [n] \to C$ with $\{\ell\} \in \ker(\mathbf{i}|_W)$ for some $W \in \ker(\mathbf{t})$.

Note that (i) is the special case of (ii) and (iii) for $C = \{c\}$. We illustrate the singleton factorization property from (iii) by an example.

**Example 3.2.2.** Let $C = \{1, 2\}$ and consider the pair of sequences $\{(x_{c,n})_{n\in\mathbb{N}} \mid c \in C\} \subset \mathcal{A}$. For simplicity, we assume $x_{c,n} = x_{c,n}^*$ for all $n \in \mathbb{N}$ and $c \in C$. Moreover, we will write $x_{c,n}$ as $x_n^{(c)}$ for clarity of the notation. Consider the moment

$$\varphi(x_1^{(2)} x_3^{(1)} x_1^{(1)} x_3^{(1)} x_1^{(2)}).$$

One can easily read off from this moment the explicit form of the index functions $\mathbf{i}\colon [5] \to \mathbb{N}$ and $\mathbf{t}\colon [5] \to C$ such that

$$\begin{aligned}\ker(\mathbf{i}) &= \{\{1,3,5\},\{2,4\}\} \in \mathcal{P}([5])\} \\ \ker(\mathbf{t}) &= \{\{1,5\},\{2,3,4\}\} =: \{W_1, W_2\} \in \mathcal{P}([5]) \\ \ker(\mathbf{i}|_{W_1}) &= \{\{1,5\}\} \in \mathcal{P}(\{1,5\}) \\ \ker(\mathbf{i}|_{W_2}) &= \{\{2,4\},\{3\}\} \in \mathcal{P}(\{2,3,4\})\end{aligned}$$

Since $\ker(\mathbf{i}|_{W_2})$ contains a singleton the validity of the $C$-separate SFP implies that the considered moment factorizes as

$$\varphi(x_1^{(2)} x_3^{(1)} x_1^{(1)} x_3^{(1)} x_1^{(2)}) = \varphi(x_1^{(1)}) \varphi(x_1^{(2)} x_3^{(1)} x_3^{(1)} x_1^{(2)}).$$

**Definition 3.2.3.** Let $(\mathcal{A}, \varphi)$ be a *-algebraic probability space and let $C$ be a fixed non-empty set. The family of sequences $\mathcal{X} \equiv \{(x_{c,n})_{n=1}^{\infty} \mid c \in C\} \subset \mathcal{A}$ is said to have the

(i) *$C$-joint singleton vanishing property (SVP)* if, for any $n \in \mathbb{N}$, for every $\boldsymbol{\varepsilon}\colon [n] \to \{1, *\}$, $\mathbf{i}\colon [n] \to \mathbb{N}$, and $\mathbf{t}\colon [n] \to C$,

$$\varphi\Big(x_{\mathbf{t}(1),\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\Big) = 0$$

whenever there exists a singleton $\{\ell\} \in \ker(\mathbf{i})$ for some $\ell \in [n]$;

(ii) *$C$-separate singleton vanishing property (SVP)* if, for any $n \in \mathbb{N}$, for every $\boldsymbol{\varepsilon}\colon [n] \to \{1, *\}$, $\mathbf{i}\colon [n] \to \mathbb{N}$, and $\mathbf{t}\colon [n] \to C$,

$$\varphi\Big(x_{\mathbf{t}(1),\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\Big) = 0$$

whenever there exists a singleton $\{\ell\} \in \ker(\mathbf{i}|_W)$ for some $\ell \in W$ and some block $W \in \ker(\mathbf{t})$.

If $C = \{c\}$, the $C$-joint (or $C$-separate) SVP of the sequence $\mathcal{X}$ is just called *SVP*.

**Lemma 3.2.4.** *Consider for the family of sequences $\mathcal{X}$ as given in Definition 3.2.3:*

*(a) $\mathcal{X}$ has the $C$-joint SVP;*

*(b) $\mathcal{X}$ has the $C$-separate SVP.*

*Then one has '(b) $\Longrightarrow$ (a)', but the converse implication may fail to be true.*

*Proof.* '(b) $\Longrightarrow$ (a)': Consider the moment

$$\varphi\Big(x_{\mathbf{t}(1),\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\Big)$$

for some $\boldsymbol{\varepsilon}\colon [n] \to \{1, *\}$, $\mathbf{i}\colon [n] \to \mathbb{N}$, $\mathbf{t}\colon [n] \to C$, and put $\pi := \ker(\mathbf{i})$. Suppose $\{\ell\} \in \pi$ for some $\ell \in [n]$. Then $\{\ell\} \in W$ for some block $W \in \ker(\mathbf{t})$. Consequently, $\{\ell\} \in \pi_{|W}$. As we have assumed the $C$-separate SVP of $\mathcal{X}$, we conclude

$$\varphi\Big(x_{\mathbf{t}(1),\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\Big) = 0.$$

As this is true whenever $\pi$ contains a singleton, it follows that $\mathcal{X}$ enjoys the $C$-joint SVP. The failure of '(a) $\Longrightarrow$ (b)' is inferred from Example 3.2.5 or Example 3.2.6. $\square$

Let us illustrate the failure of the implication '(a) $\Longrightarrow$ (b)'.

**Example 3.2.5.** Let $(\mathcal{B}, \psi)$ be a *-algebraic probability space and consider the *-algebraic infinite tensor product probability space $(\mathcal{A}, \varphi) = \bigotimes_{n\in\mathbb{N}}(\mathcal{B}, \psi)$. Let $C = \{1, 2\}$ and $b_c \in \mathcal{B}$ with $b_c = b_c^*$ (for simplicity) be fixed and satisfy $\psi(b_c) = 0$ for $c \in C$ . Let $x_{c,n} \in \mathcal{A}$ denote the canonical embedding of $b_c \in \mathcal{B}$ into the $n$-th factor of the infinite tensor product of $\mathcal{B}$ with itself. Then the pair of sequences $\{(x_{c,n})_{n=1}^{\infty} \mid c \in C\}$ has the $C$-joint SVP, but may fail to have the $C$-separate SVP. For example, one has

$$\varphi(x_{1,1}x_{2,1}) = \psi(b_1b_2),$$

which gives the partition $\pi = \{\{1, 2\}\}$ and $\ker(\mathbf{t}) = \{\{1\}, \{2\}\} =: \{W_1, W_2\}$, and the reduced partitions $\pi_{|W_1} = \{\{1\}\}$ and $\pi_{|W_2} = \{\{2\}\}$.

Now the $C$-separate SVP of $\mathcal{X}$ implies $\varphi(x_{1,1}x_{2,1}) = 0$, whereas this usually can't be concluded from the $C$-joint SVP. To see the latter, just take $b = b_1 = b_2$. Then one has $\varphi(x_{1,1}x_{2,1}) = \psi(b^2)$ which may be non-zero. For a concrete example of this failure, consider $\mathcal{B} := C(\mathbb{R})$ to be equipped with the state $\psi$ that is given by $\psi(f) := \frac{1}{2}(f(0) + f(1))$. Choosing $b = f - \psi(f)$, one has

$$\varphi(x_{1,1}x_{2,1}) = 0 \quad \iff \quad \psi(b^2) = \psi(f^2) - \psi(f)^2 = 0 \quad \iff \quad f(0) = f(1).$$

So the corresponding concrete pair of sequences $\mathcal{X}$ has the $C$-separate SVP, but fails to have the $C$-joint SVP if $f(0) \neq f(1)$.

**Example 3.2.6.** Let $(\mathcal{B}, \psi) = (\mathbb{M}_n(\mathbb{C}), \psi)$ with $\psi(b) = \operatorname{Tr}(Db)$ for some density matrix $D \in \mathbb{M}_n(\mathbb{C})$ in the construction of the infinite tensor product probability space $(\mathcal{A}, \varphi)$ from Example 3.2.5. Now choose $b_c \in \mathbb{M}_n(\mathbb{C})$ with $\operatorname{Tr}(Db_c) = 0$ for $c \in C$. Now the pair of sequences $\mathcal{X}$ (as introduced Example 3.2.5) satisfies again the $C$-separate SVP. Furthermore, one observes:

$$\mathcal{X} \text{ satisfies the } C\text{-joint SVP} \quad \iff \quad \operatorname{Tr}(Db_1b_2) = \operatorname{Tr}(Db_1)\operatorname{Tr}(Db_2).$$

So far we have illustrated these two multivariate versions of the SVP by examples, which are going essentially along factorization properties of random variables as one knows them from classical stochastic independence or, more generally tensor independence, or free independence (see for example [NS06, Lecture 5]).

## 3.3 CLTs for Exchangeable Sequences

Our approach in this section follows [BS96, KN20] in the framework of *-algebraic probability spaces, and generalizes exchangeability to $C$-jointly exchangeable and $C$-separately exchangeable sequences for some 'color set' $C$. We introduced already in Definition 2.3.1 that a sequence $(x_n)_{n=1}^{\infty}$ is *exchangeable* if its joint moments are invariant under any permutations of the random variables.

**Definition 3.3.1.** Let $(\mathcal{A}, \varphi)$ be a *-algebraic probability space and let $C$ be a fixed non-empty set. The family of sequences $\mathcal{X} \equiv \{(x_{c,n})_{n=1}^{\infty} \mid c \in C\} \subset \mathcal{A}$ is said to be

(i) *$C$-jointly exchangeable* if, for any $n \in \mathbb{N}$, for every $\boldsymbol{\varepsilon}\colon [n] \to \{1, *\}$, $\mathbf{i}, \mathbf{j}\colon [n] \to \mathbb{N}$, and 'color' function $\mathbf{t}\colon [n] \to C$,

$$\varphi\Big(x_{\mathbf{t}(1),\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\Big) = \varphi\Big(x_{\mathbf{t}(1),\mathbf{j}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{t}(n),\mathbf{j}(n)}^{\boldsymbol{\varepsilon}(n)}\Big)$$

whenever $\mathbf{i} \sim \mathbf{j}$;

(ii) *$C$-separately exchangeable* if, for any $n \in \mathbb{N}$, for every $\boldsymbol{\varepsilon}\colon [n] \to \{1, *\}$, $\mathbf{i}, \mathbf{j}\colon [n] \to \mathbb{N}$, and 'color' function $\mathbf{t}\colon [n] \to C$,

$$\varphi\Big(x^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),\mathbf{i}(1)} \cdots x^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),\mathbf{i}(n)}\Big) = \varphi\Big(x^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),\mathbf{j}(1)} \cdots x^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),\mathbf{j}(n)}\Big)$$

whenever $\mathbf{i}|_W \sim \mathbf{j}|_W$ for every block $W \in \ker(\mathbf{t})$.

If $C = \{c\}$, a $C$-jointly (or $C$-separately) exchangeable sequence $\mathcal{X}$ is just said to be *exchangeable*.

We will frequently refer to the set $C$ as the 'color' set, and to its elements as 'colors'.

**Lemma 3.3.2.** *Let the family $\mathcal{X}$ be given as in Definition 3.3.1 and consider the following two properties:*

*(a) $\mathcal{X}$ is $C$-jointly exchangeable;*

*(b) $\mathcal{X}$ is $C$-separately exchangeable.*

*Then one has '(b) $\Longrightarrow$ (a)', but the converse implication may fail to be true.*

*Proof.* '(b) $\Longrightarrow$ (a)': Consider the index functions $\mathbf{i}, \mathbf{j}\colon [n] \to \mathbb{N}$ and $\mathbf{t}\colon [n] \to C$, and let $W \in \mathrm{Ker}(\mathbf{t})$. We note that $\mathbf{i}|_W \sim \mathbf{j}|_W$ if and only if there exists a permutation $\tau_W \in S_\infty$ such that $\mathbf{i}|_W = \tau_W \circ (\mathbf{j}|_W)$. On the other hand, $\mathbf{i} \sim \mathbf{j}$ if and only if there exists a permutation $\tau \in S_\infty$ such that $\mathbf{i} = \tau \circ \mathbf{j}$. Since $\mathbf{i}|_W = (\tau \circ \mathbf{j})|_W = \tau \circ (\mathbf{j}|_W)$, taking $\tau_W := \tau$ for all $W \in \ker(\mathbf{t})$, we conclude that $C$-separate exchangeability implies $C$-joint exchangeability.
The failure of the converse implication '(a) $\Longrightarrow$ (b)' is manifested in Example 3.3.4. □

Let us illustrate these two notions of exchangeability.

**Example 3.3.3.** Let $C = \{1, 2\}$ and consider the two sequences $(x_{1,n})_{n=1}^\infty$ and $(x_{2,n})_{n=1}^\infty$, where we will write $x_n^{(c)}$ for $x_{n,c}$ for simplicity of notation.

1. $C$-joint exchangeability ensures only

$$\varphi(x_1^{(1)} x_1^{(2)}) = \varphi(x_k^{(1)} x_k^{(2)})$$

for all $k \in \mathbb{N}$. But $C$-separate exchangeability implies

$$\varphi(x_1^{(1)} x_1^{(2)}) = \varphi(x_k^{(1)} x_\ell^{(2)})$$

for any $k, \ell \in \mathbb{N}$.

2. $C$-joint exchangeability implies

$$\varphi\Big(x_{k_1}^{(1)}x_{\ell_1}^{(2)}x_{k_2}^{(1)}x_{\ell_2}^{(2)}\Big) = \varphi\Big(x_{\sigma(k_1)}^{(1)}x_{\sigma(\ell_1)}^{(2)}x_{\sigma(k_2)}^{(1)}x_{\sigma(\ell_2)}^{(2)}\Big)$$

for any $k_1, k_2, \ell_1, \ell_2 \in \mathbb{N}$ and any $\sigma \in S_\infty$. But $C$-separate exchangeability allows us to conclude

$$\varphi\Big(x_{k_1}^{(1)}x_{\ell_1}^{(2)}x_{k_2}^{(1)}x_{\ell_2}^{(2)}\Big) = \varphi\Big(x_{\sigma(k_1)}^{(1)}x_{\tau(\ell_1)}^{(2)}x_{\sigma(k_2)}^{(1)}x_{\tau(\ell_2)}^{(2)}\Big)$$

for any $k_1, k_2, \ell_1, \ell_2 \in \mathbb{N}$ and any $\sigma, \tau \in S_\infty$.

We illustrate next that $C$-joint exchangeability may not imply $C$-separate exchangeability.

**Example 3.3.4.** Let $(\mathcal{B}, \psi)$ be a *-algebraic probability space and consider the *-algebraic infinite tensor product probability space $(\mathcal{A}, \varphi) = \bigotimes_{n\in\mathbb{N}}(\mathcal{B}, \psi)$. Let $C = \{1, 2\}$ and $b_c \in \mathcal{B}$ be fixed for $c \in C$. Let $x_{c,n} \in \mathcal{A}$ denote the canonical embedding of $b_c \in \mathcal{B}$ into the $n$-th factor of the infinite tensor product of $\mathcal{B}$ with itself. In the following, we consider the pair of sequences $\mathcal{X} = \{(x_{c,n})_{n=1}^\infty \mid c \in C\}$ and write again $x_{c,n}$ as $x_n^{(c)}$.

1. $\mathcal{X}$ is $C$-jointly exchangeable, but may fail to be $C$-separately exchangeable. For example, one has

   $$\varphi(x_1^{(1)}x_1^{(2)}) = \psi(b_1b_2) = \varphi(x_k^{(1)}x_k^{(2)})$$

   for all $k \in \mathbb{N}$. But
   $$\varphi(x_1^{(1)}x_1^{(2)}) = \varphi(x_k^{(1)}x_\ell^{(2)})$$
   is valid for any $k, \ell \in \mathbb{N}$ if and only if $\psi(b_1b_2) = \psi(b_1)\psi(b_2)$. Actually, one can show that $\mathcal{X}$ is $C$-separately exchangeable if and only if

   $$\psi(b_1^ib_2^j) = \psi(b_1^i)\psi(b_2^j)$$

   for all $i, j \in \mathbb{N}$. This clearly fails to be the case for $b = b_1 = b_2 \in \mathcal{B}$ unless the state $\psi$ on $\mathcal{B}$ is multiplicative, i.e. $\psi(b^n) = \psi(b)^n$ for all $n \in \mathbb{N}$. (A canonical example for a unital *-algebra $\mathcal{B}$ equipped with a multiplicative state $\psi$ is $\mathcal{B} := C(\mathbb{R})$ and $\psi(f) := f(0)$.)

2. Even the additional assumption of the $C$-separate SVP for $\mathcal{X}$ is insufficient to strengthen the $C$-joint exchangeability of $\mathcal{X}$ to $C$-separate exchangeability. As before,

   $$\varphi(x_1^{(1)}x_1^{(1)}x_1^{(2)}x_1^{(2)}) = \psi(b_1b_1b_2b_2) = \varphi(x_k^{(1)}x_k^{(1)}x_k^{(2)}x_k^{(2)})$$

   for all $k \in \mathbb{N}$. But

   $$\varphi(x_1^{(1)}x_1^{(1)}x_1^{(2)}x_1^{(2)}) = \varphi(x_k^{(1)}x_k^{(1)}x_\ell^{(2)}x_\ell^{(2)})$$

is valid for any $k, \ell \in \mathbb{N}$ if and only if $\psi(b_1b_1b_2b_2) = \psi(b_1b_1)\psi(b_2b_2)$. Note that this argument does not involve the consideration of any singletons. As before, it is easy to find examples such that $\psi(b_1b_1b_2b_2) \neq \psi(b_1b_1)\psi(b_2b_2)$. This shows that the $C$-separate SVP is insufficient to strengthen $C$-joint exchangeability to $C$-separate exchangeability.

We provide next a concrete example for $C$-separate exchangeability.

**Example 3.3.5.** Let the *-algebraic probability space $(\mathcal{A}, \varphi) = \bigotimes_{n\in\mathbb{N}}(\mathcal{B}, \psi)$ and the pair of sequences $\{(x_{c,n})_{n=1}^{\infty} \mid c \in C\}$ be given as in the previous Example 3.3.4. Additionally, assume that $(\mathcal{B}, \psi) = (\mathcal{B}_1, \psi_1) \otimes (\mathcal{B}_2, \psi_2)$ for two *-algebraic probability spaces $(\mathcal{B}_i, \psi_i)$ and choose $b_1 := \widetilde{b}_1 \otimes 1_{\mathcal{B}_2}$ and $b_2 := 1_{\mathcal{B}_1} \otimes \widetilde{b}_2$ for some $\widetilde{b}_i \in \mathcal{B}_i$ $(i = 1, 2)$. Then a straightforward computation shows that the pair of sequences $\{(x_{c,n})_{n=1}^{\infty} \mid c \in C\}$ is $C$-separately exchangeable.

**Remark 3.3.6.** (1) If the 'color' set is given by $C = \mathbb{N}$, the family of sequences $\mathcal{X} \equiv \{(x_{m,n})_{n=1}^{\infty} \mid m \in \mathbb{N}\}$ is an array of random variables $\mathcal{X} \equiv (x_{m,n})_{m,n=1}^{\infty}$. Symmetries and invariance principles for arrays of random variables and, more generally, set-indexed processes are well-studied in classical probability theory, see for example [Ka92, Ka05]. Beyond sequences of random variables, there are various notions of exchangeability available for set-indexed families of random variables. In particular, there are established notions for '(joint) exchangeability' and 'separate exchangeability' for random arrays which should not be confused with our notions of '$C$-joint exchangeability' and '$C$-separate exchangeability'. The 'color' set $C$ as index set and the lower index set $\mathbb{N}$ of random variables play different roles in the context of CLTs, in contrast to the index set $\mathbb{N}^2$ in the framework of random arrays.
(2) Families of sequences of random variables have already been considered for general CLTs in the published literature. These early results did usually stipulate more general conditions than '$C$-joint exchangeability', see for example [SW94]. So far, multivariate versions of (noncommutative) CLTs under the stronger assumption of '$C$-separate exchangeability' seem to be not explicitly treated in the published literature.

The next result will be used in Theorem 3.3.10, a multivariate version of the CLT for $C$-jointly exchangeable families of random variables. Recall from Lemma 3.2.4 that the $C$-separate SVP implies the $C$-joint SVP, but the converse may fail.

**Lemma 3.3.7.** *Suppose $\mathcal{X}$ is a $C$-separately exchangeable family of sequences as stated in Definition 3.3.1. Then the following are equivalent:*

*(a) $\mathcal{X}$ satisfies the $C$-joint SVP;*

*(b) $\mathcal{X}$ satisfies the $C$-separate SVP.*

*In general, this equivalence is not valid for a $C$-jointly exchangeable family $\mathcal{X}$.*

*Proof.* Consider the function $\mathbf{i}\colon [n] \to \mathbb{N}$ with $\pi = \ker(\mathbf{i})$. Suppose $\pi$ contains the singleton $\{\widetilde{k}\}$. As $\mathbf{t}(\widetilde{k}) \in \widetilde{W}$ for some $\widetilde{W} \in \ker(\mathbf{t})$, we conclude that the restricted partition $\pi_{|\widetilde{W}}$ contains also the singleton $\{\widetilde{k}\}$. Conversely, suppose $\pi_{|\widetilde{W}}$ contains the singleton $\{\widetilde{k}\}$ for some $\widetilde{W} \in \ker(\mathbf{t})$. Now we use the $C$-separate exchangeability of $\mathcal{X}$ to conclude that there exists for any $\sigma \in S_\infty$ a function $\mathbf{j}\colon [n] \to \mathbb{N}$ with $\ker(\mathbf{j}) = \ker(\mathbf{i})$ such that $\mathbf{j}|_W = \mathbf{i}|_W$ for all $W \in \ker(\mathbf{t})\backslash\{\widetilde{W}\}$ and $\mathbf{j}|_{\widetilde{W}} = \sigma \circ \mathbf{i}|_{\widetilde{W}}$. In particular, we can choose the permutation $\sigma \in S_\infty$ such that $\ker(\mathbf{j})$ contains the singleton $\{\widetilde{k}\}$.

Altogether, these arguments show that, for some $W \in \ker(\mathbf{t})$, the function $\mathbf{i}\colon [n] \to \mathbb{N}$ has a restricted kernel set partition $\ker(\mathbf{i}|_W)$, which contains a singleton if and only if there exists a function $\mathbf{j}\colon [n] \to \mathbb{N}$ with $\varphi_{\ker(\mathbf{j}),\mathbf{t}} = \varphi_{\ker(\mathbf{i}),\mathbf{t}}$, which has a singleton in its kernel set partition $\ker(\mathbf{j})$.

'(a) $\Rightarrow$ (b)': Suppose that $\ker(\mathbf{i}|_W)$ contains a singleton for some $W \in \ker(\mathbf{t})$. By the previous arguments, we find a function $\mathbf{j}\colon [n] \to \mathbb{N}$ which contains a singleton and satisfies $\varphi_{\ker(\mathbf{j}),\mathbf{t}} = \varphi_{\ker(\mathbf{i}),\mathbf{t}}$. Thus the validity of the $C$-joint SVP implies $\varphi_{\ker(\mathbf{i}),\mathbf{t}} = \varphi_{\ker(\mathbf{j}),\mathbf{t}} = 0$.
The converse '(b) $\Rightarrow$ (a)' is ensured by Lemma 3.2.4 and, essentially, we repeat next the arguments of its proof. Suppose $\ker(\mathbf{i})$ contains a singleton. Then $\ker(\mathbf{i}|_W)$ contains a singleton for some $W \in \ker(\mathbf{t})$. Thus we conclude $\varphi_{\ker(\mathbf{i}),\mathbf{t}} = 0$ from the validity of the $C$-joint SVP.

We are left to exemplify the failure of this equivalence in the setting of a $C$-jointly exchangeable family $\mathcal{X}$. For this purpose, consider the $C$-jointly exchangeable family $\mathcal{X}$ as constructed in Example 3.3.4, when using $b = b_1 = b_2 \in \mathcal{B}$ such that $\psi(b) = 0$ and $\psi(b^2) \neq 0$. Then $\mathcal{X}$ satisfies the $C$-joint SVP, but fails to satisfy the $C$-separate SVP. $\square$

**Example 3.3.8.** Suppose $\mathcal{X}$ is $C$-separately exchangeable, and write again $x_{c,n}$ as $x_n^{(c)}$, consider the moment

$$\varphi_{\pi,\mathbf{t}} = \varphi\big(x_1^{(1)}x_2^{(2)}x_2^{(2)}x_1^{(1)}x_1^{(1)}x_1^{(2)}\big),$$

which has $\pi := \ker(\mathbf{i}) = \{\{1,4,5,6\},\{2,3\}\}$ and $\ker(\mathbf{t}) = \{\{1,4,5\},\{2,3,6\}\} =: \{W_1, W_2\}$. We conclude from the $C$-separate exchangeability that

$$\varphi\big(x_1^{(1)}x_2^{(2)}x_2^{(2)}x_1^{(1)}x_1^{(1)}x_1^{(2)}\big) = \varphi\big(x_1^{(1)}x_2^{(2)}x_2^{(2)}x_1^{(1)}x_1^{(1)}x_3^{(2)}\big),$$

where we have replaced the last factor $x_1^{(2)}$ by $x_3^{(2)}$. So we have found a function $\mathbf{j}$ such that $\varphi_{\ker(\mathbf{i}),\mathbf{t}} = \varphi_{\ker(\mathbf{j}),\mathbf{t}}$ and $\mathbf{j}|_{W_1} = \mathbf{i}|_{W_1}$, but $\mathbf{j}|_{W_2} = \sigma \circ \mathbf{i}|_{W_2}$ for some $\sigma \in S_\infty$. Note that the partition $\ker(\mathbf{j}) = \{\{1,4,5\},\{6\},\{2,3\}\}$ contains a singleton, but $\ker(\mathbf{i}) = \{\{1,4,5,6\},\{2,3\}\}$ contains no singleton.

Altogether, this example indicates how we can turn a singleton of the $W$-restricted partition $\pi_{|W} = \ker(\mathbf{i}|_W)$ into a singleton of the partition $\ker(\mathbf{j})$ while maintaining the equality $\varphi_{\ker(\mathbf{i}),\mathbf{t}} = \varphi_{\ker(\mathbf{j}),\mathbf{t}}$.

We recall from Lemma 2.1.9 that two functions $\mathbf{i}, \mathbf{j}\colon [n] \to \mathbb{N}$ satisfy $\mathbf{i} \sim \mathbf{j}$ if and only if $\ker(\mathbf{i}) = \ker(\mathbf{j})$. This ensures that the function introduced next is well-defined in the context of $C$-joint exchangeability.

**Definition 3.3.9.** Let $(\mathcal{A}, \varphi)$ be a *-algebraic probability space and let $\mathcal{X} \equiv \{(x_{c,n})_{n=1}^{\infty} \mid c \in C\} \subset \mathcal{A}$ be a $C$-jointly exchangeable family. Then the map

$$\varphi_{\bullet,\bullet,\bullet}\colon \bigsqcup_{n=1}^{\infty} \mathcal{P}(n) \times \{\mathbf{t}\colon [n] \to C\} \times \{\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}\} \to \mathbb{C},$$

given by

$$\varphi_{\pi,\mathbf{t},\boldsymbol{\varepsilon}} := \varphi\Big(x_{\mathbf{t}(1),\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\Big)$$

for any $\boldsymbol{\varepsilon}\colon [n] \to \{1, *\}$, $\mathbf{t}\colon [n] \to C$, and $\mathbf{i}\colon [n] \to \mathbb{N}$ with $\ker(\mathbf{i}) = \pi$, is called the *function on partitions associated to* $\mathcal{X}$. If the family $\mathcal{X}$ satisfies $x_{c,n} = x_{c,n}^*$ for all $c \in C$ and $n \in \mathbb{N}$, then the function on partitions associated to $\mathcal{X}$ simplifies to the map

$$\varphi_{\bullet,\bullet}\colon \bigsqcup_{n=1}^{\infty} \mathcal{P}(n) \times \{\mathbf{t}\colon [n] \to C\} \to \mathbb{C}$$

that is given by

$$\varphi_{\pi,\mathbf{t}} := \varphi\Big(x_{\mathbf{t}(1),\mathbf{i}(1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}\Big).$$

Note that, canonically identifying the function $\mathbf{t}\colon [n] \to C$ and the $n$-tuple $(\mathbf{t}(1), \ldots, \mathbf{t}(n)) \in C^n$, the function on partitions associated to $\mathcal{X}$ can be also addressed as

$$\varphi_{\bullet,\bullet}\colon \bigsqcup_{n=1}^{\infty} \mathcal{P}(n) \times C^n \times \{\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}\} \to \mathbb{C}.$$

We are ready to formulate a multivariate CLT for $C$-jointly exchangeable families of random variables.

**Theorem 3.3.10.** *Let $(\mathcal{A}, \varphi)$ be a *-algebraic probability space and suppose the family $\mathcal{X} \equiv \{(x_{c,n})_{n=1}^{\infty} \mid c \in C\} \subset \mathcal{A}$ satisfies the following conditions:*

*(i) $\mathcal{X}$ is $C$-jointly exchangeable;*

*(ii) $\mathcal{X}$ satisfies the $C$-joint SVP.*

*Then one has for*

$$S_{c,N} := \frac{x_{c,1} + \cdots + x_{c,N}}{\sqrt{N}}$$

*that, for all $n \in \mathbb{N}$, $\boldsymbol{\varepsilon}\colon [n] \to \{1, *\}$, and $\mathbf{t}\colon [n] \to C$,*

$$\lim_{N\to\infty} \varphi(S_{\mathbf{t}(1).N}^{\boldsymbol{\varepsilon}(1)} \cdots S_{\mathbf{t}(n),N}^{\boldsymbol{\varepsilon}(n)}) = \sum_{\pi \in \mathcal{P}_2(n)} \varphi_{\pi,\mathbf{t},\boldsymbol{\varepsilon}}.$$

*Moreover, if $\mathcal{X}$ is $C$-separately exchangeable or enjoys the $C$-separate SVP then for all $n \in \mathbb{N}$, $\boldsymbol{\varepsilon}\colon [n] \to \{1,*\}$, and $\mathbf{t}\colon [n] \to C$,*

$$\lim_{N\to\infty} \varphi(S_{\mathbf{t}(1),N}^{\boldsymbol{\varepsilon}(1)} \cdots S_{\mathbf{t}(n),N}^{\boldsymbol{\varepsilon}(n)}) = \sum_{\substack{\pi\in\mathcal{P}_2(n)\\ \pi\le\ker(\mathbf{t})}} \varphi_{\pi,\mathbf{t},\boldsymbol{\varepsilon}}.$$

**Remark.** In principle, we could define $\mathcal{P}_2(n,\mathbf{t}) := \{\pi \in \mathcal{P}_2(n) \mid \pi \le \ker(\mathbf{t})\}$. This definition could be further specified to include a direction map: $\mathcal{P}_2(n,\mathbf{t},\boldsymbol{\varepsilon}) := \{\pi \in \mathcal{P}_2(n,\mathbf{t}) \mid \boldsymbol{\varepsilon}(\min V) \neq \boldsymbol{\varepsilon}(\max V)$ for all $V \in \pi\}$.

Another version of this theorem will be provided below as Theorem 3.3.13, specifically to the context of $C$-separate exchangeability and $C$-separate SVP, where $\varphi_{\bullet,\bullet,\bullet}$, the function of partitions for $\mathcal{X}$, is replaced by $\varphi_{\bullet_|,\bullet,\bullet}$, a certain function of Cartesian products of restricted partitions of $\mathcal{X}$ (see Definition 3.3.12).

*Proof.* We need to calculate, for some fixed $\mathbf{t}\colon [n] \to \mathbb{N}$ and $\boldsymbol{\varepsilon}\colon [n] \to \{*,1\}$,

$$\begin{aligned}\varphi\big(S_{\mathbf{t}(1),N}^{\boldsymbol{\varepsilon}(1)} \cdots S_{\mathbf{t}(n),N}^{\boldsymbol{\varepsilon}(n)}\big) &= \frac{1}{N^{n/2}} \sum_{\mathbf{i}\colon [n]\to[N]} \varphi\big(x_{\mathbf{t}(1),\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\big)\\ &= \frac{1}{N^{n/2}} \sum_{\pi\in\mathcal{P}(n)} A_{|\pi|}^{(N)} \varphi_{\pi,\mathbf{t},\boldsymbol{\varepsilon}}\end{aligned}$$

for $N \to \infty$. Here we have used that $A_{|\pi|}^{(N)}$ is the cardinality of the set $\{\mathbf{i}\colon [n] \to [N] \mid \ker(\mathbf{i}) = \pi\}$ for a partition $\pi \in \mathcal{P}(n)$, or more explicitly:

$$A_{|\pi|}^{(N)} = N\cdot(N-1)\cdots(N-|\pi|+1) = \binom{N}{|\pi|}|\pi|!.$$

Now the $C$-joint SVP implies $\varphi_{\pi,\mathbf{t},\boldsymbol{\varepsilon}} = 0$ for any $\pi \in \mathcal{P}(n)$ which contains a singleton. Moreover, one has

$$\lim_{N\to\infty} \frac{1}{N^{n/2}} A_{|\pi|}^{(N)} = 0$$

for any $\pi \in \mathcal{P}(n)$ with $|\pi| < n/2$, based on the combinatorial arguments already deployed in the proof of Theorem 3.1.1. Thus one arrives at

$$\lim_{N\to\infty} \varphi\big(S_{\mathbf{t}(1),N}^{\boldsymbol{\varepsilon}(1)} \cdots S_{\mathbf{t}(n),N}^{\boldsymbol{\varepsilon}(n)}\big) = \sum_{\pi\in\mathcal{P}_2(n)} \varphi_{\pi,\mathbf{t},\boldsymbol{\varepsilon}}.$$

We are left to show that the $C$-separate exchangeability or the $C$-separate SVP of $\mathcal{X}$ allows us to restrict the summation further to pair partitions $\pi \in \mathcal{P}_2(n)$ satisfying $\pi \le \ker(\mathbf{t})$. We know from Lemma 3.3.7 that $C$-separate exchangeability implies the equivalence of the $C$-joint SVP and the $C$-separate SVP. Thus it suffices to show that the strengthening of $C$-joint SVP to $C$-separate SVP implies the vanishing of these further pair partitions. For this purpose, consider some $\mathbf{i}\colon [n] \to [N]$ with $\pi = \ker(\mathbf{i}) \in \mathcal{P}_2(n)$ and let $W \in \ker(\mathbf{t})$. Then $\ker(\mathbf{i}|_W) \in \mathcal{P}(W)$ has blocks of size 2 at most. Now the $C$-separate SVP implies that only summands of those partitions $\pi = \ker(\mathbf{i})$ contribute where $\ker(\mathbf{i}|_W) \in \mathcal{P}(W)$ is a pair partition. As this argument is valid for any block $W \in \ker(\mathbf{t})$, it follows that only those summands contribute for which $\pi = \ker(\mathbf{i}) \in \mathcal{P}_2(n)$ satisfies $\pi \leq \ker(\mathbf{t})$. $\square$

We had introduced for a $C$-jointly exchangeable family $\mathcal{X}$ a function on partitions associated to $\mathcal{X}$ in Definition 3.3.9. Next we provide a definition of such a function as appropriate for a $C$-separately exchangeable family $\mathcal{X}$. Recall for this purpose the notion of a restricted partition from Definition 2.1.6.

**Notation 3.3.11.** Let the 'color' function $\mathbf{t}\colon [n] \to C$ be fixed. We denote by

$$\prod_{W \in \ker(\mathbf{t})} \mathcal{P}(W) = \Big\{ f\colon \ker(\mathbf{t}) \mapsto \bigsqcup_{W \in \ker(\mathbf{t})} \mathcal{P}(W) \Big\}$$

the Cartesian product of the partitions of the blocks of $\ker(\mathbf{t})$ with coordinate projections $\ker(\mathbf{t}) \ni W \mapsto f(W) \in \mathcal{P}(W)$. Similarly, let $\prod_{W \in \ker(\mathbf{t})} \mathcal{P}_2(W)$ denote the Cartesian product of the pair partitions of the blocks of $\ker(\mathbf{t})$.

For a given partition $\pi \in \mathcal{P}(n)$, we denote by

$$\pi_{|\bullet} \in \prod_{W \in \ker(\mathbf{t})} \mathcal{P}(W)$$

the function which has, for each $W \in \ker(\mathbf{t})$, the $W$-restriction $\pi_{|W}$ as a coordinate. Furthermore, we will write

$$\pi_{|\bullet} \in \prod_{W \in \ker(\mathbf{t})} \mathcal{P}_2(W)$$

if $\pi_{|W} \in \mathcal{P}_2(W)$ for all $W \in \ker(\mathbf{t})$.

Similar to the situation of $C$-joint exchangeability, we infer from Lemma 2.1.9 again that two functions $\mathbf{i}, \mathbf{j}\colon [n] \to \mathbb{N}$ satisfy $\mathbf{i}|_W \sim \mathbf{j}|_W$ for all $W \in \ker(\mathbf{t})$ if and only if $\ker(\mathbf{i}|_W) = \ker(\mathbf{j}|_W)$ for all $W \in \ker(\mathbf{t})$. This ensures that the function introduced next is well-defined in the context of $C$-separate exchangeability.

**Definition 3.3.12.** Let $(\mathcal{A}, \varphi)$ be a *-algebraic probability space and let $\mathcal{X} \equiv \{(x_{c,n})_{n=1}^{\infty} \mid c \in C\} \subset \mathcal{A}$ be a $C$-separately exchangeable family. Then the map

$$\varphi_{\bullet_|,\bullet,\bullet}\colon \bigsqcup_{n=1}^{\infty} \Big( \prod_{W \in \ker(\mathbf{t})} \mathcal{P}(W) \Big) \times \{\mathbf{t}\colon [n] \to C\} \times \{\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}\} \to \mathbb{C},$$

given by

$$\varphi_{\pi_|,\mathbf{t},\boldsymbol{\varepsilon}} := \varphi\Big( x_{\mathbf{t}(1),\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)} \Big)$$

for any $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$, $\mathbf{t}\colon [n] \to C$, $\mathbf{i}\colon [n] \to \mathbb{N}$ with $\ker(\mathbf{i}) = \pi$ and $\ker(\mathbf{i}|_W) = \pi_{|W} \in \mathcal{P}(W)$ for any $W \in \ker \mathbf{t}$, is called the *function on reduced partitions associated to* $\mathcal{X}$. If the family $\mathcal{X}$ satisfies $x_{c,n} = x^*_{c,n}$ for all $c \in \mathbb{C}$ and $n \in \mathbb{N}$, then the function on partitions associated to $\mathcal{X}$ simplifies to the map

$$\varphi_{\bullet_{|},\bullet}\colon \bigsqcup_{n=1}^{\infty} \Big( \prod_{W \in \ker(\mathbf{t})} \mathcal{P}(W) \Big) \times \{\mathbf{t}\colon [n] \to C\} \to \mathbb{C}$$

that is given by

$$\varphi_{\pi_{|},\mathbf{t}} := \varphi\Big( x_{\mathbf{t}(1),\mathbf{i}(1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)} \Big).$$

**Theorem 3.3.13.** *Let $(\mathcal{A}, \varphi)$ be a *-algebraic probability space and suppose the the family of sequences $\mathcal{X} \equiv \{(x_{c,n})_{n=1}^{\infty} \mid c \in C\} \subset \mathcal{A}$ satisfies:*

*(i) $\mathcal{X}$ is $C$-separately exchangeable;*

*(ii) $\mathcal{X}$ satisfies the $C$-separate SVP.*

*Then one has for*

$$S_{c,N} = \frac{x_{c,1} + \cdots + x_{c,N}}{\sqrt{N}}$$

*that, for all $n \in \mathbb{N}$, $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$, and $\mathbf{t}\colon [n] \to C$,*

$$\lim_{N\to\infty} \varphi(S^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),N} \cdots S^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),N}) = \sum_{\substack{\pi \in \mathcal{P}_2(n) \\ \pi \leq \ker(\mathbf{t})}} \varphi_{\pi,\mathbf{t},\boldsymbol{\varepsilon}} = \sum_{\pi_{|\bullet} \in \prod_{W \in \ker(\mathbf{t})} \mathcal{P}_2(W)} \varphi_{\pi_{|},\mathbf{t},\boldsymbol{\varepsilon}}.$$

*Proof.* As Theorem 3.3.10 applies for a family $\mathcal{X}$ which is $C$-separately exchangeable and satisfies the $C$-separate SVP, we are left to verify that

$$\sum_{\substack{\pi \in \mathcal{P}_2(n) \\ \pi \leq \ker(\mathbf{t})}} \varphi_{\pi,\mathbf{t},\boldsymbol{\varepsilon}} = \sum_{\pi_{|\bullet} \in \prod_{W \in \ker(\mathbf{t})} \mathcal{P}_2(W)} \varphi_{\pi_{|},\mathbf{t},\boldsymbol{\varepsilon}}.$$

But this is immediate from the following two observations. First of all, one has for a $C$-separately exchangeable family $\mathcal{X}$ that

$$\varphi_{\pi,\mathbf{t},\boldsymbol{\varepsilon}} = \varphi\Big( x^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),\mathbf{i}(1)} \cdots x^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),\mathbf{i}(n)} \Big) = \varphi_{\pi_{|},\mathbf{t},\boldsymbol{\varepsilon}}$$

for any $\pi \in \mathcal{P}(n)$, $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$, and $\mathbf{t}\colon [n] \to C$. Secondly, the map

$$\big\{ \widetilde{\pi} \in \mathcal{P}_2(n) \big| \widetilde{\pi} \leq \ker(\mathbf{t}) \big\} \ni \pi \mapsto \pi_{|\bullet} \in \prod_{W \in \ker(\mathbf{t})} \mathcal{P}_2(W) \tag{3.7}$$

is bijective. Note that this map is well-defined as the condition $\widetilde{\pi} \leq \ker(\mathbf{t})$ implies that each pair block $\widetilde{V} \in \widetilde{\pi}$ satisfies $\widetilde{V} \in \widetilde{\pi}_{|W}$ for some $W \in \ker(\mathbf{t})$, and thus $\widetilde{\pi}_{|W} \in \mathcal{P}_2(W)$ for all $W \in \ker(\mathbf{t})$.

We verify next that the map defined by (3.7) is injective. Suppose $\pi, \widetilde{\pi} \in \{\pi \in \mathcal{P}_2(n) \big| \pi \leq \ker(\mathbf{t})\}$ satisfy $\pi_{|_\bullet} = \widetilde{\pi}_{|_\bullet}$. We conclude from $\pi_{|W} = \widetilde{\pi}_{|W}$ for all 'coordinates' $W \in \ker(\mathbf{t})$ that $V \in \pi$ if and only if $V \in \widetilde{\pi}$. Thus one has $\pi = \widetilde{\pi}$ which ensures the injectivity of the map.

Finally, we need to verify the surjectivity of the map defined by (3.7). The function $f \in \prod_{W \in \ker(\mathbf{t})} \mathcal{P}_2(W)$ has the coordinates $f(W) \in \mathcal{P}_2(W)$ such that $f(W) \cap f(\widetilde{W}) = \emptyset$ for $W \neq \widetilde{W}$ and $\bigcup_{W \in \ker(\mathbf{t})} f(W) = \mathcal{P}(n)$. Thus $\pi := \bigcup_{W \in \ker(\mathbf{t})} f(W)$ defines a partition in $\mathcal{P}(n)$ with $\pi \leq \ker(\mathbf{t})$ which is mapped to $f$. This ensures the surjectivity of the map. $\square$

We address next how multivariate settings for exchangeable sequences can be constructed from a single exchangeable sequence. We illustrate such a procedure in the following, in particular to show that this procedure yields separate exchangeability.

**Lemma 3.3.14.** *Let $C$ be a countable set and let $J\colon C \times \mathbb{N} \to \mathbb{N}$ be an injective function. Given the exchangeable sequence $\mathbf{x} \equiv (x_n)_{n=1}^{\infty} \subset \mathcal{A}$ and putting*

$$x_{c,n} := x_{J(c,n)},$$

*the family of sequences $\mathcal{X} \equiv \{(x_{c,n})_{n=1}^{\infty} \mid c \in C\} \subset \mathcal{A}$ is $C$-separately exchangeable.*

*Proof.* We need to show that, for any $\mathbf{t}\colon [n] \to C$, $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$,

$$\varphi\Big(x_{\mathbf{t}(1),\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\Big) = \varphi\Big(x_{\mathbf{t}(1),\mathbf{j}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{t}(n),\mathbf{j}(n)}^{\boldsymbol{\varepsilon}(n)}\Big)$$

whenever $\mathbf{i}|_W \sim \mathbf{j}|_W$ for every block $W \in \ker(\mathbf{t})$. Note that $\mathbf{i}|_W \sim \mathbf{j}|_W$ if and only if there exists some permutation $\tau_W$ such that $\mathbf{j}|_W = \tau_W \circ (\mathbf{i}|_W)$. So we need to show for

$$\varphi\Big(x_{\mathbf{t}(1),\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\Big) = \varphi\big(x_{J(\mathbf{t}(1),\mathbf{i}(1))}^{\boldsymbol{\varepsilon}(1)} \cdots x_{J(\mathbf{t}(n),\mathbf{i}(n))}^{\boldsymbol{\varepsilon}(n)}\big)$$

that a factor $x_{J(c,\mathbf{i}(w))}$ with $w \in W$ and $\mathbf{t}(w) = c$ can be replaced by a factor $x_{J(c,\tau_W \circ \mathbf{i}(w))}$ without altering the value of the joint moment. But this amounts to showing that $\mathbf{i}(w)$ can be replaced by $\tau_W \circ \mathbf{i}(w)$ without changing the joint moment. This corresponds to replacing $J(\mathbf{t}(w), \mathbf{i}(w))$ by $J(\mathbf{t}(w), \tau_W \circ \mathbf{i}(w))$. Since the function $J$ is injective, we always find a permutation $\sigma_W$ such that $J(c, \tau_W \circ \mathbf{i}(w)) = \sigma_W \circ J(c, \mathbf{i}(w))$ for all $w \in W$. Now we conclude from the exchangeability of the underlying sequence $\mathbf{x}$ that

$$\varphi\big(x_{J(\mathbf{t}(1),\mathbf{i}(1))}^{\boldsymbol{\varepsilon}(1)} \cdots x_{J(\mathbf{t}(n),\mathbf{i}(n))}^{\boldsymbol{\varepsilon}(n)}\big) = \varphi\big(x_{J(\mathbf{t}(1),\tilde{\mathbf{j}}(1))}^{\boldsymbol{\varepsilon}(1)} \cdots x_{J(\mathbf{t}(n),\tilde{\mathbf{j}}(n))}^{\boldsymbol{\varepsilon}(n)}\big)$$

where we have $\tilde{\mathbf{j}}|_{[n]\setminus W} = \mathbf{i}|_{[n]\setminus W}$ and $\tilde{\mathbf{j}}|_W = \tau_W \circ \mathbf{i}|_W$. A finite iteration on the blocks of $\ker(\mathbf{t})$ yields

$$\varphi\big(x_{J(\mathbf{t}(1),\mathbf{i}(1))}^{\boldsymbol{\varepsilon}(1)} \cdots x_{J(\mathbf{t}(n),\mathbf{i}(n))}^{\boldsymbol{\varepsilon}(n)}\big) = \varphi\big(x_{J(\mathbf{t}(1),\mathbf{j}(1))}^{\boldsymbol{\varepsilon}(1)} \cdots x_{J(\mathbf{t}(n),\mathbf{j}(n))}^{\boldsymbol{\varepsilon}(n)}\big)$$

whenever $\mathbf{j}|_W = \tau_W \circ (\mathbf{i}|_W)$ for some $\tau_W \in S_\infty$. $\square$

Since $C$-separate exchangeability implies $C$-joint exchangeability, the following result is actually a corollary to the previous result. Nevertheless, we state it here and give an explicit proof without appealing to $C$-separate exchangeability.

**Lemma 3.3.15.** *Let $C$ be a countable set and let $J\colon C \times \mathbb{N} \to \mathbb{N}$ be an injective function. Given the exchangeable sequence $\mathbf{x} \equiv (x_n)_{n=1}^{\infty} \subset \mathcal{A}$ and putting*

$$x_{c,n} := x_{J(c,n)},$$

*the family of sequences $\mathcal{X} \equiv \{(x_{c,n})_{n=1}^{\infty} \mid c \in C\} \subset \mathcal{A}$ is $C$-jointly exchangeable.*

*Proof.* We need to show that

$$\varphi\Big(x^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),\mathbf{i}(1)} \cdots x^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),\mathbf{i}(n)}\Big) = \varphi\Big(x^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),\mathbf{j}(1)} \cdots x^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),\mathbf{j}(n)}\Big)$$

whenever $\mathbf{i} \sim \mathbf{j}$, or that, equivalently by Lemma 2.1.9,

$$\varphi\Big(x^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),\mathbf{i}(1)} \cdots x^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),\mathbf{i}(n)}\Big) = \varphi\Big(x^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),\tau\circ\mathbf{i}(1)} \cdots x^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),\tau\circ\mathbf{i}(n)}\Big)$$

for all $\tau \in S_\infty$. Using the definition of the $x_{c,n}$'s, the latter translates to verify

$$\varphi\big(x^{\boldsymbol{\varepsilon}(1)}_{J(\mathbf{t}(1),\mathbf{i}(1))} \cdots x^{\boldsymbol{\varepsilon}(n)}_{J(\mathbf{t}(n),\mathbf{i}(n))}\big) = \varphi\big(x^{\boldsymbol{\varepsilon}(1)}_{J(\mathbf{t}(1),\tau\circ\mathbf{i}(1))} \cdots x^{\boldsymbol{\varepsilon}(n)}_{J(\mathbf{t}(n),\tau\circ\mathbf{i}(n))}\big)$$

for all $\tau \in S_\infty$. But this amounts to showing for the index functions $J(\mathbf{t},\mathbf{i})\colon [n] \to \mathbb{N}$ and $J(\mathbf{t},\tau\circ\mathbf{i})\colon [n] \to \mathbb{N}$ (for any fixed $\tau \in S_\infty$) that the two kernel set partitions

$$\ker(J(\mathbf{t},\mathbf{i})) \in \mathcal{P}(n) \quad \text{and} \quad \ker(J(\mathbf{t},\tau\circ\mathbf{i})) \in \mathcal{P}(n)$$

are the same. For this purpose, consider a block $V \in \ker(J(\mathbf{t},\mathbf{i}))$. Then $\mathbf{i}(V) \subset \mathbb{N}$ and $\tau \circ \mathbf{i}(V) \subset \mathbb{N}$ have the same cardinality. Thus $A := \{(\mathbf{t}(v), \mathbf{i}(v)) \mid v \in V\} \subset [d] \times \mathbb{N}$ and $A_\tau := \{(\mathbf{t}(v), \tau\circ\mathbf{i}(v)) \mid v \in V\} \subset [d] \times \mathbb{N}$ have the same cardinality. Consequently, by the injectivity of $J$, the two sets $J(A) \subset \mathbb{N}$ and $J(A_\tau) \subset \mathbb{N}$ have the same cardinality. Now our assumption on $V$ (being a level set of the function $J(\mathbf{t},\mathbf{i})\colon [n] \to \mathbb{N}$) ensures that the set $J(A)$ is a singleton set. But the previous argument forces then that also $J(A_\tau)$ is a singleton set. Consequently, $V$ is also a level set of the function $J(\mathbf{t},\tau\circ\mathbf{i})\colon [n] \to \mathbb{N}$. In other words, we have concluded that $V \subset V_\sigma \in \ker(J(\mathbf{t},\tau\circ\mathbf{i})$. The converse inclusion $V_\sigma \subset V \in \ker(J(\mathbf{t},\mathbf{i})$ follows by the same arguments, reversing now the roles of $\mathbf{i}$ and $\tau\circ\mathbf{i}$, such that $\tau^{-1} \circ (\tau \circ \mathbf{i}) = \mathbf{i}$. Altogether, this ensures $\ker(J(\mathbf{t},\mathbf{i})) = \ker(J(\mathbf{t},\tau\circ\mathbf{i}))$. □

**Example 3.3.16.** We have provided above a procedure which turns a single exchangeable sequence into a $C$-separate exchangeable tuple of sequences. This was done with the help of an injective function $J\colon C \times \mathbb{N} \to \mathbb{N}$. Choosing for the 'color' set $C = [d]$, a particular choice for this function is

$$J(t,n) = (n-1)d + t \tag{3.8}$$

which yields for the ansatz of the multivariate CLT:

$$\begin{aligned} S_{1,N} &= \frac{1}{\sqrt{N}}(x_1 + x_{d+1} + \ldots + x_{(N-1)d+1}), \\ S_{2,N} &= \frac{1}{\sqrt{N}}(x_2 + x_{d+2} + \ldots + x_{(N-1)d+2}), \\ &\vdots \\ S_{d,N} &= \frac{1}{\sqrt{N}}(x_d + x_{2d} + \ldots + x_{Nd}). \end{aligned}$$

**Remark 3.3.17.** Given the single exchangeable sequence $\mathbf{x} \equiv (x_n)_{n=1}^{\infty} \subset \mathcal{A}$, another possible choice for the ansatz of a multivariate CLT is

$$\widetilde{S}_{t,N} = \frac{1}{\sqrt{N}}(x_{(t-1)N+1} + \ldots + x_{(t-1)N+N}).$$

Here we will have to replace the role of the single injective function $J\colon [d] \times \mathbb{N} \to \mathbb{N}$ in Lemma 3.3.14 (and Lemma 3.3.15) by the family of injective functions $\{J_M\}_M\colon [d] \times [M] \to [dM]$, where we have

$$J_M(t,n) = (t-1)M + n \qquad (t \in [d], n \in [M]).$$

Adapting the notions of $C$-joint exchangeability and $C$-separate exchangeability to families of finite sequences, one can show as before that, for each fixed $M \in \mathbb{N}$, the $d$-tuple of finite sequences $(\widetilde{x}_{1,n})_{n=1}^{M}, \ldots (\widetilde{x}_{d,n})_{n=1}^{M}$, given by

$$\widetilde{x}_{t,n} := x_{J_M(t,n)},$$

is $[d]$-separately exchangeable, i.e. one has

$$\varphi\Big(\widetilde{x}_{\mathbf{t}(1),\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots \widetilde{x}_{\mathbf{t}(n),\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\Big) = \varphi\Big(\widetilde{x}_{\mathbf{t}(1),\mathbf{j}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots \widetilde{x}_{\mathbf{t}(n),\mathbf{j}(n)}^{\boldsymbol{\varepsilon}(n)}\Big)$$

for any $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$, any $\mathbf{i}, \mathbf{j}\colon [n] \to [M]$ whenever $\mathbf{i}|_W \sim \mathbf{j}|_W$ for every block $W \in \ker(\mathbf{t})$. Similarly, one obtains $[d]$-joint exchangeability for this $d$-tuple of finite sequences.

The multivariate CLT for this choice of $[d]$-separately exchangeable sequences is again as before. To be more precise, one obtains for any $n \in \mathbb{N}$

$$\lim_{N\to\infty} \varphi\big(\widetilde{S}_{\mathbf{t}(1),N}^{\boldsymbol{\varepsilon}(1)} \cdots \widetilde{S}_{\mathbf{t}(n),N}^{\boldsymbol{\varepsilon}(n)}\big) = \sum_{\substack{\pi \in \mathcal{P}_2(n) \\ \pi \le \ker(\mathbf{t})}} \varphi_{\pi,\mathbf{t},\boldsymbol{\varepsilon}},$$

where

$$\varphi_{\pi,\mathbf{t},\boldsymbol{\varepsilon}} = \varphi\Big(\widetilde{x}_{\mathbf{t}(1),\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots \widetilde{x}_{\mathbf{t}(n),\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\Big)$$

is well-defined for any $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$, any $\mathbf{i}\colon [n] \to [n]$ with $\pi = \ker(\mathbf{i})$.

Note that there exists a permutation $\sigma \in S_\infty$ such that $\sigma(i) = i$ for $n > dM$ and $J_M(t,i) = \sigma(J(t,i))$ for all $t \in [d]$ and all $i \in [n]$, where the function $J\colon [d] \times \mathbb{N} \to \mathbb{N}$ is as in (3.8), and thus $x_{t,i} = x_{J(t,i)}$. Due to the exchangeability of the underlying sequence $\mathbf{x}$, this implies

$$\varphi\Big(\widetilde{x}^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),\mathbf{i}(1)} \cdots \widetilde{x}^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),\mathbf{i}(n)}\Big) = \varphi\Big(x^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),\mathbf{i}(1)} \cdots x^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),\mathbf{i}(n)}\Big)$$

for all $\mathbf{i}\colon [n] \to [n]$, $\mathbf{t}\colon [n] \to [d]$, and $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$. We illustrate this for $d = 3$ and $n = 6$ by the following example, where we write $x^{(c)}_n$ for $x_{c,n}$ and $\widetilde{x}^{(c)}_n$ for $\widetilde{x}_{c,n}$ for notational convenience. Let

$$\begin{array}{llllll}
\widetilde{x}^{(1)}_1 = x_1, & \widetilde{x}^{(2)}_1 = x_7, & \widetilde{x}^{(3)}_1 = x_{13}, & x^{(1)}_1 = x_1, & x^{(2)}_1 = x_2, & x^{(3)}_1 = x_3,\\
\widetilde{x}^{(1)}_2 = x_2, & \widetilde{x}^{(2)}_2 = x_8, & \widetilde{x}^{(3)}_2 = x_{14}, & x^{(1)}_2 = x_4, & x^{(2)}_2 = x_5, & x^{(3)}_2 = x_6,\\
\widetilde{x}^{(1)}_3 = x_3, & \widetilde{x}^{(2)}_3 = x_9, & \widetilde{x}^{(3)}_3 = x_{15}, & x^{(1)}_3 = x_7, & x^{(2)}_3 = x_8, & x^{(3)}_3 = x_9,\\
\widetilde{x}^{(1)}_4 = x_4, & \widetilde{x}^{(2)}_4 = x_{10}, & \widetilde{x}^{(3)}_4 = x_{16}, & x^{(1)}_4 = x_{10}, & x^{(2)}_4 = x_{11}, & x^{(3)}_4 = x_{12},\\
\widetilde{x}^{(1)}_5 = x_5, & \widetilde{x}^{(2)}_5 = x_{11}, & \widetilde{x}^{(3)}_5 = x_{17}, & x^{(1)}_5 = x_{13}, & x^{(2)}_5 = x_{14}, & x^{(3)}_5 = x_{15},\\
\widetilde{x}^{(1)}_6 = x_6, & \widetilde{x}^{(2)}_6 = x_{12}, & \widetilde{x}^{(3)}_6 = x_{18}, & x^{(1)}_6 = x_{16}, & x^{(2)}_6 = x_{17}, & x^{(3)}_6 = x_{18}.
\end{array}$$

One easily verifies

$$\begin{aligned}
\varphi\Big(\widetilde{x}^{(2)}_3 \widetilde{x}^{(1)}_1 \widetilde{x}^{(2)}_3 \widetilde{x}^{(1)}_1 \widetilde{x}^{(3)}_5 \widetilde{x}^{(3)}_3\Big) &= \varphi(x_9 x_1 x_9 x_1 x_{17} x_{15})\\
&= \varphi(x_8 x_1 x_8 x_1 x_{15} x_9) = \varphi\Big(x^{(2)}_3 x^{(1)}_1 x^{(2)}_3 x^{(1)}_1 x^{(3)}_5 x^{(3)}_3\Big).
\end{aligned}$$

Here we have used exchangeability for the second equality.

## 3.4 CLTs for Spreadable Sequences

We arrive at the principle objective of this chapter, which is reviewing and deeply investigating the limit distribution of CLT for spreadable sequences as $\mathbb{N} \to \infty$ by adjusting alternative definitions. Similar to distributional invariance principles for arrays of random variables, there are different notions available for spreadability when considering families of sequences of random variables. Here we will focus on two notions of spreadability, called '$C$-joint spreadability' and '$C$-separate spreadability', and provide the corresponding multivariate CLTs. Our approach in this section is in the framework of *-algebraic probability spaces.

**Definition 3.4.1.** Let $(\mathcal{A}, \varphi)$ be a *-algebraic probability space and let $C$ be a fixed set. The family of sequences $\mathcal{X} \equiv \{(x_{c,n})_{n=1}^\infty \mid c \in C\} \subset \mathcal{A}$ is said to be

(i) *$C$-jointly spreadable* if, for any $n \in \mathbb{N}$ and every $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$ and $\mathbf{t}\colon [n] \to C$,

$$\varphi\Big(x^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),\mathbf{i}(1)} \cdots x^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),\mathbf{i}(n)}\Big) = \varphi\Big(x^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),\mathbf{j}(1)} \cdots x^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),\mathbf{j}(n)}\Big)$$

whenever $\mathbf{i} \sim_{\mathcal{O}} \mathbf{j}$;

(ii) *$C$-separately spreadable* if, for any $n \in \mathbb{N}$ and every $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$ and $\mathbf{t}\colon [n] \to C$,

$$\varphi\Big(x^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),\mathbf{i}(1)} \cdots x^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),\mathbf{i}(n)}\Big) = \varphi\Big(x^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),\mathbf{j}(1)} \cdots x^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),\mathbf{j}(n)}\Big)$$

whenever $\mathbf{i}|_W \sim_{\mathcal{O}} \mathbf{j}|_W$ for every block $W \in \ker(\mathbf{t})$.

If $C = \{c\}$, a $C$-jointly ( or $C$-separately) spreadable sequence $\mathcal{X}$ is just said to be *spreadable.*

As already done in the context of exchangeability, we will also refer to the set $C$ as the 'color' set, and to its elements as 'colors'. $C$-separate spreadability means that one has spreadability separately for each 'color'. We will see next that $C$-joint spreadability is weaker than $C$-separate spreadability as it may not allow us to spread lower indices separately for each color.

**Lemma 3.4.2.** *Let the family $\mathcal{X}$ be given as in Definition 3.4.1 and consider the following two properties:*

*(a) $\mathcal{X}$ is $C$-jointly spreadable;*

*(b) $\mathcal{X}$ is $C$-separately spreadable.*

*Then one has '(b) $\Longrightarrow$ (a)', but the converse implication may fail to be true.*

*Proof.* '(b) $\Longrightarrow$ (a)': Consider the two index functions $\mathbf{i}, \mathbf{j}\colon [n] \to \mathbb{N}$ and the 'color' function $\mathbf{t}\colon [n] \to C$, and let $W \in \mathrm{Ker}(\mathbf{t})$. By Lemma 2.1.21, $C$-joint spreadability ensures $\tau \circ \mathbf{i} = \sigma \circ \mathbf{j}$ for some permutations $\sigma, \tau \in S_\infty$ with order preserving restrictions $\sigma|_{\mathrm{Ran}\,\mathbf{j}}$ and $\tau|_{\mathrm{Ran}\,\mathbf{i}}$. We consider next the restrictions $\mathbf{i}|_W$ and $\mathbf{j}|_W$ for $W \in \ker(\mathbf{t})$. Putting $\sigma_W := \sigma$ and $\tau_W := \tau$ for all $W \in \ker(\mathbf{t})$, we conclude that $\tau_W \circ \mathbf{i}|_W = \sigma_W \circ (\mathbf{j}|_W)$. Furthermore, $\tau_W|_{\mathrm{Ran}\,\mathbf{i}|_W}$ and $\sigma_W|_{\mathrm{Ran}\,\mathbf{j}|_W}$ are order preserving. Thus $C$-joint spreadability can be modelled as a special case of $C$-separate spreadability.
'(a) $\not\Longrightarrow$ (b)': $C$-joint/separate exchangeability implies $C$-joint/separate spreadability, respectively. Now an inspection of Example 3.3.4 shows that all arguments transfer immediately from the setting of exchangeability to that of spreadability. Thus Example 3.3.4 establishes the failure of the converse implication. ☐

We illustrate next these two notions of spreadability.

**Example 3.4.3.** Let $C = \{1, 2\}$ and consider the two sequences $(x_{1,n})_{n=1}^\infty$ and $(x_{2,n})_{n=1}^\infty$. Writing $x_{c,n}$ as $x_n^{(c)}$ for notational simplicity, $C$-separate spreadability implies

$$\varphi\Big(x^{(1)}_{k_1} x^{(2)}_{\ell_1} x^{(1)}_{k_2} x^{(2)}_{\ell_2}\Big) = \varphi\Big(x^{(1)}_{k_1'} x^{(2)}_{\ell_1'} x^{(1)}_{k_2'} x^{(2)}_{\ell_2'}\Big)$$

whenever $(k_1, k_2) \sim_{\mathcal{O}} (k_1', k_2')$ and $(\ell_1, \ell_2) \sim_{\mathcal{O}} (\ell_1', \ell_2')$. $C$-joint spreadability only allows us to conclude that

$$\varphi\Big(x^{(1)}_{k_1} x^{(2)}_{\ell_1} x^{(1)}_{k_2} x^{(2)}_{\ell_2}\Big) = \varphi\Big(x^{(1)}_{k_1'} x^{(2)}_{\ell_1'} x^{(1)}_{k_2'} x^{(2)}_{\ell_2'}\Big)$$

whenever $(k_1, \ell_1, k_2, \ell_2) \sim_{\mathcal{O}} (k_1', \ell_1', k_2', \ell_2')$. Note that $(k_1, k_2, \ell_1, \ell_2) \sim_{\mathcal{O}} (k_1', k_2', \ell_1', \ell_2')$ implies $(k_1, k_2) \sim_{\mathcal{O}} (k_1', k_2')$ and $(\ell_1, \ell_2) \sim_{\mathcal{O}} (\ell_1', \ell_2')$. But the converse may fail as one has, for example, $(1,3) \sim_{\mathcal{O}} (1,2)$ and $(1,3) \sim_{\mathcal{O}} (1,4)$, but $(1,1,3,3) \not\sim_{\mathcal{O}} (1,1,2,4)$.

We indicate next how a $C$-jointly spreadable family of random variables can be obtained from certain sequences of embeddings of a unital *-algebra into a larger unital *-algebra.

**Example 3.4.4.** Let $(\mathcal{A}, \varphi)$ and $(\mathcal{C}, \chi)$ be *-algebraic probability spaces and let

$$\lambda \equiv (\lambda_n)_{n\in\mathbb{N}} \colon \mathcal{C} \to \mathcal{A}$$

be injective *-homomorphisms such that $\varphi \circ \lambda_n = \chi$ for all $n \in \mathbb{N}$. Suppose one has that, for any $n \in \mathbb{N}$ and $c_1, c_2, \ldots, c_n \in \mathcal{C}$,

$$\varphi\big(\lambda_{\mathbf{i}(1)}(c_1) \cdots \lambda_{\mathbf{i}(n)}(c_n)\big) = \varphi\big(\lambda_{\mathbf{j}(1)}(c_1) \cdots \lambda_{\mathbf{j}(n)}(c_n)\big)$$

for all $\mathbf{i}, \mathbf{j} \colon [n] \to \mathbb{N}$ with $\mathbf{i} \sim_{\mathcal{O}} \mathbf{j}$. Upon putting $x_{c,n} := \lambda_n(c)$, one obtains the $C$-jointly spreadable family $\mathcal{X} \equiv \{(x_{c,n})_{n=1}^{\infty} \mid c \in C\}$.

We provide next a simple example of how to obtain a $C$-separately spreadable family from spreadable sequences.

**Example 3.4.5.** Let $C = [d]$ for some $d \in \mathbb{N}$. For each $c \in C$, suppose $(\mathcal{A}^{(c)}, \varphi^{(c)})$ is a *-algebraic probability space which contains the spreadable sequence $(y_n^{(c)})_{n=1}^{\infty}$. Let $x_n^{(c)}$ denote the canonical embedding of $y_n^{(c)}$ into the $c$-th factor of the tensor product $\bigotimes_{c\in C} \mathcal{A}^{(c)}$. Then $\mathcal{X} \equiv \{x_n^{(c)} \mid n \in \mathbb{N}, c \in C\}$ defines a $C$-separately spreadable family for the *-algebraic probability space $(\mathcal{A}, \varphi) := \bigotimes_{c\in C}(\mathcal{A}^{(c)}, \varphi^{(c)})$. Note that the $C$-separate spreadability is immediately inferred from the spreadability of each underlying sequence and the factorization

$$\varphi(y_1 \otimes y_2 \otimes \cdots \otimes y_d) = \varphi^{(1)}(y_1)\varphi^{(2)}(y_2) \cdots \varphi^{(d)}(y_d)$$

for all $y_c \in \mathcal{A}^{(c)}$ and $c = 1, 2, \ldots, d$.

The next result is parallel to Lemma 3.3.7 from the framework of exchangeability, and it will be used in Theorem 3.4.9.

**Lemma 3.4.6.** *Suppose $\mathcal{X}$ is a $C$-separately spreadable family of sequences as stated in Definition 3.4.1(ii). Then the following are equivalent:*

*(a) $\mathcal{X}$ satisfies the $C$-joint SVP;*

*(b) $\mathcal{X}$ satisfies the $C$-separate SVP.*

*In general, this equivalence is not valid for a $C$-jointly spreadable family $\mathcal{X}$.*

*Proof.* Consider the function $\mathbf{i}\colon [n] \to \mathbb{N}$ with $\pi = \ker_{\mathcal{O}}(\mathbf{i})$. Suppose $\pi$ contains the singleton $\{\widetilde{k}\}$. As $\mathbf{t}(\widetilde{k}) \in \widetilde{W}$ for some $\widetilde{W} \in \ker(\mathbf{t})$, we conclude that the restricted partition $\pi_{|_{\widetilde{W}}}$ also contains the singleton $\{\widetilde{k}\}$. Conversely, suppose $\pi_{|_{\widetilde{W}}}$ contains the singleton $\{\widetilde{k}\}$ for some $\widetilde{W} \in \ker(\mathbf{t})$. Now we use the $C$-separate spreadability of $\mathcal{X}$ to conclude that there exists for any $\sigma_W, \tau_W \in S_\infty$ a function $\mathbf{j}\colon [n] \to \mathbb{N}$ with $\ker_{\mathcal{O}}(\mathbf{j}) = \ker_{\mathcal{O}}(\mathbf{i})$ such that $\mathbf{j}|_W = \mathbf{i}|_W$ for all $W \in \ker(\mathbf{t})\backslash\{\widetilde{W}\}$ and $\tau_W \circ \mathbf{i}|_W = \sigma_W \circ (\mathbf{j}|_W)$ and $\tau_W|_{\mathrm{Ran}\,\mathbf{i}|_W}$ and $\sigma_W|_{\mathrm{Ran}\,\mathbf{j}|_W}$ are order preserving. In particular, we can choose the permutation $\sigma \in S_\infty$ such that $\ker(\mathbf{j})$ contains the singleton $\{\widetilde{k}\}$.

Altogether, these arguments show that, for some $W \in \ker(\mathbf{t})$, the function $\mathbf{i}\colon [n] \to \mathbb{N}$ has a restricted ordered kernel set partition $\ker_{\mathcal{O}}(\mathbf{i}|_W)$, which contains a singleton if and only if there exists a function $\mathbf{j}\colon [n] \to \mathbb{N}$ with $\varphi_{\ker_{\mathcal{O}}(\mathbf{j}),\mathbf{t}} = \varphi_{\ker_{\mathcal{O}}(\mathbf{i}),\mathbf{t}}$ which has a singleton in its ordered kernel set partition $\ker(\mathbf{j})$.

'(a) $\Rightarrow$ (b)': Suppose that $\ker_{\mathcal{O}}(\mathbf{i}|_W)$ contains a singleton for some $W \in \ker(\mathbf{t})$. By the previous arguments, we find a function $\mathbf{j}\colon [n] \to \mathbb{N}$ which contains a singleton and satisfies $\varphi_{\ker_{\mathcal{O}}(\mathbf{j}),\mathbf{t}} = \varphi_{\ker_{\mathcal{O}}(\mathbf{i}),\mathbf{t}}$. Thus the validity of the $C$-joint SVP implies $\varphi_{\ker_{\mathcal{O}}(\mathbf{i}),\mathbf{t}} = \varphi_{\ker_{\mathcal{O}}(\mathbf{j}),\mathbf{t}} = 0$.
'(b) $\Rightarrow$ (a)': Suppose $\ker_{\mathcal{O}}(\mathbf{i})$ contains a singleton. Then $\ker_{\mathcal{O}}(\mathbf{i}|_W)$ contains a singleton for some $W \in \ker(\mathbf{t})$. Thus we conclude $\varphi_{\ker_{\mathcal{O}}(\mathbf{i}),\mathbf{t}} = 0$ from the validity of the $C$-joint SVP.

We are left to show the failure of this equivalence in the setting of a $C$-jointly spreadable family $\mathcal{X}$. But this is immediate from Lemma 3.3.7, where it is shown that this equivalence already fails for a $C$-jointly exchangeable family $\mathcal{X}$. □

We introduce next in the spreadable setting the counterpart of the function on partitions associated to a $C$-jointly exchangeable family $\mathcal{X}$ . We recall from Lemma 2.1.21 that the following are equivalent for two functions $\mathbf{i}, \mathbf{j}\colon [n] \to \mathbb{N}$:

$$\mathbf{i} \sim_{\mathcal{O}} \mathbf{j} \quad \Longleftrightarrow \quad \ker_{\mathcal{O}}(\mathbf{i}) = \ker_{\mathcal{O}}(\mathbf{j}).$$

This ensures that the function introduced next is well-defined.

**Definition 3.4.7.** Let $(\mathcal{A}, \varphi)$ be a *-algebraic probability space and let $\mathcal{X} \equiv \{(x_{c,n})_{n=1}^\infty \mid c \in C\} \subset \mathcal{A}$ be a $C$-jointly spreadable family. Then the map

$$\varphi^{\mathcal{O}}_{\bullet,\bullet,\bullet}\colon \bigsqcup_{n=1}^{\infty} \mathcal{OP}(n) \times \{\mathbf{t}\colon [n] \to C\} \times \{\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}\} \to \mathbb{C},$$

given by

$$\varphi^{\mathcal{O}}_{\pi,\mathbf{t},\boldsymbol{\varepsilon}} := \varphi\Big(x^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),\mathbf{i}(1)} \cdots x^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),\mathbf{i}(n)}\Big)$$

for any $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$, $\mathbf{t}\colon [n] \to C$, and $\mathbf{i}\colon [n] \to \mathbb{N}$ with $\ker_{\mathcal{O}}(\mathbf{i}) = \pi$, is called the *function on ordered partitions associated to* $\mathcal{X}$. If the sequence $\mathcal{X}$ satisfies $x_{c,n} = x_{c,n}^*$ for all $c \in \mathbb{C}$ and $n \in \mathbb{N}$, then the function on partitions associated to $\mathcal{X}$ simplifies to

$$\varphi^{\mathcal{O}}_{\bullet,\bullet}\colon \bigsqcup_{n=1}^{\infty} \mathcal{OP}(n) \times \{\mathbf{t}\colon [n] \to C\} \to \mathbb{C}$$

that is given by

$$\varphi^{\mathcal{O}}_{\pi,\mathbf{t}} := \varphi\Big(x_{\mathbf{t}(1),\mathbf{i}(1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}\Big).$$

Note that, canonically identifying the function $\mathbf{t}\colon [n] \to C$ and the $n$-tuple $(\mathbf{t}(1), \ldots, \mathbf{t}(n)) \in C^n$, the function on partitions associated to $\mathcal{X}$ can be also addressed as

$$\varphi^{\mathcal{O}}_{\bullet,\bullet}\colon \bigsqcup_{n=1}^{\infty} \mathcal{OP}(n) \times C^n \times \{\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}\} \to \mathbb{C}.$$

**Notation 3.4.8.** We use $0! = 1$ and, for convenience of notation, $0 \neq (n/2)! = \frac{1}{2}\frac{3}{2} \cdots \frac{n}{2}\sqrt{\pi}$ for odd $n \in \mathbb{N}$. Furthermore, we use the convention that summation over the empty set is zero, for example $\sum_{i \in \emptyset} A_i = 0$.

**Theorem 3.4.9.** *Let $(\mathcal{A}, \varphi)$ be a *-algebraic probability space and suppose the family of sequences $\mathcal{X} \equiv \{(x_n^{(c)})_{n=1}^{\infty} \mid c \in C\} \subset \mathcal{A}$ satisfies the following conditions:*

*(i) $\mathcal{X}$ is $C$-jointly spreadable;*

*(ii) $\mathcal{X}$ satisfies the $C$-joint SVP.*

*Then one has for*

$$S_{c,N} = \frac{x_{c,1} + \cdots + x_{c,N}}{\sqrt{N}}$$

*that, for all $n \in \mathbb{N}$, $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$, and $\mathbf{t}\colon [n] \to C$,*

$$\lim_{N\to\infty} \varphi(S^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),N} \cdots S^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),N}) = \frac{1}{(n/2)!} \sum_{\pi \in \mathcal{OP}_2(n)} \varphi^{\mathcal{O}}_{\pi,\mathbf{t},\boldsymbol{\varepsilon}}.$$

*If $\mathcal{X}$ is $C$-separately spreadable or satisfies the $C$-separate SVP, then one has that, for all $n \in \mathbb{N}$, $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$, and $\mathbf{t}\colon [n] \to C$,*

$$\lim_{N\to\infty} \varphi(S^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),N} \cdots S^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),N}) = \frac{1}{(n/2)!} \sum_{\substack{\pi \in \mathcal{OP}_2(n) \\ \overline{\pi} \leq \ker(\mathbf{t})}} \varphi^{\mathcal{O}}_{\pi,\mathbf{t},\boldsymbol{\varepsilon}}.$$

*Here $\overline{\pi} \in \mathcal{P}(n)$ denotes the partition canonically assigned to $\pi \in \mathcal{OP}(n)$.*

**Remark.** In principle, we could define $\mathcal{OP}_2(n,\mathbf{t}) := \{\pi \in \mathcal{OP}_2(n) \mid \overline{\pi} \leq \ker(\mathbf{t})\}$. This definition can be further specified to include a direction map: $\mathcal{OP}_2(n,\mathbf{t},\boldsymbol{\varepsilon}) := \{\pi \in \mathcal{OP}_2(n,\mathbf{t}) \mid \boldsymbol{\varepsilon}(\min V) \neq \boldsymbol{\varepsilon}(\max V) \text{ for all } V \in \pi\}$.

*Proof.* We need to compute, for some fixed $n \in \mathbb{N}$ and $\mathbf{t}(1), \ldots, \mathbf{t}(n)$, the large $N$-limit of

$$\begin{aligned}
\varphi(S^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),N} \cdots S^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),N}) &= \frac{1}{N^{n/2}} \sum_{\mathbf{i}\colon [n]\to[N]} \varphi\big(x^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),\mathbf{i}(1)} \cdots x^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),\mathbf{i}(n)}\big) \\
&= \frac{1}{N^{n/2}} \sum_{\pi \in \mathcal{OP}(n)} \sum_{\substack{\mathbf{i}\colon [n]\to[N] \\ \ker_{\mathcal{O}}(\mathbf{i})=\pi}} \varphi\big(x^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),\mathbf{i}(1)} \cdots x^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),\mathbf{i}(n)}\big) \\
&= \frac{1}{N^{n/2}} \sum_{\pi \in \mathcal{OP}(n)} B^{(N)}_{|\pi|} \varphi^{\mathcal{O}}_{\pi,\mathbf{t},\boldsymbol{\varepsilon}},
\end{aligned}$$

where

$$B^{(N)}_{|\pi|} = \frac{N \cdot (N-1) \cdots (N-|\pi|+1)}{1 \cdot 2 \cdots |\pi|} = \binom{N}{|\pi|}$$

is the cardinality of the set $\{\mathbf{i}\colon [n] \to [N] \mid \ker_{\mathcal{O}}(\mathbf{i}) = \pi\}$.

Here we have used for the second equality that the summation over all functions $\mathbf{i}\colon [n] \to [N]$ can be reorganized as summation over the order equivalence classes of these index functions:

$$\sum_{\mathbf{i}\colon [n]\to[N]} = \sum_{\pi \in \mathcal{OP}(n)} \sum_{\substack{\mathbf{i}\colon [n]\to[N] \\ \ker_{\mathcal{O}}(\mathbf{i})=\pi}}.$$

Note for the next arguments that one has

$$B^{(N)}_{|\pi|} = A^{(N)}_{|\pi|}/|\pi|!,$$

where $A^{(N)}_{|\pi|} = N \cdot (N-1) \cdots (N-|\pi|+1)$ denotes the cardinality of the set $\{\mathbf{i}\colon [n] \to N \mid \ker(\mathbf{i}) = \bar{\pi}\}$, as it has been used in the proof of the classical CLT, Theorem 3.1.1.

Whenever the ordered partition $\pi$ has a singleton, then the SVP implies that $\varphi^{\mathcal{O}}_{\pi,\mathbf{t},\boldsymbol{\varepsilon}} = 0$. Therefore, we are left to consider those ordered partitions $\pi$ for which each block contains at least two elements. This implies $k \leq n/2$. Similar to the arguments from Step 2 in the proof of Theorem 3.1.1, we deduce in the case $|\pi| < n/2$ that

$$\frac{B^{(N)}_{|\pi|}}{N^{n/2}} = \frac{A^{(N)}_{|\pi|}}{N^{n/2}} \cdot \frac{1}{|\pi|!}$$

vanishes in the limit $N \to \infty$. Hence, we are only left with the case $k = n/2$. In other words, only ordered pair partitions contribute to the limit:

$$\begin{aligned}\lim_{N\to\infty} B_k^{(N)}/N^{n/2} &= \lim_{N\to\infty} \frac{N(N-1)\cdots(N-k+1)}{N^k}\cdot\frac{1}{k!}\\ &= \lim_{N\to\infty} \cdot\frac{N}{N}\cdot\frac{N-1}{N}\cdots\frac{N-k+1}{N}\cdot\frac{1}{k!} = \frac{1}{k!}.\end{aligned}$$

Altogether, we have shown that only ordered pair partitions $\pi \in \mathcal{OP}_2(n)$ can contribute as summands in the large $N$-limit such that

$$\lim_{N\to\infty} \varphi(S^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),N}\cdots S^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),N}) = \frac{1}{(n/2)!}\sum_{\pi\in\mathcal{OP}_2(n)} \varphi^{\mathcal{O}}_{\pi,\mathbf{t},\boldsymbol{\varepsilon}}.$$

As we argued in Theorem 3.3.10, we are left to show that the $C$-separate spreadability or the $C$-separate SVP of $\mathcal{X}$ allows us to restrict the summation further to ordered pair partitions $\pi \in \mathcal{OP}_2(n)$ satisfying $\overline{\pi} \le \ker(\mathbf{t})$. We know from Lemma 3.4.6 that $C$-separate spreadability implies the equivalence of the $C$-joint SVP and the $C$-separate SVP. Thus it suffices to showing that strengthening of the $C$-joint SVP to the $C$-separate SVP implies the vanishing of further ordered pair partitions. For this purpose, consider some $\mathbf{i}\colon [n] \to [N]$ with $\pi = \ker_{\mathcal{O}}(\mathbf{i}) \in \mathcal{OP}_2(n)$ and let $W \in \ker(\mathbf{t})$. Then $\ker_{\mathcal{O}}(\mathbf{i}|_W) \in \mathcal{OP}(W)$ has blocks of size 2 at most. Now the $C$-separate SVP implies that only summands of those partitions $\pi = \ker_{\mathcal{O}}(\mathbf{i})$ contribute where $\ker_{\mathcal{O}}(\mathbf{i}|_W) \in \mathcal{OP}(W)$ is an ordered pair partition. As this argument is valid for any block $W \in \ker(\mathbf{t})$, it follows that only those summands contribute for which $\pi = \ker_{\mathcal{O}}(\mathbf{i}) \in \mathcal{OP}_2(n)$ satisfies $\overline{\pi} \le \ker(\mathbf{t})$. □

**Remark 3.4.10.** $C$-joint spreadability of a family of random variables is a slightly stronger distributional invariance principle as the one which is stipulated by Speicher and von Waldenfels for their main result in [SW94], a general algebraic CLT. $C$-separate spreadability is a stronger distributional invariance principle than $C$-joint spreadability, and it has so far not been formulated explicitly in the context of *-algebraic CLTs.

Next we provide some notation and a definition, as needed for the formulation of Theorem 3.4.13 below.

**Notation 3.4.11.** Let the 'color' function $\mathbf{t}\colon [n] \to C$ be fixed. We denote by

$$\prod_{W\in\ker(\mathbf{t})} \mathcal{OP}(W) = \Big\{ f\colon \ker(\mathbf{t}) \mapsto \bigsqcup_{W\in\ker(\mathbf{t})} \mathcal{OP}(W) \Big\}$$

the Cartesian product of the ordered partitions of the blocks of $\ker(\mathbf{t})$ with coordinate projections $\ker(\mathbf{t}) \ni W \mapsto f(W) \in \mathcal{OP}(W)$. Similarly, let $\prod_{W\in\ker(\mathbf{t})} \mathcal{OP}_2(W)$ denote the Cartesian product of the ordered pair partitions of the blocks of $\ker(\mathbf{t})$.

For a given ordered partition $\pi \in \mathcal{OP}(n)$, we denote by

$$\pi_{|\bullet} \in \prod_{W \in \ker(\mathbf{t})} \mathcal{OP}(W)$$

the function which has, for each $W \in \ker(\mathbf{t})$, the $W$-restriction $\pi_{|W}$ as a coordinate. Furthermore, we will write

$$\pi_{|\bullet} \in \prod_{W \in \ker(\mathbf{t})} \mathcal{OP}_2(W)$$

if $\pi_{|W} \in \mathcal{OP}_2(W)$ for all $W \in \ker(\mathbf{t})$.

Clearly the equivalence

$$\mathbf{i} \sim_{\mathcal{O}} \mathbf{j} \quad \Longleftrightarrow \quad \ker_{\mathcal{O}}(\mathbf{i}) = \ker_{\mathcal{O}}(\mathbf{j})$$

for two functions $\mathbf{i}, \mathbf{j} \colon [n] \to \mathbb{N}$ (see Lemma 2.1.21) transfers to restrictions of their domain. More precisely, this equivalence ensures that

$$\mathbf{i}|_W \sim_{\mathcal{O}} \mathbf{j}|_W \quad \Longleftrightarrow \quad \ker_{\mathcal{O}}(\mathbf{i}|_W) = \ker_{\mathcal{O}}(\mathbf{j}|_W)$$

for the restrictions of the functions $\mathbf{i}$ and $\mathbf{j}$ to any (non-empty) subset $W \subset [n]$. This ensures that the function introduced next is well-defined.

**Definition 3.4.12.** Let $(\mathcal{A}, \varphi)$ be a *-algebraic probability space and let $\mathcal{X} \equiv \{(x_{c,n})_{n=1}^{\infty} \mid c \in C\} \subset \mathcal{A}$ be a $C$-separately spreadable family. Then the map

$$\varphi^{\mathcal{O}}_{\bullet_|, \bullet, \bullet} \colon \bigsqcup_{n=1}^{\infty} \Big( \prod_{W \in \ker(\mathbf{t})} \mathcal{OP}(W) \Big) \times \{\mathbf{t} \colon [n] \to C\} \times \{\boldsymbol{\varepsilon} \colon [n] \to \{*, 1\}\} \to \mathbb{C},$$

given by

$$\varphi^{\mathcal{O}}_{\pi_|, \mathbf{t}, \boldsymbol{\varepsilon}} := \varphi\Big( x^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1), \mathbf{i}(1)} \cdots x^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n), \mathbf{i}(n)} \Big)$$

for any $\boldsymbol{\varepsilon} \colon [n] \to \{*, 1\}$, $\mathbf{t} \colon [n] \to C$, and $\mathbf{i} \colon [n] \to \mathbb{N}$ with $\ker_{\mathcal{O}}(\mathbf{i}) = \pi$ and $\ker_{\mathcal{O}}(\mathbf{i}|_W) = \pi_{|W} \in \mathcal{OP}(W)$ for any $W \in \ker(\mathbf{t})$, is called the *function on reduced ordered partitions associated to* $\mathcal{X}$. If the family $\mathcal{X}$ satisfies $x_{c,n} = x^*_{c,n}$ for all $c \in \mathbb{C}$ and $n \in \mathbb{N}$, then the function on partitions associated to $\mathcal{X}$ simplifies to

$$\varphi^{\mathcal{O}}_{\bullet_|, \bullet} \colon \bigsqcup_{n=1}^{\infty} \Big( \prod_{W \in \ker(\mathbf{t})} \mathcal{OP}(W) \Big) \times \{\mathbf{t} \colon [n] \to C\} \to \mathbb{C}$$

that is given by

$$\varphi^{\mathcal{O}}_{\pi_|, \mathbf{t}} := \varphi\Big( x_{\mathbf{t}(1), \mathbf{i}(1)} \cdots x_{\mathbf{t}(n), \mathbf{i}(n)} \Big).$$

We are ready to generalize Theorem 3.3.13 from the setting of exchangeability to that of spreadability.

**Theorem 3.4.13.** *Let $(\mathcal{A}, \varphi)$ be a *-algebraic probability space and suppose the family of sequences $\mathcal{X} \equiv \{(x_{c,n})_{n=1}^{\infty} \mid c \in C\} \subset \mathcal{A}$ satisfies:*

*(i) $\mathcal{X}$ is $C$-separately spreadable;*

*(ii) $\mathcal{X}$ satisfies the $C$-separate SVP.*

*Then one has for*

$$S_{c,N} = \frac{x_{c,1} + \cdots + x_{c,N}}{\sqrt{N}}$$

*that, for all $n \in \mathbb{N}$, $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$, and $\mathbf{t}\colon [n] \to C$,*

$$\lim_{N\to\infty} \varphi(S_{\mathbf{t}(1),N}^{\boldsymbol{\varepsilon}(1)} \cdots S_{\mathbf{t}(n),N}^{\boldsymbol{\varepsilon}(n)}) = \prod_{W \in \ker(\mathbf{t})} \frac{1}{(|W|/2)!} \sum_{\pi_{|\bullet} \in \prod_{W \in \ker(\mathbf{t})} \mathcal{OP}_2(W)} \varphi^{\mathcal{O}}_{\pi_|, \mathbf{t}, \boldsymbol{\varepsilon}}.$$

*Proof.* The assumptions of $C$-separate spreadability and $C$-separate SVP ensure that Theorem 3.4.9 is applicable such that

$$\lim_{N\to\infty} \varphi(S_{\mathbf{t}(1),N}^{\boldsymbol{\varepsilon}(1)} \cdots S_{\mathbf{t}(n),N}^{\boldsymbol{\varepsilon}(n)}) = \frac{1}{(n/2)!} \sum_{\substack{\pi \in \mathcal{OP}_2(n) \\ \overline{\pi} \leq \ker(\mathbf{t})}} \varphi^{\mathcal{O}}_{\pi, \mathbf{t}, \boldsymbol{\varepsilon}}.$$

The proof is completed if we can establish

$$\frac{1}{(n/2)!} \sum_{\substack{\pi \in \mathcal{OP}_2(n) \\ \overline{\pi} \leq \ker(\mathbf{t})}} \varphi^{\mathcal{O}}_{\pi, \mathbf{t}, \boldsymbol{\varepsilon}} = \prod_{W \in \ker(\mathbf{t})} \frac{1}{(|W|/2)!} \sum_{\pi_{|\bullet} \in \prod_{W \in \ker(\mathbf{t})} \mathcal{OP}_2(W)} \varphi^{\mathcal{O}}_{\pi_|, \mathbf{t}, \boldsymbol{\varepsilon}}$$

or, equivalently,

$$\sum_{\substack{\pi \in \mathcal{OP}_2(n) \\ \overline{\pi} \leq \ker(\mathbf{t})}} \varphi^{\mathcal{O}}_{\pi, \mathbf{t}, \boldsymbol{\varepsilon}} = \frac{(n/2)!}{\prod_{W \in \ker(\mathbf{t})} (|W|/2)!} \sum_{\pi_{|\bullet} \in \prod_{W \in \ker(\mathbf{t})} \mathcal{OP}_2(W)} \varphi^{\mathcal{O}}_{\pi_|, \mathbf{t}, \boldsymbol{\varepsilon}}.$$

We infer from Definition 3.4.7 and Definition 3.4.12 that

$$\varphi^{\mathcal{O}}_{\pi, \mathbf{t}, \boldsymbol{\varepsilon}} = \varphi\Big(x_{\mathbf{t}(1),\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\Big) = \varphi^{\mathcal{O}}_{\pi_|, \mathbf{t}, \boldsymbol{\varepsilon}}$$

for any $\pi \in \mathcal{OP}_2(n)$ with $\overline{\pi} \leq \ker(\mathbf{t})$, since one has for any $\pi \in \mathcal{OP}_2(n)$:

$$\overline{\pi} \leq \ker(\mathbf{t}) \qquad \Longrightarrow \qquad \pi_{|\bullet} \in \prod_{W \in \ker(\mathbf{t})} \mathcal{OP}_2(W).$$

Thus we are left to show that, for any fixed $f \in \prod_{W \in \ker(\mathbf{t})} \mathcal{OP}_2(W)$,

$$\#\big\{\pi \in \mathcal{OP}_2(n) \big| \overline{\pi} \leq \ker(\mathbf{t}), \pi_{|\bullet} = f\big\} = \frac{(n/2)!}{\prod_{W \in \ker(\mathbf{t})}(|W|/2)!}.$$

But this claim is immediate upon its reformulation as a standard combinatorial problem. Let $n = 2p$ and $|W_1| = 2p_1, \ldots, |W_d| = 2p_d$ for $d = |\ker(\mathbf{t})|$ such that $p = p_1 + p_2 + \ldots + p_d$. For each $c \in [d]$, suppose there are $p_c$ ordered pairs of the color $c$, given by $f(W_c) \in \mathcal{OP}_2(W_c)$. Then the number of possibilities to distribute these $p$ ordered pairs to obtain an ordered pair partition $\pi \in \mathcal{OP}_2(2p)$ such that $\pi_{|W_c} = f(W_c)$ for all $c \in [d]$ is given by

$$\frac{p!}{p_1!\, p_2! \cdots p_d!}.$$

□

We illustrate the combinatorial argument used in the proof of Theorem 3.4.13.

**Example 3.4.14.** Consider the 'color' function $\mathbf{t}\colon [12] \to [3]$ with

$$\ker(\mathbf{t}) = \{\{1,3,6,10\},\{2,5\},\{4,7,8,9,11,12\}\} =: \{W_1, W_2, W_3\}$$

and suppose that $f \in \prod_{W \in \ker(\mathbf{t})} \mathcal{OP}_2(W)$ with

$$f(W_1) = (\{1,6\},\{3,10\}), \quad f(W_2) = (\{3,5\}), \quad f(W_3) = (\{4,9\},\{7,8\},\{11,12\}).$$

Let $2p_i := |W_i| =$ for $i = 1,2,3$ and $2p = 12$. More explicitly, one has

$$p_1 = 2, \qquad p_2 = 1, \qquad p_3 = 3, \qquad p = 6.$$

Upon the (unique) order preserving identification of $\mathcal{OP}_2(W_i)$ and $\mathcal{OP}_2(2p_i)$ for each $W_i \in \ker(\mathbf{t})$, we may instead consider the function $\widetilde{f} \in \prod_{W \in \ker(\mathbf{t})} \mathcal{OP}_2(2p_i)$ with

$$\widetilde{f}(W_1) = (\{1,3\},\{2,4\}), \quad \widetilde{f}(W_2) = (\{1,2\}), \quad \widetilde{f}(W_3) = (\{1,4\},\{2,3\},\{5,6\}).$$

We are interested in finding all ordered pair partitions $\pi = (V_1, V_2, V_3, V_4, V_5, V_6) \in \mathcal{OP}_2(12)$ such that $\overline{\pi} = \{V_1, V_2, V_3, V_4, V_5, V_6\} \leq \ker(\mathbf{t})$. Note that the condition $\overline{\pi} \leq \ker(\mathbf{t})$ specifies the 'color' of each pair of the partition $\overline{\pi} \in \mathcal{P}_2(12)$. More explicitly, there are

$$\frac{p!}{p_1!\, p_2!\, p_3!} = \frac{6!}{2!\, 1!\, 3!} = 60$$

possibilities to distribute 3 'colors' on $p = 6$ pairs such that $p_1 = 2$ pairs have the first 'color', $p_2 = 1$ pair has the second 'color', and $p_3 = 3$ pairs have the third 'color'. Having specified this distribution of 'colors' on pairs, the ordered pair partition $\pi$ is completely specified by its $W$-restricted ordered partitions

$$\pi_{|W_1} = \widetilde{f}(W_1), \qquad \pi_{|W_2} = \widetilde{f}(W_2), \qquad \pi_{|W_3} = \widetilde{f}(W_3).$$

We discuss next how multivariate settings for spreadable sequences can be constructed from a single spreadable sequence. Given the ordered set $(C, <)$ and the natural numbers $(\mathbb{N}, <)$ (equipped with the natural order relation), we equip the Cartesian product $C \times \mathbb{N}$ with the following two order relations:

$$
\begin{aligned}
(c, m) <_j (d, n) \quad &:\Longleftrightarrow \quad (m < n) \text{ or } (c < d \text{ if } m = n),\\
(c, m) <_s (d, n) \quad &:\Longleftrightarrow \quad (c < d) \text{ or } (m < n \text{ if } c = d).
\end{aligned}
$$

We will see that the order relation $<_j$ is of relevance for the construction of a $C$-joint spreadable family $\mathcal{X}$, and the the order relation $<_s$ will be of relevance for the construction of a $C$-separate spreadable family $\mathcal{X}$.

**Lemma 3.4.15.** *Let $C$ be a countable set and let $J\colon C \times \mathbb{N} \to \mathbb{N}$ be an injective function such that, for all $c, \widetilde{c} \in C$ and $n, \widetilde{n} \in \mathbb{N}$,*

$$
(c, n) <_j (\widetilde{c}, \widetilde{n}) \quad \Longrightarrow \quad J(c, n) < J(\widetilde{c}, \widetilde{n}).
$$

*Given a spreadable sequence $\mathbf{x} \equiv (x_n)_{n=1}^{\infty} \subset \mathcal{A}$ and putting*

$$
x_{c,n} := x_{J(c,n)},
$$

*the family of sequences $\mathcal{X} \equiv \{(x_{c,n})_{n=1}^{\infty} \mid c \in C\} \subset \mathcal{A}$ is $C$-jointly spreadable. Moreover, if the sequence $\mathbf{x} \subset \mathcal{A}$ has the SVP, then $\mathcal{X}$ has the $C$-separate SVP, and thus also the $C$-joint SVP.*

*Proof.* We need to show that, for any $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$,

$$
\varphi\Big(x_{\mathbf{t}(1),\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{t}(n),\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\Big) = \varphi\Big(x_{\mathbf{t}(1),\mathbf{j}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{t}(n),\mathbf{j}(n)}^{\boldsymbol{\varepsilon}(n)}\Big)
$$

whenever $\mathbf{i} \sim_{\mathcal{O}} \mathbf{j}$. By Lemma 2.1.21, this is equivalent to establishing

$$
\varphi\Big(x_{\mathbf{t}(1),\tau\circ\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{t}(n),\tau\circ\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\Big) = \varphi\Big(x_{\mathbf{t}(1),\sigma\circ\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots x_{\mathbf{t}(n),\sigma\circ\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\Big)
$$

or, using the definition of the $x_{c,n}$'s,

$$
\varphi\Big(x_{J(\mathbf{t}(1),\tau\circ\mathbf{i}(1))}^{\boldsymbol{\varepsilon}(1)} \cdots x_{J(\mathbf{t}(n),\tau\circ\mathbf{i}(n))}^{\boldsymbol{\varepsilon}(n)}\Big) = \varphi\Big(x_{J(\mathbf{t}(1),\sigma\circ\mathbf{i}(1))}^{\boldsymbol{\varepsilon}(1)} \cdots x_{J(\mathbf{t}(n),\sigma\circ\mathbf{i}(n))}^{\boldsymbol{\varepsilon}(n)}\Big)
$$

for $\sigma, \tau \in S_\infty$ with order preserving restrictions $\sigma|_{[n]}$ and $\tau|_{[n]}$. For this purpose, it suffices to consider lower index pairs $(k, \ell) \in [n]^2$ with $k \neq \ell$ and to show that $(k, \ell) \sim_{\mathcal{O}} \big(\sigma(k), \sigma(\ell)\big)$ for $\sigma \in S_\infty$ with order preserving restriction $\sigma|_{[n]}$ if and only if $\big(J(c, k), J(d, \ell)\big) \sim_{\mathcal{O}} \big(J(c, \sigma(k)), J(d, \sigma(\ell))\big)$ for all $c, d \in C$ and $\sigma \in S_\infty$ with order preserving restriction $\sigma|_{[n]}$.

Suppose $1 \leq k < \ell \leq n$. Then one has $(c, k) <_j (d, \ell)$ for any $c, d \in C$ if and only if $J(c, k) < J(d, \ell)$ for any $c, d \in C$. Since the restriction $\sigma \in S_\infty$ to $[n]$ is assumed to be order preserving and due to the order preserving assumptions on $J$, we conclude further that

$$k < \ell \quad \Longleftrightarrow \quad \sigma(k) < \sigma(\ell) \quad \Longleftrightarrow \quad J(c, \sigma(k)) < J(d, \sigma(\ell)) \quad (c, d \in C).$$

Exchanging the roles of $k$ and $\ell$ establishes to equivalence of the converse inequalities. Altogether, we arrive at the conclusion that the spreadability of the sequence $\mathbf{x}$ implies the $C$-joint spreadability of $\mathcal{X}$.

Finally, the $C$-joint SVP of $\mathcal{X}$ follows immediately from the SVP of the sequence $\mathbf{x}$, since the function $J$ is injective. Suppose now the (ordered) partition $\pi \in (\mathcal{O})\mathcal{P}(n)$ has a $W$-restricted (ordered) partition $\pi_{|W}$, which contains a singleton $\{\ell\}$. Then this singleton is contained in a block $V \in \pi$. If $V = \{\ell\}$ then the injectivity of $J$ implies again that $C$-separate SVP is valid in this case. So we are left to consider the case that $V = \{\ell\} \cup \widetilde{V}$ for some non-empty set $\widetilde{V}$. Our assumption on $\ell$ implies $\widetilde{V} \cap W = \emptyset$, as otherwise $\{\ell\}$ would not be a singleton for the $W$-restricted (ordered) partition $\pi_{|W}$. Now the injectivity of $J$ implies again that the SVP of the sequence $\mathbf{x}$ ensures the $C$-separate SVP of the family $\mathcal{X}$. □

**Example 3.4.16.** Consider $C = [d]$ to be equipped with its natural order and let $[d] \times \mathbb{N}$ be equipped with the order $<_j$. Then an injective function $J\colon [d] \times \mathbb{N} \to \mathbb{N}$ is defined by

$$J(c, n) = (n-1)d + c.$$

One easily verifies that $(c, m) <_j (\widetilde{c}, \widetilde{m})$ implies $J(c, m) < J(\widetilde{c}, \widetilde{m})$. Thus, given the spreadable sequence $\mathbf{x}$, one obtains that $x_{c,n} := x_{J(c,n)}$ defines a $C$-jointly spreadable family $\mathcal{X}$, as formulated in Lemma 3.4.15.

**Example 3.4.17** ('Color'-Interleaving Pattern)**.** This example is met again when we will study multivariate CLTs for $\omega$-sequences in Chapter 4. We have produced in Example 3.4.16 a jointly spreadable tuple of sequences from a single spreadable sequence with the help of the injective function $J\colon [d] \times \mathbb{N} \to \mathbb{N}$, given by

$$J(c, n) = (n-1)d + c. \tag{3.9}$$

This 'color'-interleaving choice of $J$ yields for the ansatz of the multivariate CLT:

$$\begin{aligned}
S_{1,N} &= \frac{1}{\sqrt{N}}(x_1 + x_{d+1} + \ldots + x_{(N-1)d+1}),\\
S_{2,N} &= \frac{1}{\sqrt{N}}(x_2 + x_{d+2} + \ldots + x_{(N-1)d+2}),\\
&\vdots\\
S_{d,N} &= \frac{1}{\sqrt{N}}(x_d + x_{2d} + \ldots + x_{Nd}).
\end{aligned}$$

If the spreadable sequence $\mathbf{x}$ has also the SVP, then one arrives by Theorem 3.4.9 at the following multivariate CLT:

$$\lim_{N\to\infty} \varphi(S^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),N} \cdots S^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),N}) = \frac{1}{(n/2)!} \sum_{\substack{\pi\in\mathcal{OP}_2(n)\\ \overline{\pi}\leq\ker(\mathbf{t})}} \varphi^{\mathcal{O}}_{\pi,\mathbf{t},\boldsymbol{\varepsilon}},$$

where

$$\varphi^{\mathcal{O}}_{\pi,\mathbf{t},\boldsymbol{\varepsilon}} = \varphi\Big(x^{\boldsymbol{\varepsilon}(1)}_{(\mathbf{i}(1)-1)d+\mathbf{t}(1)} \cdots x^{\boldsymbol{\varepsilon}(n)}_{(\mathbf{i}(n)-1)d+\mathbf{t}(n)}\Big)$$

for $\mathbf{t}\colon [n]\to C$, $\boldsymbol{\varepsilon}\colon [n]\to\{*,1\}$, and $\mathbf{i}\colon [n]\to\mathbb{N}$ with $\ker_{\mathcal{O}}(\mathbf{i}) = \pi$.

Furthermore, we rewrite $\varphi^{\mathcal{O}}_{\pi,\mathbf{t},\boldsymbol{\varepsilon}}$ for even $n$ such that an ordered pair partition $\pi\in\mathcal{OP}_2(n)$ is addressed through a pair $(\overline{\pi},\sigma)\in\mathcal{P}_2(n)\times S_{n/2}$. By Lemma 2.1.26, there is a bijective correspondence between ordered pair partitions $\pi\in\mathcal{OP}_2(n)$ and pairs $(\overline{\pi},\sigma)\in\mathcal{P}_2(n)\times S_{\frac{n}{2}}$ such that $(V_{\sigma(1)},\ldots,V_{\sigma(\frac{n}{2})}) = \pi\mapsto\overline{\pi} = \{V_1,\ldots,V_{\frac{n}{2}}\}$, where $(V_1,\ldots,V_{\frac{n}{2}})$ is the unique standard ordered partition assigned to $\pi$. Thus

$$\lim_{N\to\infty} \varphi(S^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),N} \cdots S^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),N}) = \frac{1}{(n/2)!} \sum_{\sigma\in S_{\frac{n}{2}}} \sum_{\substack{\pi\in\mathcal{P}_2(n)\\ \pi\leq\ker(\mathbf{t})}} \varphi_{\pi,\mathbf{t},\boldsymbol{\varepsilon},\sigma},$$

where, for even $n\in\mathbb{N}$,

$$\varphi_{\pi,\mathbf{t},\boldsymbol{\varepsilon},\sigma} := \varphi\big(x^{\boldsymbol{\varepsilon}(1)}_{\sigma\circ\mathbf{i}(1)} \cdots x^{\boldsymbol{\varepsilon}(n)}_{\sigma\circ\mathbf{i}(n)}\big)$$

for $\mathbf{i}\colon [n]\to[n/2]$ with $\ker(\mathbf{i}) = \pi\in\mathcal{P}_2(n)$, and $\sigma\in S_{n/2}$, and $\boldsymbol{\varepsilon}\colon [n]\to\{*,1\}$.

**Remark 3.4.18.** There exist $C$-jointly spreadable families $\mathcal{X}$ which are constructed as in Lemma 3.4.15 from a single spreadable sequence and which fail to be $C$-separately spreadable. We will meet such a phenomenon when studying certain sequences of quantum coin tosses in Chapter 4.

We turn next our attention to the framework of separate spreadability.

**Lemma 3.4.19.** *Let $C$ be a ordered countable set and let $J\colon C\times\mathbb{N}\to\mathbb{N}$ be an injective function such that, for all $c,\widetilde{c}\in C$ and $n,\widetilde{n}\in\mathbb{N}$,*

$$(c,n) <_s (\widetilde{c},\widetilde{n}) \quad\Longrightarrow\quad J(c,n) < J(\widetilde{c},\widetilde{n}).$$

*Given a spreadable sequence $\mathbf{x}\equiv(x_n)_{n=1}^\infty\subset\mathcal{A}$ and putting*

$$x_{c,n} := x_{J(c,n)},$$

*the family of sequences $\mathcal{X}\equiv\{(x_{c,n})_{n=1}^\infty \mid c\in C\}\subset\mathcal{A}$ is $C$-separately spreadable. Moreover, if the sequence $\mathbf{x}\subset\mathcal{A}$ has the SVP, then $\mathcal{X}$ has the $C$-separate SVP.*

*Proof.* We need to establish that, for any $n \in \mathbb{N}$, $\mathbf{t}\colon [n] \to C$, and $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$,

$$\varphi\Big(x^{\boldsymbol{\varepsilon}(1)}_{J(\mathbf{t}(1),\mathbf{i}(1))} \cdots x^{\boldsymbol{\varepsilon}(n)}_{J(\mathbf{t}(n),\mathbf{i}(n))}\Big) = \varphi\Big(x^{\boldsymbol{\varepsilon}(1)}_{J(\mathbf{t}(1),\mathbf{j}(1))} \cdots x^{\boldsymbol{\varepsilon}(n)}_{J(\mathbf{t}(n),\mathbf{j}(n))}\Big).$$

whenever $\mathbf{i}|_W \sim_{\mathcal{O}} \mathbf{j}|_W$ for every block $W \in \ker(\mathbf{t})$. Let $c, d \in C$ with $c \neq d$. Then one has the equivalences

$$c < d \quad \Longleftrightarrow \quad \forall k, \ell \in \mathbb{N}\colon (c,k) <_s (d,\ell) \quad \Longleftrightarrow \quad \forall k, \ell \in \mathbb{N}\colon J(c,k) < J(d,\ell)$$

and

$$c > d \quad \Longleftrightarrow \quad \forall k, \ell \in \mathbb{N}\colon (c,k) >_s (d,\ell) \quad \Longleftrightarrow \quad \forall k, \ell \in \mathbb{N}\colon J(c,k) > J(d,\ell).$$

Thus the order relation of the indices of a factor $x_{c,k}$ and $x_{d,\ell}$ is not effected for $c \neq d$ for any choices of $k, \ell \in \mathbb{N}$. So it is only possible to violate order relations between the indices of pairs of factors which are of the same color, i.e. for factor of the form $x_{c,k}$ and $x_{d,\ell}$. But here one has, for all $k, \ell \in \mathbb{N}$,

$$k < \ell \quad \Longleftrightarrow \quad \forall c \in C\colon (c,k) <_s (c,\ell) \quad \Longleftrightarrow \quad \forall c \in C\colon J(c,k) < J(c,\ell)$$

and

$$k > \ell \quad \Longleftrightarrow \quad \forall c \in C\colon (c,k) >_s (c,\ell) \quad \Longleftrightarrow \quad \forall c \in C\colon J(c,k) > J(c,\ell).$$

Thus the spreadability of the underlying sequence $\mathbf{x}$ ensures the $C$-separate spreadability of $\mathcal{X}$. Finally, the $C$-separate SVP of $\mathcal{X}$ follows from the SVP of the sequence $\mathbf{x}$ by the same arguments as made in the proof of Lemma 3.4.15. ☐

The following example relates to a multivariate version of a CLT for $C$-separately spreadable family of random variables, as we will meet it in Theorem 4.3.10.

**Example 3.4.20** ('Color'-Blockwise Pattern)**.** We recall that we have already seen in Example 3.4.16 that, for $C = [d]$, the 'interleaving pattern' of finitely many colors allows us to obtain a $C$-jointly spreadable family $\mathcal{X}$ from a single spreadable sequence. We describe next an approach which is, loosely phrasing, about a 'block pattern' of finitely many colors. More precisely, this 'block pattern' allows us to obtain a 'locally' $C$-separately spreadable family $\mathcal{X}$ from a single spreadable sequence. Here the attribute 'locally' refers to that spreadability will only be 'locally' available, in a sense which will become more clear in the following.

We replace the role of the single injective function $J\colon [d] \times \mathbb{N} \to \mathbb{N}$ in Lemma 3.4.19 (and Lemma 3.4.15) by the family of injective functions $\{J_M\}_M\colon [d] \times [M] \to [dM]$, defined by

$$J_M(t,n) := (t-1)M + n \qquad (t \in [d], n \in [M]).$$

Consider the finite family of finite sequences $\mathcal{X}_{\mathcal{M}} \equiv \big\{(\widetilde{x}_{c,n})_{n=1}^{M} \big| c \in [d]\big\}$, given by

$$\widetilde{x}_{c,n} := x_{J_M(c,n)},$$

where we have suppressed the index $\mathcal{M}$ in the definition of each $\widetilde{x}_{c,n}$. One can show that the finite family $\mathcal{X}_{\mathcal{M}}$ is 'locally' $C$-separately spreadable, i.e. one has

$$\varphi\Big(\widetilde{x}_{\mathbf{t}(1),\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots \widetilde{x}_{\mathbf{t}(n),\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\Big) = \varphi\Big(\widetilde{x}_{\mathbf{t}(1),\mathbf{j}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots \widetilde{x}_{\mathbf{t}(n),\mathbf{j}(n)}^{\boldsymbol{\varepsilon}(n)}\Big)$$

for any $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$, and for any $\mathbf{i}, \mathbf{j}\colon [n] \to [M]$ whenever $\mathbf{i}|_W \sim_{\mathcal{O}} \mathbf{j}|_W$ for every block $W \in \ker(\mathbf{t})$. Similarly, the finite family $\mathcal{X}_{\mathcal{M}}$ can be seen to be 'locally' $C$-jointly spreadable, i.e. one has

$$\varphi\Big(\widetilde{x}_{\mathbf{t}(1),\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots \widetilde{x}_{\mathbf{t}(n),\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\Big) = \varphi\Big(\widetilde{x}_{\mathbf{t}(1),\mathbf{j}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots \widetilde{x}_{\mathbf{t}(n),\mathbf{j}(n)}^{\boldsymbol{\varepsilon}(n)}\Big)$$

for any $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$, and for any $\mathbf{i}, \mathbf{j}\colon [n] \to [M]$ with $\mathbf{i} \sim_{\mathcal{O}} \mathbf{j}$.

One can also verify that the finite family $\mathcal{X}_M$ satisfies the $C$-separate SVP and (thus also) the $C$-jointly SVP if the underlying sequence $\mathbf{x}$ enjoys the SVP.

Let $C = [d]$. An inspection of Theorem 3.4.13 and its proof shows that this CLT is still valid if one replaces the $C$-separate spreadability and the $C$-separate SVP of the family $\mathcal{X}$ by their local versions for the finite families $\{\mathcal{X}_M\}_{M\in\mathbb{N}}$. Thus, for any $N \le M$, the ansatz

$$\widetilde{S}_{c,N} = \frac{1}{\sqrt{N}}\big(\widetilde{x}_{c,1} + \ldots + \widetilde{x}_{c,N}\big) = \frac{1}{\sqrt{N}}\big(x_{(c-1)N+1} + \ldots + x_{(c-1)N+N}\big)$$

yields again that, for any $n \in \mathbb{N}$ and any $\mathbf{t}\colon [n] \to C$,

$$\lim_{N\to\infty} \varphi(\widetilde{S}_{\mathbf{t}(1),N}^{\boldsymbol{\varepsilon}(1)} \cdots \widetilde{S}_{\mathbf{t}(n),N}^{\boldsymbol{\varepsilon}(n)}) = \prod_{W\in\ker(\mathbf{t})} \frac{1}{(|W|/2)!} \sum_{\pi_{|\bullet} \in \prod_{W\in\ker(\mathbf{t})} \mathcal{OP}_2(W)} \varphi_{\pi_|,\mathbf{t},\boldsymbol{\varepsilon}}^{\mathcal{O}}$$

with, for even $n \in \mathbb{N}$,

$$\varphi_{\pi_|,\mathbf{t},\boldsymbol{\varepsilon}}^{\mathcal{O}} = \varphi\Big(\widetilde{x}_{\mathbf{t}(1),\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots \widetilde{x}_{\mathbf{t}(n),\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\Big)$$

for any $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$, for any $\mathbf{i}\colon [n] \to [n/2]$ with $\pi_{|W} = \ker_{\mathcal{O}}(\mathbf{i}|_W)$ for all $W \in \ker(\mathbf{t}) \in \mathcal{P}(C)$.

**Remark 3.4.21.** In contrast to the situation for exchangeable sequences, one needs now to consider ordered 'color' sets $(C, <)$. We show next that a naive transfer of the construction for exchangeable sequences is doomed to fail.

Let $C$ be a countable set and let $J\colon C \times \mathbb{N} \to \mathbb{N}$ be an injective function such that $J(c, \bullet)$ is order preserving for each $c \in C$. Given a spreadable sequence $\mathbf{x} \subset \mathcal{A}$ and putting

$$x_{c,n} := x_{J(c,n)},$$

the family of sequences $\mathcal{X} \equiv \{(x_{c,n})_{n=1}^{\infty} \mid c \in C\} \subset \mathcal{A}$ may be neither $C$-separately spreadable nor $C$-jointly spreadable. We illustrate this by the following example.

Suppose the function $J\colon C \times \mathbb{N} \to \mathbb{N}$ satisfies that $J(c, \bullet)$ is order preserving for each $c \in \{1, 2\}$ such that

$$\begin{aligned} J(1,1) &= 1, \quad J(1,2)= 5, \quad J(1,3) = 6,\\ J(2,1) &= 2, \quad J(2,2)= 3, \quad J(2,3) = 4. \end{aligned}$$

Furthermore, let $\mathbf{x} \subset \mathcal{A}$ be a spreadable sequence such that $\varphi\big(x_1x_3\big) \neq \varphi\big(x_5x_4\big)$. Then $x_{c,n} := x_{J(c,n)}$ defines a pair of sequences $\mathcal{X} \equiv \{(x_{c,n})_{n=1}^{\infty} \mid c \in C\} \subset \mathcal{A}$ which fails to be $C$-jointly spreadable:

$$\begin{aligned} \varphi\big(x_{1,1}x_{2,2}\big) = \varphi\big(x_{J(1,1)}x_{J(2,2)}\big) &= \varphi\big(x_1x_3\big)\\ &\neq \varphi\big(x_5x_4\big) = \varphi\big(x_{J(1,2)}x_{J(2,3)}\big) = \varphi\big(x_{1,2}x_{2,3}\big). \end{aligned}$$

This calculation also shows that $\mathcal{X}$ fails to be $C$-separately spreadable.

## 3.5 Factorization properties in the context of distributional invariance principles

The singleton vanishing property (SVP), as introduced in Definition 3.2.3, has been a crucial assumption for the *-algebraic CLTs formulated in Section 3.3 and Section 3.4. We provide next some results which allow us to verify the validity of the SVP in concrete models within the framework of tracial *-algebraic probability spaces. Furthermore, any concrete identification of multivariate central limit laws depends on the availability of factorization properties for distributional invariance principles. These factorization properties take over the role of stochastic independence for the classical CLT in the noncommutative context. We provide results on such factorization properties, again for tracial *-algebraic probability spaces, at the end of this section. Here the emphasis is on providing criteria for the validity of such factorization properties, which can be verified in concrete models.

**Proposition 3.5.1.** *Let $(\mathcal{A}, \varphi)$ be a tracial *-algebraic probability space. The following are equivalent for a spreadable sequence $\mathbf{x} \equiv (x_n)_{n=1}^{\infty} \subset \mathcal{A}$:*

*(a)* $\mathbf{x}$ *satisfies the SVP;*

*(b)* $\varphi(x_1^* x_2) = 0$.

*Proof.* Clearly, (a) implies (b) since $\varphi(x_1^* x_2)$ has underlying the ordered partition $\pi = (\{1\}, \{2\})$ which contains two singletons. So we are left to prove the converse. Let $y, z \in$ *-alg$\{x_k \mid k \in \mathbb{N}, k \neq \ell\}$ for some $\ell \in \mathbb{N}$. We want to show that (b) implies $\varphi(yx_\ell z) = 0$ and $\varphi(yx_\ell^* z) = 0$. Clearly, the second equation is immediate from the first, equation, as $\varphi(yx_\ell^* z) = \overline{\varphi(z^* x_\ell y^*)}$ is valid for all $y, z \in$ *-alg$\{x_k \mid k \in \mathbb{N}, k \neq \ell\}$. Thus it suffices to establish the first equation. Recall from Definition 2.3.7 that the partial shift $\theta_{\ell+1}$ acts on $\mathbb{N}$ as

$$\theta_{\ell+1}(n) = \begin{cases} n & \text{if } n > \ell + 1, \\ n+1 & \text{if } n \leq \ell + 1. \end{cases}$$

Let $\alpha_{\ell+1}$ denote the unital *-algebra homomorphism on *-alg$\{x_n \mid n \in \mathbb{N}\}$, which is given by the $\mathbb{C}$-linear multiplicative extension of the maps $x_n \mapsto x_{\theta_{\ell+1}(n)}$ (with $n \in \mathbb{N}$). We conclude from spreadability that

$$\varphi(yx_\ell z) = \varphi\big(\alpha_{\ell+1}^N(y) x_\ell \alpha_{\ell+1}^N(z)\big) = \varphi\big(\alpha_{\ell+1}^N(y) x_{\ell+p} \alpha_{\ell+1}^N(z)\big)$$

for any $1 \leq p \leq N < \infty$. Thus, writing $y_N := \alpha_{\ell+1}^N(y)$ and $z_N := \alpha_{\ell+1}^N(z)$ for brevity, one has

$$\varphi(yx_\ell z) = \varphi\Big(y_N \big(\frac{1}{N}\sum_{p=1}^N x_{\ell+p}\big) z_N\Big)$$

for all $N \in \mathbb{N}$. Using first the traciality of $\varphi$, this can be also written as

$$\varphi(yx_\ell z) = \varphi\Big(z_N y_N \big(\frac{1}{N}\sum_{p=1}^N x_{\ell+p}\big)\Big).$$

By the Cauchy-Schwarz inequality for *-algebraic probability spaces and stationarity, we obtain the estimate

$$\begin{aligned} \Big|\varphi\Big(z_N y_N \big(\frac{1}{N}\sum_{p=1}^N x_\ell\big)\Big)\Big|^2 &\leq \varphi\Big(|y_N^* z_N^*|^2\Big)\varphi\Big(\Big|\frac{1}{N}\sum_{p=1}^N x_{\ell+p}\Big|^2\Big) \\ &= \varphi\Big(|y^* z^*|^2\Big)\varphi\Big(\Big|\frac{1}{N}\sum_{p=1}^N x_p\Big|^2\Big). \end{aligned}$$

Now the second factor can be further reduced with the help of spreadability such that we obtain, for all $N \geq 1$,

$$\begin{aligned} \varphi\Big(\Big|\frac{1}{N}\sum_{p=1}^N x_p\Big|^2\Big) &= \frac{1}{N^2}\sum_{p=1}^N \varphi\big(x_p^* x_p\big) + \frac{1}{N^2}\sum_{1\leq p<q\leq N}\varphi\big(x_p^* x_q\big) + \frac{1}{N^2}\sum_{N\geq p>q\geq 1}\varphi\big(x_p^* x_q\big) \\ &= \frac{1}{N^2}\sum_{p=1}^N \varphi\big(x_1^* x_1\big) + \frac{1}{N^2}\sum_{1\leq p<q\leq N}\varphi\big(x_1^* x_2\big) + \frac{1}{N^2}\sum_{N\geq p>q\geq 1}\varphi\big(x_2^* x_1\big) \\ &= \frac{N}{N^2}\varphi\big(x_1^* x_1\big) + \frac{N(N-1)}{N^2}\Re\varphi\big(x_1^* x_2\big) \geq 0. \end{aligned}$$

Altogether, we conclude in the large $N$-limit that

$$|\varphi(yx_\ell z)| \leq \varphi\Big(|y^*z^*|^2\Big)\Re\varphi(x_1^*x_2).$$

Thus $\varphi(x_1^*x_2) = 0$ implies $\varphi(yx_\ell z)$ and consequently the SVP. $\square$

**Corollary 3.5.2.** *Let $(\mathcal{A}, \varphi)$ be a tracial *-algebraic probability space. The following are equivalent for a $C$-jointly spreadable family $\mathcal{X} \equiv \{(x_{c,n})_{n=1}^\infty \mid c \in C\} \subset \mathcal{A}$:*

*(a) $\mathcal{X}$ satisfies the $C$-joint SVP;*

*(b) $\varphi(x_{c,1,}^* x_{c,2}) = 0$ for all $c \in C$.*

*Proof.* '(a) $\Longrightarrow$ (b)' is concluded as done in the proof of Proposition 3.5.1. The converse implication follows from the same arguments used in the proof of Proposition 3.5.1, now choosing $x_{c,\ell} \in \mathcal{A}$ for some $c \in C$ and $y, z \in$ *-alg$\{x_{c,k} \mid c \in C, k \in \mathbb{N}, k \neq \ell\}$ for some $\ell \in \mathbb{N}$, to find the estimate

$$|\varphi(yx_{c,\ell} z)| \leq \varphi\Big(|y_N^*z_N^*|^2\Big)\Re\varphi(x_{c,1}^*x_{c,2}).$$

Thus $\varphi(x_{c,1}^*x_{c,2}) = 0$ for any $c \in C$ ensures the $C$-joint SVP. $\square$

**Corollary 3.5.3.** *Let $(\mathcal{A}, \varphi)$ be a tracial *-algebraic probability space. The following are equivalent for a $C$-separately spreadable family $\mathcal{X} \equiv \{(x_{c,n})_{n=1}^\infty \mid c \in C\} \subset \mathcal{A}$:*

*(a) $\mathcal{X}$ satisfies the $C$-separate SVP;*

*(b) $\varphi(x_{c,1}^* x_{c,2}) = 0$ for all $c \in C$.*

*Proof.* By Lemma 3.4.6, one has the equivalence of the $C$-joint SVP and $C$-separate SVP for a $C$-separately spreadable family $\mathcal{X}$. Since $C$-separately spreadability implies $C$-joint spreadability, Corollary 3.5.2 applies. $\square$

Let $\mathbb{C}\langle X, Y\rangle$ denote the unital *-algebra generated by the polynomials in the non-commuting variables $X$ and $Y$.

**Proposition 3.5.4.** *Let $(\mathcal{A}, \varphi)$ be a tracial *-algebraic probability space. The following are equivalent for a spreadable sequence $\mathbf{x} \equiv (x_n)_{n=1}^\infty \subset \mathcal{A}$ and some polynomial $P(X, Y) \in \mathbb{C}\langle X, Y\rangle$:*

*(a) $\mathbf{x}$ has the vanishing property*

$$\varphi\big(yP(x_\ell, x_\ell^*)z\big) = 0$$

*for $y, z \in$ *-alg$\{x_k \mid k \in \mathbb{N}, k \neq \ell\}$ and $\ell \in \mathbb{N}$;*

*(b)* $\mathbf{x}$ *has the vanishing property*

$$\varphi\Big(P(x_1, x_1^*)^* P(x_2, x_2^*)\Big) = 0.$$

*Proof.* '(a) $\Longrightarrow$ (b)': Choose $\ell = 1$, $y = 1_{\mathcal{A}}$ and $z = P(x_2, x_2^*)$.
'(b) $\Longrightarrow$ (a)': Let $y, z \in$ *-alg$\{x_k \mid k \in \mathbb{N}, k \neq \ell\}$ and put $\widetilde{x}_\ell := P(x_\ell, x_\ell)$. We repeat the arguments in the proof of Proposition 3.5.1 to arrive at the estimate

$$|\varphi(y\widetilde{x}_\ell z)| \leq \varphi\Big(|y^* z^*|^2\Big)\Re\varphi(\widetilde{x}_1^*\widetilde{x}_2).$$

Thus $\varphi(\widetilde{x}_1^*\widetilde{x}_2) = 0$ implies the factorization property as stated in (a). □

**Corollary 3.5.5.** *Let $(\mathcal{A}, \varphi)$ be a tracial *-algebraic probability space. The following are equivalent for a spreadable sequence $\mathbf{x} \equiv (x_n)_{n=1}^\infty \subset \mathcal{A}$ and some polynomial $P(X, Y) \in \mathbb{C}\langle X, Y\rangle$:*

*(a)* $\mathbf{x}$ *has the factorization property*

$$\varphi\big(yP(x_\ell, x_\ell^*)z\big) = \varphi\big(P(x_\ell, x_\ell^*)\big)\varphi\big(yz\big)$$

*for $y, z \in$ *-alg$\{x_k \mid k \in \mathbb{N}, k \neq \ell\}$ and $\ell \in \mathbb{N}$;*

*(b)* $\mathbf{x}$ *has the factorization property*

$$\varphi\Big(P(x_1, x_1^*)^* P(x_2, x_2^*)\Big) = \varphi\Big(P(x_1, x_1^*)^*\Big)\varphi\Big(P(x_2, x_2^*)\Big).$$

*Proof.* Let $\widetilde{P}(x_\ell, x_\ell^*) := \big(P(x_\ell, x_\ell^*) - \varphi\big(P(x_\ell, x_\ell^*)\big)1_{\mathcal{A}}\big) \in \mathbb{C}\langle x_\ell, x_\ell^*\rangle$. Now the factorization property in (a) can be reformulated as the vanishing property

$$\varphi\big(y\widetilde{P}(x_\ell, x_\ell^*)z\big) = 0,$$

and the factorization property in (b) as the vanishing property

$$\varphi\Big(\widetilde{P}(x_1, x_1^*)^* \widetilde{P}(x_2, x_2^*)\Big) = 0.$$

Consequently, Proposition 3.5.4 applies. This ensures the claimed equivalence of condition (a) and condition (b). □

**Corollary 3.5.6.** *Let $(\mathcal{A}, \varphi)$ be a tracial *-algebraic probability space. The following are equivalent for a $C$-jointly spreadable family $\mathcal{X} \equiv \{(x_{c,n})_{n=1}^\infty \mid c \in C\} \subset \mathcal{A}$ and some polynomial $P(X_\bullet, Y_\bullet) \in \mathbb{C}\langle X_c, Y_c \mid c \in C\rangle$:*

*(a)* $\mathcal{X}$ *has the factorization property*

$$\varphi\big(yP(x_{\bullet,\ell}, x_{\bullet,\ell}^*)z\big) = \varphi\big(P(x_{\bullet,\ell}, x_{\bullet,\ell}^*)\big)\varphi\big(yz\big)$$

*for $y, z \in$ *-alg$\{x_{c,k} \mid c \in C, k \in \mathbb{N}, k \neq \ell\}$ and $\ell \in \mathbb{N}$;*

(b) $\mathcal{X}$ *has the factorization property*

$$\varphi\Big(P(x_{\bullet,1}, x_{\bullet,1}^*)^* P(x_{\bullet,2}, x_{\bullet,2}^*)\Big) = \varphi\Big(P(x_{\bullet,1}, x_{\bullet,1}^*)^*\Big)\varphi\Big(P(x_{\bullet,2}, x_{\bullet,2}^*)\Big).$$

*Proof.* All arguments in the proof of Proposition 3.5.4 and Corollary 3.5.5 directly transfer from the spreadable sequence $\mathbf{x}$ to the $C$-jointly spreadable family $\mathcal{X}$, after some minor modifications of notation. □

**Corollary 3.5.7.** *Let $(\mathcal{A}, \varphi)$ be a tracial *-algebraic probability space. The following are equivalent for a $C$-separately spreadable family $\mathcal{X} \equiv \{(x_{c,n})_{n=1}^{\infty} \mid c \in C\} \subset \mathcal{A}$ and some polynomial $P(X_c, Y_c) \in \mathbb{C}\langle X_c, Y_c\rangle$ for some $c \in C$:*

(a) $\mathcal{X}$ *has the factorization property*

$$\varphi\big(yP(x_{c,\ell}, x_{c,\ell}^*)z\big) = \varphi\big(P(x_{c,\ell}, x_{c,\ell}^*)\big)\varphi\big(yz\big)$$

*for* $y, z \in$ *-alg$\{x_{t,k} \mid t \in C, k \in \mathbb{N}, (t,k) \neq (c,\ell)\}$ *and* $\ell \in \mathbb{N}$, $c \in C$;

(b) $\mathcal{X}$ *has the factorization property*

$$\varphi\Big(P(x_{c,1}, x_{c,1}^*)^* P(x_{c,2}, x_{c,2}^*)\Big) = \varphi\Big(P(x_{c,1}, x_{c,1}^*)^*\Big)\varphi\Big(P(x_{c,2}, x_{c,2}^*)\Big).$$

*Proof.* The condition formulated in (b) is a special case of the more general condition (a). The converse is proven using the arguments in the proof of Proposition 3.5.4, after rewriting the factorization condition of (b) as the vanishing condition, as done in the proof of Corollary 3.5.5. □

**Remark 3.5.8.** Corollary 3.5.6 and Corollary 3.5.7 are about factorization properties. Corresponding results on vanishing properties are of course also available. We omit to writing them down explicitly because above factorization properties are more useful within the concrete identification of joint distributions for a concrete spreadable family of random variables.

## 3.6 Large $N$-Limit Models

The next abstract result shows that the joint distributions, as they emerge in the large $N$-limit of multivariate algebraic CLTs, can be modelled on *-algebraic probability spaces by the joint distribution of operators. This folklore result in *-algebraic probability theory can be also found in [Sp93], for example.

**Theorem 3.6.1.** *Suppose $(\mathcal{A}, \varphi)$ is a *-algebraic probability space and the non-empty set $C$ is (at most) countable. Let $\mathcal{X} \equiv \{x_{c,n} \mid n \in \mathbb{N}, c \in C\} \subset \mathcal{A}$ be a $C$-jointly spreadable family satisfying the $C$-joint SVP and put*

$$S_{c,N} := \frac{1}{\sqrt{N}}\big(x_{c,1} + x_{c,2} + \ldots + x_{c,N}\big).$$

*Furthermore, let* $\mathcal{M} = \mathbb{C}\langle y_c, y_c^* \mid c \in C\rangle$ *denote the unital free *-algebra over* $\mathbb{C}$ *generated by the indeterminates* $\{y_c, y_c^* \mid c \in C\}$*. A state* $\psi$ *on* $\mathcal{M}$ *is defined by* $\psi(1_{\mathcal{M}}) := 1$ *and the* $\mathbb{C}$*-linear extension of*

$$\psi\big(y_{\mathbf{c}(1)}^{\boldsymbol{\varepsilon}(1)} y_{\mathbf{c}(2)}^{\boldsymbol{\varepsilon}(2)} \cdots y_{\mathbf{c}(n)}^{\boldsymbol{\varepsilon}(n)}\big) := \lim_{N\to\infty} \varphi\Big(S_{\mathbf{c}(1),N}^{\boldsymbol{\varepsilon}(1)} S_{\mathbf{c}(2),N}^{\boldsymbol{\varepsilon}(2)} \cdots S_{\mathbf{c}(n),N}^{\boldsymbol{\varepsilon}(n)}\Big)$$

*for any* $\mathbf{c}\colon [n] \to C$*, any* $\boldsymbol{\varepsilon}\colon [n] \to \{1, *\}$*, and* $n \in \mathbb{N}$*.*

*Proof.* The $\mathbb{C}$-linearity and unitality of the map $\mathcal{M} \ni y \mapsto \psi(y) \in \mathbb{C}$ are evident from the definition of $\psi$. Since positivity is preserved under pointwise limits, the unital $\mathbb{C}$-linear functional $\psi\colon \mathcal{M} \to \mathbb{C}$ is also positive. $\square$

We discuss further this theorem in the special case where the set $C$ is the singleton set $\{c\}$ such that we can write $x_{c,n}$ just as $x_n$ and $S_{1,N}$ as $S_N$ in Theorem 3.6.1. Furthermore, suppose $x_n = x_n^*$ for all $n$. Then a tracial state $\psi$ is defined on the commutative unital *-algebra $\mathbb{C}[y]$ (generated by the indeterminate $y$ with $y^* := y$) by $\psi(1) = 1$ and the $\mathbb{C}$-linear extension of

$$\psi(y^n) := \lim_{N\to\infty} \varphi(S_N^n) \qquad (n \in \mathbb{N}).$$

We are interested in conditions under which the moment sequence $(a_n)_{n=0}^\infty \subset \mathbb{R}$, with

$$a_n := \psi(y^n),$$

can be obtained as the the moment sequence of a probability measure $\mu$ on $\mathbb{R}$. We recall the following version of the Hamburger Moment Problem (see [RS75, Theorem X.4] for example).

**Theorem 3.6.2.** *A sequence* $(a_n)_{n=0}^\infty \subset \mathbb{R}$ *with* $a_0 = 1$ *is the moment sequence of the probability measure* $\mu$ *on* $\mathbb{R}$*, i.e.*

$$a_n = \int_{\mathbb{R}} t^n \mu(dt) \qquad (n = 0, 1, 2, \ldots),$$

*if and only if, for all* $\ell \in \mathbb{N}$ *and* $\beta_0, \ldots \beta_\ell \in \mathbb{C}$*,*

$$\sum_{m,n=0}^{\ell} \overline{\beta}_n \beta_m a_{n+m} \geq 0.$$

In the following we provide an affirmative answer to this question in the context of algebraic CLTs, by ensuring that the spectral measure is defined for each operator $S_N$.

Suppose that $(\mathcal{A}_N, \varphi_N)$, for $N \in \mathbb{N}$, are *-algebraic probability spaces and $f_N\colon \mathcal{A}_N \to \mathcal{A}_{N+1}$ be injective *-homomorphisms such that $f_N(1_{\mathcal{A}_N}) = 1_{\mathcal{A}_{N+1}}$ and

$\varphi_{N+1} \circ f_N = \varphi_N$ for all $N \in \mathbb{N}$. Then $(\mathcal{A}_N, f_N)_{N\in\mathbb{N}}$ is a directed system of unital *-algebras with direct limit

$$\mathcal{A} := \lim_{\longrightarrow_N} \mathcal{A}_N$$

and $\varphi = \lim\limits_{\longrightarrow_N} \varphi_N$ defines a state on $\mathcal{A}$ such that $(\mathcal{A}_N, \varphi_N)$ is again a *-algebraic probability space. Suppose now that each $\mathcal{A}_N$ is a unital C*-algebra.

We will make use of the following general result when establishing the univariate CLT for $\omega$-sequences in Subsection 4.3.1.

**Theorem 3.6.3.** *Let $(\mathcal{A}_N, \varphi_N)$ be unital *-algebras with the *-algebraic probability space $(\mathcal{A}, \varphi)$ as inductive limit, and with $\mathcal{A}_N$ being a unital C*-algebra for each $N$, as described above. Suppose that the spreadable sequence $\mathbf{x} = (x_n)_{n=1}^{\infty} \subset \mathcal{A}$ satisfies the SVP, and $x_n^* = x_n$ for all $n \in \mathbb{N}$. Furthermore, suppose*

$$S_N := \frac{1}{\sqrt{N}}(x_1 + \ldots + x_N) \in \mathcal{A}_N$$

*for all $N \in \mathbb{N}$. Then there exists a probability measure $\mu$ on $\mathbb{R}$ such that*

$$\lim_{N\to\infty} \varphi(S_N^n) = \int_{\mathbb{R}} t^n \mu(dt) \qquad (n \in \mathbb{N}).$$

*Proof.* Let $N \in \mathbb{N}$ be fixed. As the spectral theorem applies to self-adjoint operators in a C*-algebra, there exists a probability measure (also known as spectral measure) $\mu_N$ on $\mathbb{R}$ such that

$$a_{N,n} := \varphi(S_N^n) = \int_{\mathbb{R}} t^n \mu_N(dt) \qquad (n = 0, 1, 2, \ldots).$$

Consequently, by the Hamburger Moment Problem, for all $\ell \in \mathbb{N}$ and $\beta_0, \ldots \beta_\ell \in \mathbb{C}$,

$$\sum_{m,n=0}^{\ell} \overline{\beta}_n \beta_m a_{N,n+m} \geq 0.$$

We know from the univariate algebraic CLT that $\lim_{N\to\infty} a_{N,n} = a_n$ for all $n$. Thus

$$\sum_{m,n=0}^{\ell} \overline{\beta}_n \beta_m a_{n+m} = \lim_{N\to\infty} \sum_{m,n=0}^{\ell} \overline{\beta}_n \beta_m a_{N,n+m} \geq 0.$$

Consequently, by Theorem 3.6.2, $(a_n)_{n\in\mathbb{N}}$ is the moment sequence of some probability measure $\mu$ on $\mathbb{R}$. $\square$

# Chapter 4

# Central Limit Theorems for a Class of Quantum Coin Tossings

This chapter is the main objective of this thesis. We construct a braidable sequence $\mathbf{x} \equiv (x_n)_{n=1}^{\infty}$ in the infinite algebraic tensor product of complex $2 \times 2$ matrices $\mathcal{A} = \bigotimes_{n=1}^{\infty} \mathbb{M}_2(\mathbb{C})$. Also, we extract algebraic properties of the constructed braidable sequence. In turn we use these algebraic properties to abstractly introduce $\omega$-sequences of partial isometries. We investigate some properties of $\omega$-sequences, as we will need them when establishing CLTs associated to $\omega$-sequences of partial isometries for a (tracial) *-algebraic probability space. In particular, we prove explicit combinatorial formulas for moments as they appear in the large $N$-limit of algebraic CLTs, including their multivariate versions, for $\omega$-sequences. These combinatorial formulas reveal that the moment formulas count oriented crossings of directed ordered pair partitions in the large $N$-limit and differ from those of $q$-Gaussian random variables starting the 8-th moment.

## 4.1 A Concrete Model

**The *-algebraic probability space of the model.** Let $\mathcal{B}_n = \mathcal{B} \otimes \mathcal{B} \otimes \cdots \otimes \mathcal{B}$ denote the $n$-fold algebraic tensor product of a unital *-algebra $\mathcal{B}$ with itself. Furthermore, for $n \in \mathbb{N}$, let the injective *-homomorphism $f_n \colon \mathcal{B}_n \to \mathcal{B}_{n+1}$ be given by

$$f_n(x) = x \otimes 1_{\mathcal{B}}.$$

Then $(\mathcal{B}_n, f_n)_{n \in \mathbb{N}}$ forms a direct system of unital *-algebras with the direct limit

$$\mathcal{B}_\infty := \varinjlim_n \mathcal{B}_n$$

which is again a unital *-algebra. We identify $\mathcal{B}_n$ with its canonical embedding into $\mathcal{B}_\infty$ whenever it is convenient. Therefore, $\mathcal{B}_\infty = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ with

$$\mathcal{B}_n = \underbrace{\mathcal{B} \otimes \mathcal{B} \otimes \cdots \otimes \mathcal{B}}_{n-\text{fold}} \otimes 1_{\mathcal{B}}^{\otimes \mathbb{N}} \subset \mathcal{B}_\infty.$$

We define next a state $\psi_\infty$ on the unital *-algebra $\mathcal{B}_\infty$ as follows. Let $\psi$ be a state on $\mathcal{B}$. Then

$$\psi_n := \psi \otimes \psi \otimes \cdots \otimes \psi$$

defines a tensor product state on $\mathcal{B}_n$ with

$$\psi_n(x_1 \otimes \cdots \otimes x_n) = \psi(x_1) \cdot \psi(x_2) \cdot \ldots \cdot \psi(x_n).$$

Since $\psi_n = \psi_{n+1} \circ f_n$, the limit

$$\psi_\infty := \lim_{n\to\infty} \psi_n$$

exists and defines a state on $\mathcal{B}_\infty$. Indeed, let $x \in \mathcal{B}_\infty$, then there exist some $m \in \mathbb{N}$ such that $x \in \mathcal{B}_m$. Thus

$$\lim_{n\to\infty} \psi_n(x) = \psi_m(x)$$

exists for all $x \in \mathcal{B}_\infty$. Clearly, the map $\mathcal{B}_\infty \ni x \mapsto \psi_\infty(x) \in \mathbb{C}$ is $\mathbb{C}$-linear and $\psi_\infty(1_{\mathcal{A}}) = 1$. The positivity of $\psi_\infty$ is also immediate from

$$\psi_\infty(x^*x) = \lim_{n\to\infty} \psi_n(x^*x) \geq 0.$$

Thus the pair $(\mathcal{B}_\infty, \psi_\infty)$ is a *-algebraic probability space which we will also address as $\otimes_{\mathbb{N}}(\mathcal{B}, \psi)$ in the sequel.

**A representation of the infinite braid group $B_\infty$.** Next we want to introduce a $\varphi$-preserving representation of the infinite braid group $B_\infty$ in $\mathrm{Aut}(\mathcal{B}_\infty)$, the *-automorphisms of $\mathcal{B}_\infty$. Let $T \in \mathrm{Aut}(\mathcal{B} \otimes \mathcal{B})$ such that $(\psi \otimes \psi) \circ T = \psi \otimes \psi$ and

$$(T \otimes 1_{\mathcal{B}})(1_{\mathcal{B}} \otimes T)(T \otimes 1_{\mathcal{B}}) = (1_{\mathcal{B}} \otimes T)(T \otimes 1_{\mathcal{B}})(1_{\mathcal{B}} \otimes T).$$

Furthermore, for $n \in \mathbb{N}$, let $T_n$ denote the amplification of $T$ to a *-automorphism of $\mathcal{B}_\infty$ such that

$$T_n = 1_{\mathcal{B}}^{\otimes_{n-1}} \otimes T \otimes 1_{\mathcal{B}}^{\otimes \mathbb{N}}.$$

Thus

$$T_iT_jT_i = T_jT_iT_j \quad \text{if } |i-j| = 1, \tag{4.1}$$
$$T_iT_j = T_jT_i \quad \text{if } |i-j| > 1. \tag{4.2}$$

It is easy to verify that $\rho(\sigma_k) := T_k$ extends multiplicatively to a representation $\rho$ of the braid group $B_\infty$ in the $\psi_\infty$-preserving *-automorphism of $\mathcal{B}_\infty$, which we will also address as

$$\rho\colon B_\infty \to \mathrm{Aut}(\mathcal{B}_\infty, \psi_\infty).$$

In the following, we consider the special case $\mathcal{B} = \mathbb{M}_2(\mathbb{C})$ and we take $\psi$ to be the normalized trace $\mathrm{tr}_2$ on $\mathbb{M}_2(\mathbb{C})$. Thus we have

$$\begin{aligned}\mathcal{B}_n &= \underbrace{\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C}) \otimes \cdots \otimes \mathbb{M}_2(\mathbb{C})}_{n\text{-fold}},\\ \psi_n &= \underbrace{\mathrm{tr}_2 \otimes \mathrm{tr}_2 \otimes \cdots \otimes \mathrm{tr}_2}_{n\text{-fold}}.\end{aligned}$$

Let $\omega \in \mathbb{C}$ with $|\omega| = 1$. Then

$$U = E_{11} \otimes E_{11} + E_{12} \otimes E_{21} + E_{21} \otimes E_{12} + \omega E_{22} \otimes E_{22} \in \mathbb{M}_2 \otimes \mathbb{M}_2$$

is a unitary matrix in $\mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_2(\mathbb{C})$, which we will identify with the complex $4 \times 4$ matrix

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$$

**Lemma 4.1.1.** *The unitary $U$ satisfies the braid relations*

$$(U \otimes 1_{\mathcal{B}})(1_{\mathcal{B}} \otimes U)(U \otimes 1_{\mathcal{B}}) = (1_{\mathcal{B}} \otimes U)(U \otimes 1_{\mathcal{B}})(1_{\mathcal{B}} \otimes U).$$

*Proof.* This is verified by an elementary computation. □

Now put

$$\begin{aligned} u_1 &= U \otimes 1_2 \otimes 1_2 \otimes \cdots \\ u_2 &= 1_2 \otimes U \otimes 1_2 \otimes 1_2 \otimes \cdots \\ &\vdots \\ u_n &= \underbrace{1_2 \otimes \cdots \otimes 1_2}_{(n-1)\text{-fold}} \otimes U \otimes 1_2 \otimes \cdots \end{aligned}$$

such that $u_1, u_2, \ldots \in \mathcal{B}_\infty$. Recall from Subsection 2.3.3 that $\widetilde{\sigma}_i$ denotes the $i$-th Artin generator of the braid group $B_\infty$.

**Corollary 4.1.2.** *The multiplicative extension of the map $B_\infty \ni \widetilde{\sigma}_i \mapsto u_i \in \mathcal{B}_\infty$ defines a unitary representation of the infinite braid group $B_\infty$.*

*Proof.* Clearly, $u_i u_j = u_j u_i$ for $|i - j| > 1$. The braid relations $u_i u_j u_i = u_j u_i u_j$ for $|i - j| = 1$ are immediate from the braid relations in Lemma 4.1.1. □

Let $\mathrm{Ad}_v(x) := v x v^*$ for any $x, v \in \mathcal{B}_\infty$.

**Corollary 4.1.3.** *The multiplicative extension of the map*

$$B_\infty \ni \widetilde{\sigma}_i \mapsto \rho(\widetilde{\sigma}_i) := \mathrm{Ad}_{u_i} \in \mathrm{Aut}(\mathcal{B}_\infty)$$

*defines a $\psi_\infty$-preserving representation $\rho$ of the infinite braid group $B_\infty$ in the *-automorphisms of $\mathcal{B}_\infty$. Furthermore, this representation $\rho\colon B_\infty \to \mathrm{Aut}(\mathcal{B}_\infty, \psi_\infty)$ has the generating property.*

*Proof.* Let $x \in \mathcal{B}_\infty$. Clearly,

$$\rho(\widetilde{\sigma}_i\widetilde{\sigma}_j)(x) = u_i u_j x u_j^* u_i^* = u_j u_i x u_i^* u_j^* = \rho(\widetilde{\sigma}_j\widetilde{\sigma}_i)(x)$$

for $|i-j| > 1$, and

$$\rho(\widetilde{\sigma}_i\widetilde{\sigma}_j\widetilde{\sigma}_i)(x) = u_i u_j u_i x u_i^* u_j^* u_i^* = u_j u_i u_j x u_j^* u_i^* u_j^* = \rho(\widetilde{\sigma}_j\widetilde{\sigma}_i\widetilde{\sigma}_j)(x)$$

for $|i-j| = 1$. We are left to verify the generating property. It is elementary to check that

$$\mathbb{M}_2(\mathbb{C})^{\otimes n} \otimes 1_2^{\otimes \mathbb{N}} = \mathcal{B}_n \subset \mathcal{B}_\infty^{\rho(\widetilde{\sigma}_{k+1})} = \{x \in \mathcal{B}_\infty \mid \rho(\widetilde{\sigma}_{k+1})(x) = x\}$$

for any $k \geq n$ and thus

$$\mathcal{B}_n \subset \bigcap_{k \geq n} \mathcal{B}_\infty^{\rho(\widetilde{\sigma}_{k+1})}.$$

Now

$$\mathcal{B}_\infty = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n \subset \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} \mathcal{B}_\infty^{\rho(\widetilde{\sigma}_{k+1})} \subset \mathcal{B}_\infty$$

ensures the generating property. □

**Construction of braidable sequences.**

**Proposition 4.1.4.** *Suppose the *-algebraic probability space*

$$(\mathcal{B}_\infty, \psi_\infty) = \bigotimes_{\mathbb{N}} \big(\mathbb{M}_2(\mathbb{C}), \mathrm{tr}_2\big)$$

*is equipped with the representation $\rho\colon B_\infty \to \mathrm{Aut}(\mathcal{B}_\infty, \psi_\infty)$ as given in Corollary 4.1.3 by*

$$\rho(\widetilde{\sigma}_k)x = u_k x u_k^* \qquad (k \in \mathbb{N}, x \in \mathcal{B}_\infty).$$

*Let $y \in \mathbb{M}_2(\mathbb{C})$. Then*

$$x_1 := y \otimes 1_2^{\otimes\mathbb{N}}, \qquad x_2 := u_1 x_1 u_1^*, \qquad x_{n+1} := u_n x_n u_n^*,$$

*defines a braidable sequence $\mathbf{x} \equiv (x_n)_{n=1}^\infty \subset \mathcal{B}_\infty$.*

*Proof.* Clearly $(\mathcal{B}_\infty, \psi_\infty)$ is equipped with a representation $\rho$ of the infinite braid group as required in Definition 2.3.14 such that $x_{n+1} = \rho(\widetilde{\sigma}_n \cdots \widetilde{\sigma}_1)x_1$ for all $n \geq 0$. Thus we are left to verify the localization property $\rho(\widetilde{\sigma}_k)x_1 = x_1$ for all $k \geq 2$. But this property is immediate from

$$\rho(\widetilde{\sigma}_k)x_1 = u_k x_1 u_k^* = (1_2^{\otimes k-1} \otimes u \otimes 1_2^{\otimes \mathbb{N}})(y \otimes 1_2^{\otimes \mathbb{N}})(1_2^{\otimes k-1} \otimes u \otimes 1_2^{\otimes \mathbb{N}})^* = y \otimes 1_2^{\otimes \mathbb{N}}$$

for $k \geq 2$. □

**Proposition 4.1.5.** *The braidable sequence* $\mathbf{x}$ *from Proposition 4.1.4 is spreadable. Moreover,* $\mathbf{x}$ *satisfies the SVP if* $\psi_\infty(x_1) = 0$.

*Proof.* A braidable sequence is spreadable by Theorem 2.3.16. Thus we are left to show that $\mathbf{x}$ has the SVP whenever $\psi_\infty(x_1) = 0$. Note that the centredness of the first random variable $x_1$ implies $\psi_\infty(x_n) = 0$ for all $n \in \mathbb{N}$, as a spreadable sequence is stationary. Due to Proposition 3.5.4, it suffices to check that the vanishing condition

$$\psi_\infty(x_1^* x_2) = 0$$

is valid. But this equation can be verified by a direct computation. □

Let the braidable sequence $\mathbf{x}$ from Proposition 4.1.4 be implemented by the operator $y \in \mathbb{M}_2(\mathbb{C})$ such that

$$x_1 = y \otimes 1_2^{\otimes \mathbb{N}}.$$

We discuss next various choices of $y \in \mathbb{M}_2(\mathbb{C})$. Consider the three Pauli spin matrices, and the identity matrix,

$$\sigma_x := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_y := \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \qquad \sigma_z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad 1_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which form a basis of $\mathbb{M}_2(\mathbb{C})$. We omit the proof of the following elementary result.

**Lemma 4.1.6.** *The following are equivalent for the sequence* $\mathbf{x}$ *from Proposition 4.1.4:*

*(a)* $\mathbf{x}$ *satisfies the SVP;*

*(b)* $\mathrm{tr}_2(y) = 0$*;*

*(c)* $y = a_x\sigma_x + a_y\sigma_y + a_z\sigma_z$ *for some* $a_x, a_y, a_z \in \mathbb{C}$.

Thus we can restrict our attention to the $\mathbb{C}$-linear span of the Pauli spin matrices when choosing $y \in \mathbb{M}_2(\mathbb{C})$ for the study of corresponding CLTs. We will not investigate further the choice $y = \sigma_z$, as the following lemma implies that the corresponding CLT is the classical one, as all random variables are self-adjoint and commute.

**Lemma 4.1.7.** *Let $x_1 := \sigma_z \otimes 1_2^{\otimes \mathbb{N}}$ and let $\omega \in \mathbb{C}$ with $|\omega| = 1$. The braidable sequence* $\mathbf{x}$ *(as constructed in Proposition 4.1.4) is exchangeable and satisfies $x_i x_j = x_j x_i$ for $i \neq j$.*

*Proof.* This follows from the fact that $U(A \otimes B)U^* = B \otimes A$ for any two diagonal matrices $A, B \in \mathbb{M}_2(\mathbb{C})$. □

In the following, we will focus our investigations onto the choice $y = \sigma_x$, as it will provide us with a rich source of spreadable sequences which are not exchangeable. We could have considered equally well the Paul-spin matrix $\sigma_y$ for this purpose, or more generally, a linear combination of both. For the sake of clarity of our results, we refrain from this slight generalization, see also Remark 4.1.10.

**Lemma 4.1.8.** *Let $x_1 := \sigma_x \otimes 1_2^{\otimes \mathbb{N}}$ and $\omega \in \mathbb{C}$ with $|\omega| = 1$. Then the braidable sequence* $\mathbf{x}$ *(as constructed in Proposition 4.1.4) is spreadable but not exchangeable if and only if $\omega \neq \pm 1$.*

*Proof.* Consider the following two mixed moments:

$$\psi_\infty(x_1x_2x_3x_1x_2x_3) = \frac{1}{8}(2\omega^3 + 6\overline{\omega})$$

and

$$\psi_\infty(x_1x_3x_2x_1x_3x_2) = \frac{1}{8}(2\overline{\omega}^3 + 6\omega).$$

Since $\omega^3 + 3\overline{\omega} = \overline{\omega}^3 + 3\omega$ if and only if $\omega = \pm 1$, it follows that $\mathbf{x}$ is not exchangeable for $\omega \neq \pm 1$. Conversely, $\omega = \pm 1$ implies that

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}$$

satisfies $U^2 = 1_2 \otimes 1_2$. Consequently, the unitaries $u_n$ implement a representation of the infinite symmetric group $S_\infty$, such that the sequence $\mathbf{x}$ is exchangeable. □

**A quantum decomposition.** We are interested in studying the joint distribution of the braidable sequence $\mathbf{x}$ for the choice $x_1 = \sigma_x \otimes 1_2^{\otimes \mathbb{N}}$, as required for computing the moments as they appear within *-algebraic CLTs. For example, we will need to understand how a joint moment of the form

$$\psi_\infty(x_1x_2x_3x_1x_2x_3)$$

can be computed systematically. As the random variables $x_n$ do not commute for $|\omega| \neq 1$, brute force computations are only suitable for obtaining explicit formulas for moments of lower order in the considered *-algebraic CLTs. Key to the study of the distribution of the braidable sequence $\mathbf{x}$ with

$$\begin{aligned} x_1 &= \sigma_x \otimes 1_2 \otimes 1_2 \otimes \cdots, \\ x_2 &= u_1 x_1 u_1^*, \\ x_3 &= u_2 x_2 u_2^*, \\ &\vdots \\ x_{n+1} &= u_n x_n u_n^* \end{aligned} \tag{4.3}$$

is the elementary 'quantum decomposition'

$$\sigma_x = a + a^* \qquad \text{with } a := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This 'quantum decomposition' can be done of course for each $x_n$ such that one has

$$x_n = a_n + a_n^*,$$

where the operators $a_n$ are as specified in the next proposition.

**Proposition 4.1.9.** *Let $\omega \in \mathbb{C}$ with $|\omega| = 1$ be fixed and consider the sequence $\mathbf{a} \equiv (a_n)_{n=1}^{\infty} \subset \mathcal{B}_\infty$ defined inductively by*

$$a_1 := a \otimes 1_2^{\otimes \mathbb{N}} \qquad \textit{and} \qquad a_{n+1} := u_n a_n u_n^*.$$

*Then one has, for any $1 \leq i < j < \infty$,*

$$\begin{aligned} a_i a_j &= \omega a_j a_i, & a_i a_i &= 0 \\ a_i a_j^* &= \bar{\omega} a_j^* a_i, & a_i a_i^* + a_i^* a_i &= 1. \end{aligned}$$

*Moreover, the sequence $\mathbf{a}$ has the following properties:*

*(i)* $\mathbf{a}$ *is spreadable;*

*(ii)* $\mathbf{a}$ *has the SVP*

$$\{\ell\} \in \ker(\mathbf{i}) \quad \Longrightarrow \quad \psi_\infty\big(a_{\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots a_{\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\big) = 0$$

*for any $\mathbf{i}\colon [n] \to \mathbb{N}$ and $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$, and $n \in \mathbb{N}$;*

*(iii)* $\mathbf{a}$ *has the factorization properties*

$$\psi_\infty\big(x a_\ell a_\ell^* z\big) = \psi_\infty\big(x a_\ell^* a_\ell z\big) = \frac{1}{2}\psi_\infty(xz)$$

*for any $x, z \in$ *-alg$\big\{a_i \mid i \in \mathbb{N}\backslash\{\ell\}\big\}$ and $\ell \in \mathbb{N}$.*

*Proof.* The relations $a_i a_i = 0$ and $a_i a_i^* + a_i^* a_i = 1$ are clear from $a^2 = 0$ and $aa^* + a^*a = 1_2$. Furthermore, the two relations

$$a_1 a_2 = \omega a_2 a_1, \qquad a_1 a_2^* = \bar{\omega} a_2^* a_1$$

are checked by an elementary computation in the $4 \times 4$ matrices. All remaining relations are inferred from these relations for $a_1$ and $a_2$ by spreadability as follows. Clearly, the sequence $(a_n)_{n=1}^{\infty}$ is spreadable by construction and it is immediate from Theorem 2.3.22 and Corollary 4.1.3 that the unitaries $u_n$ implement partial shifts $\alpha_n \in \operatorname{End}(\mathcal{B}_\infty, \psi_\infty)$ such that

$$\alpha_n(a_m) = \begin{cases} a_m & \text{if } n > m, \\ a_{m+1} & \text{if } n \le m. \end{cases}$$

Thus acting on the equation $a_1 a_2 = \omega a_2 a_1$ with the partial shift $\alpha_1$ gives

$$\begin{aligned} \alpha_1(a_1 a_2) = \alpha_1(\omega a_2 a_1) \quad &\Longleftrightarrow \quad \alpha_1(a_1)\alpha_1(a_2) = \omega \alpha_1(a_2)\alpha_1(a_1) \\ &\Longleftrightarrow \quad a_2 a_3 = \omega a_3 a_2. \end{aligned}$$

Repeatedly doing so, we arrive at

$$\alpha_1^{i-1}(a_1 a_2) = \alpha_1^{i-1}(\omega a_2 a_1) \quad \Longleftrightarrow \quad a_i a_{i+1} = \omega a_{i+1} a_i.$$

Next we continue with hitting the last equation by $\alpha_{i+1}$, to obtain

$$\alpha_{i+1}(a_i a_{i+1}) = \alpha_{i+1}(\omega a_{i+1} a_i) \quad \Longleftrightarrow \quad a_i a_{i+2} = \omega a_{i+2} a_i.$$

Again repeatedly doing so, we arrive at

$$\alpha_{i+1}^{j-(i+1)}(a_i a_{i+1}) = \alpha_{i+1}^{j-(i+1)}(\omega a_{i+1} a_i) \quad \Longleftrightarrow \quad a_i a_j = \omega a_j a_i.$$

The same procedure yields $a_i a_j^* = \bar{\omega} a_j^* a_i$. We are left to verify the claimed three properties. (i) is clear by construction. (ii) The SVP is immediate from Proposition 3.5.1 and the elementary computation

$$\psi_\infty(a_1^* a_2) = \operatorname{tr}_2 \otimes \operatorname{tr}_2\Big((a^* \otimes 1_2)U(a \otimes 1_2)U^*\Big) = 0,$$

with

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}.$$

(iii) A direct computation in the $4 \times 4$ matrices shows that

$$\psi_\infty(a_1 a_1^* a_2 a_2^*) = \psi_\infty(a_1 a_1^*)\psi_\infty(a_2 a_2^*) = \frac{1}{2} \cdot \frac{1}{2}$$

and
$$\psi_\infty(a_1^*a_1a_2^*a_2) = \psi_\infty(a_1^*a_1)\psi_\infty(a_2^*a_2) = \frac{1}{2}\cdot\frac{1}{2}.$$
Using stationarity of the sequence $\mathbf{a}$, another simple calculation in the $2\times 2$ matrices shows $\psi_\infty(a_\ell a_\ell^*) = \psi_\infty(a_\ell^* a_\ell) = \frac{1}{2}$ for all $\ell \in \mathbb{N}$. Now we infer the claimed factorization properties from an application of Corollary 3.5.5 for the noncommutative polynomial $P(X,Y) = XY$ in the indeterminants $X$ and $Y$. $\square$

**Remark 4.1.10.** Suppose we would have taken the Pauli spin matrix $\sigma_y$ instead of $\sigma_x$ for the construction of the braidable sequence $\mathbf{x}$ in Lemma 4.1.8. This would have resulted in the quantum decomposition $\sigma_y = -\mathrm{i}\,a + \mathrm{i}\,a^*$ instead of $\sigma_x = a + a^*$. Defining $b = -\mathrm{i}\,a$ results in operators $b_n := -\mathrm{i}\,a_n$, which satisfy the same set of relations as the $a_n$'s do for $1 \le i < j < \infty$ (see Proposition 4.1.9):
$$b_ib_j = \omega b_jb_i, \qquad b_ib_j^* = \bar\omega b_j^*b_i, \qquad b_ib_i = 0, \qquad b_ib_i^* + b_i^*b_i = 1.$$
More generally, considering self-adjoint linear combinations of $a$ and $a^*$ such that
$$y = \kappa a + \overline{\kappa} a^* \qquad (\kappa \in \mathbb{C}^*),$$
and, more general as before, now defining $b_n := \kappa a_n$ for all $n \ge 1$, one obtains the relations:
$$b_ib_j = \omega b_jb_i, \qquad b_ib_j^* = \bar\omega b_j^*b_i, \qquad b_ib_i = 0, \qquad b_ib_i^* + b_i^*b_i = |\kappa|^2.$$
Note also that all $b_n$'s are centered, i.e. $\psi_\infty(b_n) = 0$, and have the variance
$$\psi_\infty(b_n^*b_n) = \psi_\infty(b_nb_n^*) = \frac{|\kappa|^2}{2}.$$
As a review of our main results shows, this generalization from $\kappa = 1$ to some $\kappa \in \mathbb{C}$ causes mainly a rescaling of mixed moments in the CLTs, according to the covariance of the underlying operator $b_1$.

## 4.2 Combinatorics of the Model

We introduce $\omega$-sequences of partial isometries, to provide an abstract algebraic model for the concrete matrix model studied in Section 4.1. The main result of this section is Theorem 4.2.9, which is about the distributional invariance properties, vanishing and factorization properties, as well as some combinatorial properties of such sequences.

**Definition 4.2.1.** Let $(\mathcal{A},\varphi)$ be a tracial *-algebraic probability space and $\omega \in \mathbb{C}$ with $|\omega| = 1$. A sequence $\mathbf{a} \equiv (a_n)_{n=1}^\infty \subset \mathcal{A}$ satisfying the relations
$$a_ia_j = \omega a_ja_i, \qquad a_ia_j^* = \bar\omega a_j^*a_i \quad \text{for } 1 \le i < j < \infty, \tag{4.4}$$
$$a_ia_i = 0, \qquad a_ia_i^* + a_i^*a_i = 1 \quad \text{for } 1 \le i < \infty. \tag{4.5}$$
is called an *$\omega$-sequence of partial isometries.*

We have already provided a concrete realization for an interesting class of such sequences.

**Remark 4.2.2.** An element $a \in \mathcal{A}$ is called a partial isometry if $p := a^*a$ and $q := aa^*$ are (orthogonal) projections. This is ensured by the relations (4.5) in Definition 4.2.1, as

$$p^2 = a^*aa^*a = a^*(1-a^*a)a = a^*a = p,$$
$$q^2 = aa^*aa^* = a(1-aa^*)a^* = aa^* = q.$$

(The 'orthogonality' $p^* = p$ and $q^* = q$ is automatic.)

**Remark 4.2.3.** The defining relations of an $\omega$-sequence can be also written as

$$a_ia_j - \omega_{i,j}a_ja_i = 0, \qquad a_ia_j^* - \overline{\omega}_{i,j}a_j^*a_i = \delta_{i,j} \qquad (1 \leq i,j < \infty),$$

where $\omega_{i,j} = \omega$ for $i < j$ and $\omega_{i,j} = -1$ for $i = j$ and $\omega_{i,j} = \overline{\omega}_{j,i}$ for $i > j$.

**Proposition 4.2.4.** *Let $(\mathcal{A}, \varphi) = \bigotimes_{\mathbb{N}} \big(\mathbb{M}_2(\mathbb{C}), \mathrm{tr}_2\big)$. The sequence $\mathbf{a} \subset \mathcal{A}$, as introduced in Proposition 4.1.9 for some $\omega \in \mathbb{C}$ with $|\omega| = 1$, is an $\omega$-sequence of partial isometries.*

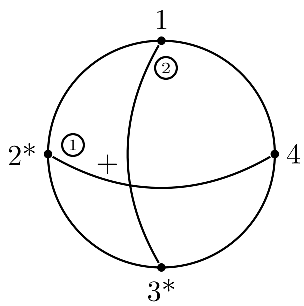
*Proof.* All defining properties of an $\omega$-sequence of partial isometries are verified in Proposition 4.1.9. □

We have already shown that such concrete $\omega$-sequences of partial isometries (as considered in Proposition 4.1.9) are spreadable and have the SVP, as well as certain factorization properties. Here we will show that this distributional invariance property, as well as the related SVP and factorization property are enjoyed by any $\omega$-sequences of partial isometries. Next we illustrate some of these properties by elementary examples before studying them in full generality. We will constrain ourselves in these examples to apply mainly algebraic and geometric arguments, which can be later deployed in full generality during the proof of our main result, Theorem 4.2.9.

**Example 4.2.5.** We compute the moment $\varphi(a_3a_1^*a_3^*a_2a_1)$. The relations (4.5) imply $a_2 = a_2(a_2a_2^* + a_2^*a_2) = a_2a_2^*a_2$. Since $a_2^*a_2a_i = a_ia_2^*a_2$ for $i \neq 2$ and $\varphi$ is a trace, the factor $(a_2^*a_2)$ can be moved cyclically around until it appears on the left side of $a_2$ such that

$$\varphi\big(a_3a_1^*a_3^*a_2a_1\big) = \varphi\big(a_3a_1^*a_3^*a_2(a_2^*a_2)a_1\big) = \varphi\big(a_3a_1^*a_3^*(a_2^*a_2)a_2a_1\big) = 0.$$

Here the last equality is immediate from $a_2a_2 = 0$. Thus we have verified the SVP for this particular moment.



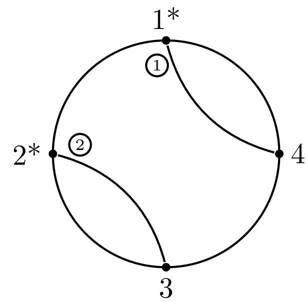




Figure 4.1: $\boldsymbol{\varepsilon}$-restricted ordered partition $\big(V_{\sigma(1)}, V_{\sigma(2)}\big) = (\{2,4\},\{1,3\}) \in \mathcal{OP}_2(4,\boldsymbol{\varepsilon})$ with blocks $V_1 = \{1,3\}$ and $V_2 = \{2,4\}$, direction data $\boldsymbol{\varepsilon}(1) = \boldsymbol{\varepsilon}(4) = 1$ and $\boldsymbol{\varepsilon}(2) = \boldsymbol{\varepsilon}(3) = *$, and permutation $\sigma = (1,2)$ (left), and $\widetilde{\boldsymbol{\varepsilon}}$-restricted ordered partition $\big(\widetilde{V}_{\widetilde{\sigma}(1)}, \widetilde{V}_{\widetilde{\sigma}(2)}\big) = (\{1,4\},\{2,3\}) \in \mathcal{OP}_2(4,\widetilde{\boldsymbol{\varepsilon}})$ with blocks $\widetilde{V}_1 = \{1,4\}$ and $\widetilde{V}_2 = \{2,3\}$, direction data $\widetilde{\boldsymbol{\varepsilon}}(1) = \widetilde{\boldsymbol{\varepsilon}}(2) = *$ and $\widetilde{\boldsymbol{\varepsilon}}(3) = \widetilde{\boldsymbol{\varepsilon}}(4) = 1$, and permutation $\widetilde{\sigma} = e$ (right).

**Example 4.2.6.** Consider the moment $\varphi(a_3a_1^*a_3^*a_2a_2^*a_1)$. We use the relations (4.4) and the trace property of $\varphi$ to move the factor $a_2^*$ cyclically around until it appears on the left side of the factor $a_2$. This will of course cause that factors of $\omega$ or $\overline{\omega}$ are picked up while applying the relevant relations. But as we observe from the particular structure of this moment, whenever we use relations for $a_2^*a_i$ to obtain $\omega^{\pm 1}a_ia_2^*$ for $i \neq 2$, we will also have to apply them to $a_2^*a_i^*$, to obtain $\omega^{\mp 1}a_i^*a_2^*$. Since $\omega^{\pm 1}\omega^{\mp 1} = 1$, we arrive at

$$\varphi(a_3a_1^*a_3^*(a_2a_2^*)a_1) = \varphi(a_3a_1^*a_3^*a_2^*a_2a_1) = \frac{1}{2}\varphi(a_3a_1^*a_3^*a_1).$$

Here the last equality is immediate from $a_2a_2^* + a_2a_2^* = 1$. Thus we have verified the following factorization property for this particular moment:

$$\varphi(a_3a_1^*a_3^*(a_2a_2^*)a_1) = \varphi\big(a_2a_2^*\big)\varphi\big(a_3a_1^*a_3^*a_1\big).$$

**Example 4.2.7.** Consider the moment $\varphi(a_2a_1^*a_2^*a_1)$. We find by algebraic manipulations that

$$\begin{aligned}
\varphi(a_2a_1^*a_2^*a_1) &= \omega\varphi(a_1^*a_2a_2^*a_1) && \text{(moving } a_2 \text{ to the left of } a_2^* \text{ with (4.4))}\\
&= \omega\varphi(a_1^*a_2^*a_2a_1) && \text{(moving } a_2 \text{ to the right with relations,}\\
& && \text{ using traciality and again relations)}\\
&= \frac{\omega}{2}\varphi\big(a_1^*(a_2a_2^* + a_2^*a_2)a_1\big)\\
&= \frac{\omega}{2}\varphi\big(a_1^*a_1\big) = \frac{1}{2^2}\omega && \text{(by relations (4.5) and traciality).}
\end{aligned}$$

On the other hand, the geometric realization of directed ordered pair partitions provides us with the notion of oriented crossings (see Subsection 2.1.4) such that

$$\varphi(a_2a_1^*a_2^*a_1) = \frac{1}{2^2}\omega^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}.$$

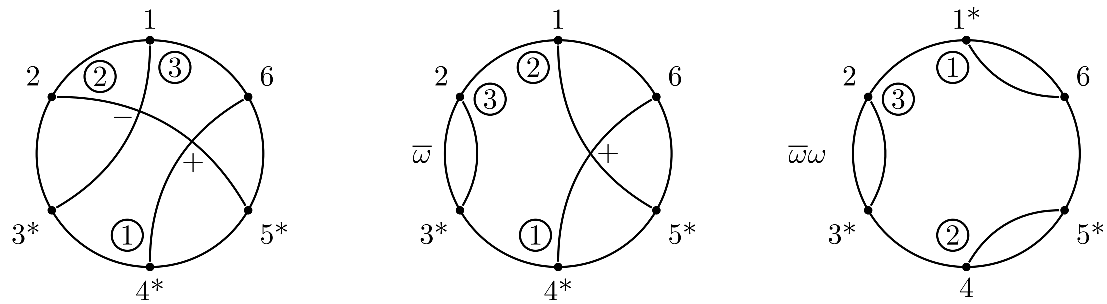



Figure 4.2: Graphical computation of $\varphi(a_3a_2a_3^*a_1^*a_2^*a_1)$. The middle diagram is obtained from the left by moving the target of the 3-labelled directed arc from 1 to 2, and the target of the 2-labelled directed arc from 2 to 1. As this removes a crossing with negative orientation, the middle diagram is multiplied by the factor $\overline{\omega}$. The right diagram is obtained from the middle diagram by moving the source of the 2-labelled directed arc from 1 to 4 and the target of the 1-labelled directed arc from 4* to 1*. As this removes a crossing with positive orientation, the right diagram is multiplied with the factor $\omega$. Finally, removing a non-crossing arc from the diagram produces a factor $\frac{1}{2}$. Removing all three non-crossing arcs we arrive at the empty circle multiplied by the factor $\frac{1}{2^3}\omega\overline{\omega}$.

Here $\mathrm{cr}_+(\pi, \boldsymbol{\varepsilon}, \sigma)$ denotes the number of crossings with positive orientation, given the standard ordered pair partition $\pi = \{\{1,3\},\{2,4\}\} \in \mathcal{P}(4, \boldsymbol{\varepsilon})$, the direction map $\boldsymbol{\varepsilon}\colon [4] \to \{*, 1\}$ with $\boldsymbol{\varepsilon}(2) = \boldsymbol{\varepsilon}(3) = *$ and $\boldsymbol{\varepsilon}(1) = \boldsymbol{\varepsilon}(4) = 1$, and permutation $\sigma = (1,2) \in S_2$. We have illustrated the directed ordered pair partitions underlying the equation $\varphi(a_2a_1^*a_2^*a_1) = \omega\varphi(a_1^*a_2a_2^*a_1)$ in Figure 4.1. Note that one has also

$$\varphi\big(a_2a_1^*a_2^*a_1\big) = \varphi\big(a_{\mathbf{i}(1)}a_{\mathbf{i}(2)}^*a_{\mathbf{i}(3)}^*a_{\mathbf{i}(4)}\big)$$

for any index map $\mathbf{i}\colon [4] \to \mathbb{N}$ with $\mathbf{i}(1) = \mathbf{i}(3) > \mathbf{i}(2) = \mathbf{i}(4)$, as the same algebraic manipulations and geometric realization can be carried out.

**Example 4.2.8.** We compute the moment $\varphi(a_3a_2a_3^*a_1^*a_2^*a_1)$ using the algebraic method. First we move the factor $a_3$ to the right of $a_3^*$ with the help of the relations (4.4). Next we apply a factorization result similar to Example 4.2.6, to obtain

$$\varphi(a_3a_2a_3^*a_1^*a_2^*a_1) = \overline{\omega}\varphi(a_2a_3a_3^*a_1^*a_2^*a_1) = \frac{1}{2}\overline{\omega}\varphi(a_2a_1^*a_2^*a_1).$$

Now we can use the result from Example 4.2.7 to fully compute this moment as

$$\varphi(a_3a_2a_3^*a_1^*a_2^*a_1) = \frac{1}{2^3}\omega\overline{\omega}.$$

The right-hand side simplifies of course further to $\frac{1}{2^3}$, due to the unimodularity of $\omega$. But we resist this final algebraic simplification as it obscures the counting of oriented crossings in the evaluation of such moments. Using the geometric method of counting oriented crossings, above formula reads as

$$\varphi(a_3 a_2 a_3^* a_1^* a_2^* a_1) = \frac{1}{2^3}\omega^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\overline{\omega}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)},$$

where the balanced standard ordered pair partition $\pi = \{V_1, V_2, V_3\} \in \mathcal{P}_2(6, \boldsymbol{\varepsilon})$ is given by $V_1 = \{1, 3\}$, $V_2 = \{2, 5\}$ and $V_3 = \{4, 6\}$. The balanced direction map $\boldsymbol{\varepsilon}\colon [6] \to \{*, 1\}$ is given by $\boldsymbol{\varepsilon}(1) = \boldsymbol{\varepsilon}(2) = \boldsymbol{\varepsilon}(6) = 1$ and $\boldsymbol{\varepsilon}(3) = \boldsymbol{\varepsilon}(4) = \boldsymbol{\varepsilon}(5) = *$, and finally the permutation $\sigma \in S_3$ is given by $\sigma(1) = 3$, $\sigma(2) = 2$ and $\sigma(3) = 1$. One verifies $\mathrm{cr}_+(\pi, \boldsymbol{\varepsilon}, \sigma) = 1$ and $\mathrm{cr}_-(\pi, \boldsymbol{\varepsilon}, \sigma) = 1$ with the help of the right-hand rule for oriented crossings. We have also provided a graphical computation of the value of $\varphi(a_3 a_2 a_3^* a_1^* a_2^* a_1)$ in Figure 4.2. Note that the steps of this graphical computation corresponds to algebraic manipulations, which are slightly different than those done above.

**Theorem 4.2.9.** *Let $(\mathcal{A}, \varphi)$ be a tracial *-algebraic probability space and $\omega \in \mathbb{C}$ with $|\omega| = 1$. An $\omega$-sequence of partial isometries $\mathbf{a} \equiv (a_n)_{n=1}^\infty \subset \mathcal{A}$ has the following properties:*

*(i)* $\mathbf{a}$ *is spreadable;*

*(ii)* $\mathbf{a}$ *has the SVP*

$$\{\ell\} \in \ker(\mathbf{i}) \quad \Longrightarrow \quad \varphi\big(a_{\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots a_{\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\big) = 0$$

*for any $n \in \mathbb{N}$, $\mathbf{i}\colon [n] \to \mathbb{N}$, and $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$;*

*(iii)* $\mathbf{a}$ *has the factorization property*

$$\varphi\big(y a_\ell a_\ell^* z\big) = \varphi\big(y a_\ell^* a_\ell z\big) = \frac{1}{2}\varphi(yz)$$

*for any $y, z \in$ *-alg$\,\{a_i \mid i \in \mathbb{N}\backslash\{\ell\}\}$ and $\ell \in \mathbb{N}$;*

*(iv)* $\mathbf{a}$ *has the balanced pair distribution*

$$\varphi\big(a_{\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots a_{\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\big) = \frac{1}{2^k}\,\omega^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\,\overline{\omega}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}$$

*for any $n \in \mathbb{N}$, $\mathbf{i}\colon [n] \to \mathbb{N}$ with $\ker(\mathbf{i}) \in \mathcal{P}_2(n, \boldsymbol{\varepsilon})$, and $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$. Here the pair partition $\pi = \{V_1, V_2, \ldots, V_k\} \in \mathcal{P}_2(n, \boldsymbol{\varepsilon})$, the permutation $\sigma \in S_k$ and $k \in \mathbb{N}$ are uniquely determined by $\ker(\mathbf{i}) = \pi$ and $\ker_{\mathcal{O}}(\mathbf{i}) = (V_{\sigma(1)}, \ldots, V_{\sigma(k)}) \in \mathcal{OP}_2(n, \boldsymbol{\varepsilon})$ with $n = 2k$.*

*Proof.* As known for partial isometries, we conclude from the relations (4.5) that, for all $p > 1$,

$$(a_i a_i^*)^{p+1} = (a_i a_i^*)^p (1 - a_i^* a_i) = (a_i a_i^*)^p$$

and, similarly,

$$(a_i^*a_i)^{p+1} = (a_i^*a_i)^p(1 - a_ia_i^*) = (a_i^*a_i)^p.$$

Thus one has that, for all $i, p \in \mathbb{N}$,

$$(a_ia_i^*)^p = a_ia_i^*, \qquad (a_i^*a_i)^p = a_i^*a_i \tag{4.6}$$

$$(a_ia_i^*)^pa_i = a_i, \qquad (a_i^*a_i)^pa_i^* = a_i^*. \tag{4.7}$$

Furthermore, the relations (4.4) imply the commutation relations

$$(a_ia_i^*)a_j = a_j(a_ia_i^*), \qquad (a_i^*a_i)a_j = a_j(a_i^*a_i) \qquad (i \neq j). \tag{4.8}$$

We start with proving the SVP as claimed in (ii). Let $y, z \in$ *-alg$\{a_i \mid i \in \mathbb{N}\backslash\{k\}\}$ for some $k \in \mathbb{N}$. We compute

$$\varphi(ya_kz) = \varphi(ya_ka_k^*a_kz) = \varphi(ya_kza_k^*a_k) = \varphi(a_k^*a_kya_kz) = \varphi(ya_k^*a_ka_kz) = 0,$$

where each equality is valid due to (4.7), the commutation relations from (4.8), traciality, again the commutation relations from (4.8), and finally (4.5). A similar computation establishes $\varphi(ya_k^*z) = 0$. Thus we have shown the SVP.
We show next the factorization property stated in (iii). The equation

$$\varphi\big(ya_\ell a_\ell^*z\big) = \varphi\big(ya_\ell^*a_\ell z\big) = \frac{1}{2}\varphi(yz)$$

is valid if we can show the first equality, as the second equality is immediate from $a_\ell a_\ell^* + a_\ell^*a_\ell = 1$. Thus, using traciality, it suffices to verify

$$\varphi(xa_\ell a_\ell^*) = \varphi(a_\ell^*xa_\ell) = \varphi(xa_\ell^*a_\ell)$$

for elements $x \in$ *-alg$\big\{a_i \mid i \in \mathbb{N}\backslash\{\ell\}\big\}$. Furthermore, using the relations (4.4), (4.5) and the equations (4.6) and (4.7), it is sufficient to consider elements $x$ which are of the normal form

$$x = x_{i_1}x_{i_2}\cdots x_{i_n} \tag{NF}$$

with $x_i \in \{a_i, a_i^*, a_ia_i^*, a_i^*a_i\}$ for $i \in \mathbb{N}$ and $1 \leq i_1 < i_2 < \cdots < i_n$ for some $n \in \mathbb{N}$ and $i_k \neq \ell$ for all $k = 1, 2, \ldots, n$. Using the SVP from (ii), one has $\varphi(xa_\ell a_\ell^*) = 0 = \varphi(x)$ if $x$ contains a factor of the form $x_{i_k} = a_{i_k}$ or $x_{i_k} = a_{i_k}^*$ for some $k \in [n]$. Thus we are left to consider elements $x$ where all factors are of the balanced form $x_j = a_ja_j^*$ or $x_j = a_j^*a_j$. Using the commutation relations (4.8), we calculate

$$\varphi(xa_\ell a_\ell^*) = \varphi(a_\ell^*xa_\ell) = \varphi(xa_\ell^*a_\ell)$$

for elements $x$ which are of this balanced form. This ensures the validity of the factorization property stated in (iii).
We show next (i), the spreadability of the sequence. Let $n \in \mathbb{N}$ and $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$ be given. Suppose $\mathbf{i}, \mathbf{j}\colon [n] \to \mathbb{N}$ satisfy $\mathbf{i} \sim_{\mathcal{O}} \mathbf{j}$. We need to show

$$\varphi\big(a_{\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)}\cdots a_{\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\big) = \varphi\big(a_{\mathbf{j}(1)}^{\boldsymbol{\varepsilon}(1)}\cdots a_{\mathbf{j}(n)}^{\boldsymbol{\varepsilon}(n)}\big).$$

We infer from the order equivalence of the two functions $\mathbf{i}$ and $\mathbf{j}$ that, for any $k, \ell \in [n]$.

$$a_{\mathbf{i}(k)}^{\varepsilon(k)} a_{\mathbf{i}(\ell)}^{\varepsilon(\ell)} = \omega a_{\mathbf{i}(\ell)}^{\varepsilon(\ell)} a_{\mathbf{i}(k)}^{\varepsilon(k)} \quad \Longleftrightarrow \quad a_{\mathbf{j}(k)}^{\varepsilon(k)} a_{\mathbf{j}(\ell)}^{\varepsilon(\ell)} = \omega a_{\mathbf{j}(\ell)}^{\varepsilon(\ell)} a_{\mathbf{j}(k)}^{\varepsilon(k)}$$

and

$$a_{\mathbf{i}(k)}^{\varepsilon(k)} a_{\mathbf{i}(\ell)}^{\varepsilon(\ell)} = \overline{\omega} a_{\mathbf{i}(\ell)}^{\varepsilon(\ell)} a_{\mathbf{i}(k)}^{\varepsilon(k)} \quad \Longleftrightarrow \quad a_{\mathbf{j}(k)}^{\varepsilon(k)} a_{\mathbf{j}(\ell)}^{\varepsilon(\ell)} = \overline{\omega} a_{\mathbf{j}(\ell)}^{\varepsilon(\ell)} a_{\mathbf{j}(k)}^{\varepsilon(k)}.$$

We can apply these relations to bring $a_{\mathbf{i}(1)}^{\varepsilon(1)} \cdots a_{\mathbf{i}(n)}^{\varepsilon(n)}$ and $a_{\mathbf{j}(1)}^{\varepsilon(1)} \cdots a_{\mathbf{j}(n)}^{\varepsilon(n)}$ simultaneously into a normal form (NF) (as we have already used it for the proof of (iii)). Thus we may assume without loss of generality that both $a_{\mathbf{i}(1)}^{\varepsilon(1)} \cdots a_{\mathbf{i}(n)}^{\varepsilon(n)}$ and $a_{\mathbf{j}(1)}^{\varepsilon(1)} \cdots a_{\mathbf{j}(n)}^{\varepsilon(n)}$ are in normal form with $\mathbf{i} \sim_{\mathcal{O}} \mathbf{j}$. More explicitly,

$$a_{\mathbf{i}(1)}^{\varepsilon(1)} \cdots a_{\mathbf{i}(n)}^{\varepsilon(n)} = x_{i_1} x_{i_2} \cdots x_{i_k} \qquad \text{for } 1 \le i_1 < i_2 < \cdots < i_k,$$

and

$$a_{\mathbf{j}(1)}^{\varepsilon(1)} \cdots a_{\mathbf{j}(n)}^{\varepsilon(n)} = x_{j_1} x_{j_2} \cdots x_{j_k} \qquad \text{for } 1 \le j_1 < j_2 < \cdots < j_k,$$

where $(i_1, i_2, \ldots, i_k) \sim_{\mathcal{O}} (j_1, j_2, \ldots, j_k)$ and, for all $1 \le r \le k$,

$$\begin{aligned} x_{i_r} &= a_{i_r} &\Longleftrightarrow\quad x_{j_r} &= a_{j_r}, \\ x_{i_r} &= a_{i_r}^* &\Longleftrightarrow\quad x_{j_r} &= a_{j_r}^*, \\ x_{i_r} &= a_{i_r} a_{i_r}^* &\Longleftrightarrow\quad x_{j_r} &= a_{j_r} a_{j_r}^*, \\ x_{i_r} &= a_{i_r}^* a_{i_r} &\Longleftrightarrow\quad x_{j_r} &= a_{j_r}^* a_{j_r}. \end{aligned}$$

Now we infer from the SVP (see (ii)) or repeatedly applying the factorization property (iii) that either

$$\varphi(x_{i_1} x_{i_2} \cdots x_{i_k}) = 0 = \varphi(x_{j_1} x_{j_2} \cdots x_{j_k})$$

or

$$\varphi(x_{i_1} x_{i_2} \cdots x_{i_k}) = \frac{1}{2^k} = \varphi(x_{j_1} x_{j_2} \cdots x_{j_k}).$$

Consequently, the sequence $\mathbf{a}$ is spreadable.

(iv) We are left to prove the concrete formula for balanced pair distributions in terms of oriented crossings. As the sequence $\mathbf{a}$ is spreadable (see (i)) and $\ker(\mathbf{i}) \subset \mathcal{P}_2(n, \varepsilon)$, we may assume $\mathbf{i}\colon [n] \to [k]$ with $n = 2k$ without loss of generality. We will show the validity of the formula

$$\varphi\big(a_{\mathbf{i}(1)}^{\varepsilon(1)} \cdots a_{\mathbf{i}(2k)}^{\varepsilon(2k)}\big) = \frac{1}{2^k}\, \omega^{\mathrm{cr}_+(\pi,\varepsilon,\sigma)}\, \overline{\omega}^{\mathrm{cr}_-(\pi,\varepsilon,\sigma)} \tag{M$_k$}$$

by finite induction on $k$. Note that (M$_k$) is valid if and only if (M$_k$) is valid for a particular (and thus every) cyclic permutation

$$\varphi\big(a_{\mathbf{i}(1)}^{\varepsilon(1)} \cdots a_{\mathbf{i}(2k)}^{\varepsilon(2k)}\big) = \varphi\big(a_{\mathbf{i}(\ell)}^{\varepsilon(\ell)} \cdots a_{\mathbf{i}(2k)}^{\varepsilon(2k)} a_{\mathbf{i}(1)}^{\varepsilon(1)} \cdots a_{\mathbf{i}(\ell-1)}^{\varepsilon(\ell-1)}\big) \qquad (1 \le \ell \le 2k).$$

Clearly, $(\mathrm{M}_k)$ is true for $k=1$ since

$$\varphi(a_{\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)}a_{\mathbf{i}(2)}^{\boldsymbol{\varepsilon}(2)})=\frac{1}{2}$$

for $\mathbf{i}\colon[2]\to[1]$ with $\ker(\mathbf{i})\in\mathcal{P}_2(2,\boldsymbol{\varepsilon})$ and $\boldsymbol{\varepsilon}\colon[2]\to\{*,1\}$ balanced. Assume now that $(\mathrm{M}_{k-1})$ is true for $k\geq 2$. By traciality, it suffices to consider moments of the form

$$\varphi\big(a_k a_k^* a_{\mathbf{i}(3)}^{\boldsymbol{\varepsilon}(3)}\cdots a_{\mathbf{i}(2k)}^{\boldsymbol{\varepsilon}(2k)}\big)\quad\text{with } \mathbf{i}(1)=\mathbf{i}(2)=k \tag{A}$$

or

$$\varphi\big(a_k a_{\mathbf{i}(2)}^{\boldsymbol{\varepsilon}(2)}\cdots a_{\mathbf{i}(2k-1)}^{\boldsymbol{\varepsilon}(2k-1)}a_k^*\big)\quad\text{with } \mathbf{i}(1)=\mathbf{i}(2k)=k. \tag{B}$$

or

$$\varphi\big(a_k a_{\mathbf{i}(2)}^{\boldsymbol{\varepsilon}(2)}\cdots a_{\mathbf{i}(r-1)}^{\boldsymbol{\varepsilon}(r-1)}a_k^* a_{\mathbf{i}(r+1)}^{\boldsymbol{\varepsilon}(r+1)}\cdots a_{\mathbf{i}(2k)}^{\boldsymbol{\varepsilon}(2k)}\big)\quad\begin{array}{c}\text{with } \mathbf{i}(1)=\mathbf{i}(r)=k\\ \text{and } 3\leq r<2k,\end{array} \tag{C}$$

**Case** (A). The factorization property (iii) applies such that

$$\varphi\big(a_k a_k^* a_{\mathbf{i}(3)}^{\boldsymbol{\varepsilon}(3)}\cdots a_{\mathbf{i}(2k)}^{\boldsymbol{\varepsilon}(2k)}\big)=\frac{1}{2}\varphi\big(a_{\mathbf{i}(3)}^{\boldsymbol{\varepsilon}(3)}\cdots a_{\mathbf{i}(2k)}^{\boldsymbol{\varepsilon}(2k)}\big).$$

Clearly, the number of crossings with positive or negative orientation is unchanged upon removing the factor $a_k a_k^*$ under the trace. Consequently, $(\mathrm{M}_{k-1})$ implies $(\mathrm{M}_k)$ in this case.

**Case** (B). This case is reduced to Case (A) by traciality and interchanging the roles of $a_k$ and $a_k^*$. Thus we have also concluded in this case that $(\mathrm{M}_{k-1})$ implies $(\mathrm{M}_k)$.

**Case** (C). One has

$$a_k a_{\mathbf{i}(s)}=\overline{\omega}a_k a_{\mathbf{i}(s)}\qquad\text{and}\qquad a_k a_{\mathbf{i}(s)}^*=\omega a_k a_{\mathbf{i}(s)}^*$$

since $k>\mathbf{i}(s)$ for all $3\leq s<r$. Using these algebraic relations we obtain

$$\begin{aligned}&\varphi\big(a_k a_{\mathbf{i}(2)}^{\boldsymbol{\varepsilon}(2)}\cdots a_{\mathbf{i}(r-1)}^{\boldsymbol{\varepsilon}(r-1)}a_k^* a_{\mathbf{i}(r+1)}^{\boldsymbol{\varepsilon}(r+1)}\cdots a_{\mathbf{i}(2k)}^{\boldsymbol{\varepsilon}(2k)}\big)\\ &\qquad=\omega^{r_+}\overline{\omega}^{r_-}\varphi\big(a_{\mathbf{i}(2)}^{\boldsymbol{\varepsilon}(2)}\cdots a_{\mathbf{i}(r-1)}^{\boldsymbol{\varepsilon}(r-1)}a_k a_k^* a_{\mathbf{i}(r+1)}^{\boldsymbol{\varepsilon}(r+1)}\cdots a_{\mathbf{i}(2k)}^{\boldsymbol{\varepsilon}(2k)}\big)\end{aligned}$$

with $r_+:=\operatorname{card}\boldsymbol{\varepsilon}^{-1}_{|\{2,\ldots,r-1\}}(\{*\})$ and $r_-:=\operatorname{card}\boldsymbol{\varepsilon}^{-1}_{|\{2,\ldots,r-1\}}(\{1\})$.

Let $(\overline{\pi},\boldsymbol{\varepsilon},\sigma)\in\mathcal{P}_2(2k,\boldsymbol{\varepsilon})\times\big\{f\colon[2k]\to\{*,1\}\big\}\times S_k$ be the triple uniquely associated to $\pi=\ker_{\mathcal{O}}(\mathbf{i})\in\mathcal{OP}_2(2k,\boldsymbol{\varepsilon})$ (see Corollary 2.1.34 and Definition 2.1.35).

Now let the index map $\widetilde{\mathbf{i}}\colon [2k] \to [k]$ and the direction map $\widetilde{\boldsymbol{\varepsilon}}\colon [2k] \to \{*, 1\}$ be given by

$$\big(\widetilde{\mathbf{i}}(\ell), \widetilde{\boldsymbol{\varepsilon}}(\ell)\big) := \begin{cases} \big(\mathbf{i}(\ell+1), \boldsymbol{\varepsilon}(\ell+1)\big) & \text{for } 1 \le \ell \le r-2, \\ \big(\mathbf{i}(1), 1\big) & \text{for } \ell = r-1, \\ \big(\mathbf{i}(r), *\big) & \text{for } \ell = r, \\ \big(\mathbf{i}(\ell), \boldsymbol{\varepsilon}(\ell)\big) & \text{for } r < \ell \le 2k, \end{cases}$$

such that

$$\varphi\big(a_{\widetilde{\mathbf{i}}(1)}^{\widetilde{\boldsymbol{\varepsilon}}(1)} \cdots a_{\widetilde{\mathbf{i}}(2k)}^{\widetilde{\boldsymbol{\varepsilon}}(2k)}\big) = \varphi\big(a_{\mathbf{i}(2)}^{\boldsymbol{\varepsilon}(2)} \cdots a_{\mathbf{i}(r-1)}^{\boldsymbol{\varepsilon}(r-1)} a_k a_k^* a_{\mathbf{i}(r+1)}^{\boldsymbol{\varepsilon}(r+1)} \cdots a_{\mathbf{i}(2k)}^{\boldsymbol{\varepsilon}(2k)}\big).$$

As before, let $(\overline{\widetilde{\pi}}, \widetilde{\boldsymbol{\varepsilon}}, \widetilde{\sigma}) \in \mathcal{P}_2(2k, \boldsymbol{\varepsilon}) \times \big\{f\colon [2k] \to \{*, 1\}\big\} \times S_k$ be triple uniquely associated to $\widetilde{\pi} = \ker_{\mathcal{O}}(\widetilde{\mathbf{i}}) \in \mathcal{OP}_2(2k, \boldsymbol{\varepsilon})$. We claim that

$$\mathrm{cr}_\pm(\overline{\pi}, \boldsymbol{\varepsilon}, \sigma) - \mathrm{cr}_\pm(\overline{\widetilde{\pi}}, \widetilde{\boldsymbol{\varepsilon}}, \widetilde{\sigma}) = r_\pm - r_0$$

where

$$r_0 := \mathrm{card}\{V \in \ker(\mathbf{i}) \mid V \subset [r]\} - 1 = \mathrm{card}\{V \in \ker(\widetilde{\mathbf{i}}) \mid V \subset [r]\} - 1$$

is the number of pairs of $\pi$ in the set $[2, r-1]$. (This number equals of course the number of pairs of $\widetilde{\pi}$ in the set $[1, r-2]$.) Clearly, a pair $\{a, b\}$ (with $a < b$) and the pair $\{1, r\}$ are crossing if and only if $1 < a < r < b$. Turning the ordered partition $\pi$ into $\widetilde{\pi}$ (through algebraic relations) causes that this pair $\{a-1, b\}$ and $\{r-1, r\}$ are non-crossing. Thus a crossing with positive orientation or negative orientation is removed in the geometric picture while a factor $\omega$ or $\overline{\omega}$ is created by the corresponding algebraic relations, respectively. Now suppose that the pair $\{a, b\}$ is sitting 'inside' of the pair $\{1, r\}$, i.e. $1 < a < b < r$. Thus $\{a, b\}$ and $\{1, r\}$ are non-crossing. Moving the factor $a_k$ algebraically from the 1-st position to the $(r-1)$-th position in the considered monomial, one arrives again at that the pair $\{a-1, b-1\}$ and $\{r-1, r\}$ are non-crossing. So the number of crossings with positive and negative orientation is unchanged by such a pair $\{a, b\}$ when algebraically turning $\pi$ into $\widetilde{\pi}$. But these algebraic operations produce an additional factor $\omega$ and a factor $\overline{\omega}$ which are counted by $r_+$ and $r_-$, respectively. Thus the number $r_\pm$ is larger or equal to $\mathrm{cr}_\pm(\pi, \boldsymbol{\varepsilon}, \sigma) - \mathrm{cr}_\pm(\widetilde{\pi}, \widetilde{\boldsymbol{\varepsilon}}, \widetilde{\sigma})$, the number of removed crossings with positive/negative orientation, and the difference of this two numbers equals $r_0$, the number of pairs between 1 and $r$. Altogether, we obtain

$$\omega^{r_+} \overline{\omega}^{r_-} = \omega^{r_+ - r_0} \overline{\omega}^{r_- - r_0} = \omega^{\mathrm{cr}_+(\pi, \boldsymbol{\varepsilon}, \sigma) - \mathrm{cr}_+(\widetilde{\pi}, \widetilde{\boldsymbol{\varepsilon}}, \widetilde{\sigma})} \overline{\omega}^{\mathrm{cr}_-(\pi, \boldsymbol{\varepsilon}, \sigma) - \mathrm{cr}_-(\widetilde{\pi}, \widetilde{\boldsymbol{\varepsilon}}, \widetilde{\sigma})}$$

and, using again the factorization property (iii),

$$\begin{aligned}
&\varphi\big(a_k a_{\mathbf{i}(2)}^{\varepsilon(2)} \cdots a_{\mathbf{i}(r-1)}^{\varepsilon(r-1)} a_k^* a_{\mathbf{i}(r+1)}^{\varepsilon(r+1)} \cdots a_{\mathbf{i}(2k)}^{\varepsilon(2k)}\big) \\
&\quad = \omega^{\mathrm{cr}_+(\pi,\varepsilon,\sigma)-\mathrm{cr}_+(\widetilde{\pi},\widetilde{\varepsilon},\widetilde{\sigma})}\overline{\omega}^{\mathrm{cr}_-(\pi,\varepsilon,\sigma)-\mathrm{cr}_-(\widetilde{\pi},\widetilde{\varepsilon},\widetilde{\sigma})}\varphi\big(a_{\mathbf{i}(2)}^{\varepsilon(2)} \cdots a_{\mathbf{i}(r-1)}^{\varepsilon(r-1)} a_k a_k^* a_{\mathbf{i}(r+1)}^{\varepsilon(r+1)} \cdots a_{\mathbf{i}(2k)}^{\varepsilon(2k)}\big) \\
&\quad = \frac{1}{2}\omega^{\mathrm{cr}_+(\pi,\varepsilon,\sigma)-\mathrm{cr}_+(\widetilde{\pi},\widetilde{\varepsilon},\widetilde{\sigma})}\overline{\omega}^{\mathrm{cr}_-(\pi,\varepsilon,\sigma)-\mathrm{cr}_-(\widetilde{\pi},\widetilde{\varepsilon},\widetilde{\sigma})}\varphi\big(a_{\mathbf{i}(2)}^{\varepsilon(2)} \cdots a_{\mathbf{i}(r-1)}^{\varepsilon(r-1)} a_{\mathbf{i}(r+1)}^{\varepsilon(r+1)} \cdots a_{\mathbf{i}(2k)}^{\varepsilon(2k)}\big) \\
&\quad = \frac{1}{2}\omega^{\mathrm{cr}_+(\pi,\varepsilon,\sigma)-\mathrm{cr}_+(\widehat{\pi},\widehat{\varepsilon},\widehat{\sigma})}\overline{\omega}^{\mathrm{cr}_-(\pi,\varepsilon,\sigma)-\mathrm{cr}_-(\widehat{\pi},\widehat{\varepsilon},\widehat{\sigma})}\varphi\big(a_{\widehat{\mathbf{i}}(1)}^{\widehat{\varepsilon}(1)} \cdots a_{\widehat{\mathbf{i}}(2k-2)}^{\widehat{\varepsilon}(2k-2)}\big).
\end{aligned}$$

Here the index map $\widehat{\mathbf{i}}\colon [2k-2] \to [k-1]$ and the direction map $\widehat{\varepsilon}\colon [2k-2] \to \{*, 1\}$ are defined as

$$\big(\widehat{\mathbf{i}}(\ell), \widehat{\varepsilon}(\ell)\big) := \begin{cases} \big(\mathbf{i}(\ell+1), \varepsilon(\ell+1)\big) & \text{for } 1 \le \ell \le r-2, \\ \big(\mathbf{i}(\ell-2), \varepsilon(\ell-2)\big) & \text{for } r < \ell \le 2k, \end{cases}$$

such that

$$\varphi\big(a_{\widehat{\mathbf{i}}(1)}^{\widehat{\varepsilon}(1)} \cdots a_{\widehat{\mathbf{i}}(2k-2)}^{\widehat{\varepsilon}(2k-2)}\big) = \varphi\big(a_{\mathbf{i}(2)}^{\varepsilon(2)} \cdots a_{\mathbf{i}(r-1)}^{\varepsilon(r-1)} a_{\mathbf{i}(r+1)}^{\varepsilon(r+1)} \cdots a_{\mathbf{i}(2k)}^{\varepsilon(2k)}\big).$$

Moreover, $(\overline{\widehat{\pi}}, \widehat{\varepsilon}, \widehat{\sigma}) \in \mathcal{P}_2(2k-2, \widehat{\varepsilon}) \times \big\{f\colon [2k-2] \to \{*, 1\}\big\} \times S_{k-1}$ denotes again the triple uniquely associated to $\widehat{\pi} = \ker_{\mathcal{O}}(\widehat{\mathbf{i}}) \in \mathcal{OP}_2(2k-2, \widehat{\varepsilon})$. Clearly,

$$\mathrm{cr}_\pm(\widehat{\pi}, \widehat{\varepsilon}, \widehat{\sigma}) = \mathrm{cr}_\pm(\widetilde{\pi}, \widetilde{\varepsilon}, \widetilde{\sigma}).$$

Altogether, we shown for the case (C) that $(\mathrm{M}_{k-1})$ implies $(\mathrm{M}_k)$.

Now a finite induction argument on $k$ establishes the formula

$$\varphi\big(a_{\mathbf{i}(1)}^{\varepsilon(1)} \cdots a_{\mathbf{i}(n)}^{\varepsilon(n)}\big) = \frac{1}{2^k}\, \omega^{\mathrm{cr}_+(\pi,\varepsilon,\sigma)}\, \overline{\omega}^{\mathrm{cr}_-(\pi,\varepsilon,\sigma)}$$

and thus completes the proof of (iv). □

We provide next a result which is underlying CLTs for $\omega$-sequences of partial isometries. We recall from Definition 2.1.33 that, given some $\pi \in \mathcal{P}_2(n)$, the direction map $\varepsilon\colon [n] \to \{*, 1\}$ is said to be $\pi$-*balanced* if $\varepsilon(V) = \{*, 1\}$ for every $V \in \pi$.

**Corollary 4.2.10.** *Let $(\mathcal{A}, \varphi)$ be a tracial *-algebraic probability space and $\omega \in \mathbb{C}$ with $|\omega| = 1$. Furthermore let $\mathbf{a} \subset \mathcal{A}$ be an $\omega$-sequence of partial isometries and let $\mathbf{x} \subset \mathcal{A}$ be the sequence defined by $x_n := a_n + a_n^*$. Then we have:*

*(i)* $\mathbf{x}$ *is spreadable;*

*(ii)* $\mathbf{x}$ *has the SVP*

$$\{\ell\} \in \ker(\mathbf{i}) \quad \Longrightarrow \quad \varphi\big(x_{\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots a_{\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\big) = 0$$

*for any* $n \in \mathbb{N}$, $\mathbf{i}\colon [n] \to \mathbb{N}$, *and* $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$*;*

*(iii)* $\mathbf{x}$ *has the factorization property*

$$\varphi\big(yx_\ell x_\ell z\big) = \varphi(x_\ell^2)\varphi(yz) = \varphi(yz)$$

*for any* $y, z \in {}^*\text{-alg}\,\big\{x_i \mid i \in \mathbb{N}\backslash\{\ell\}\big\}$ *and* $\ell \in \mathbb{N}$*;*

*(iv)* $\mathbf{x}$ *has the pair distribution*

$$\varphi\big(x_{\mathbf{i}(1)} \cdots x_{\mathbf{i}(n)}\big) = \frac{1}{2^k} \sum_{\substack{\boldsymbol{\varepsilon}\colon [n] \to \{*,1\} \\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} \omega^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\, \overline{\omega}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}$$

*for* $n \in \mathbb{N}$ *and* $\mathbf{i}\colon [n] \to \mathbb{N}$ *with* $\ker(\mathbf{i}) \in \mathcal{P}_2(n)$. *Here the standard ordered pair partition* $\pi = \{V_1, V_2, \ldots, V_k\} \in \mathcal{P}_2(n)$, *the permutation* $\sigma \in S_k$ *and* $k \in \mathbb{N}$ *are uniquely determined by* $\ker(\mathbf{i}) = \pi$ *and* $\ker_{\mathcal{O}}(\mathbf{i}) = (V_{\sigma(1)}, \ldots, V_{\sigma(k)}) \in \mathcal{OP}_2(n)$ *with* $n = 2k$.

*Proof.* (i) Suppose $\mathbf{i}, \mathbf{j}\colon [n] \to \mathbb{N}$ are order equivalent. By Theorem 4.2.9 (i), one has

$$\varphi\big(a_{\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots a_{\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\big) = \varphi\big(a_{\mathbf{j}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots a_{\mathbf{j}(n)}^{\boldsymbol{\varepsilon}(n)}\big)$$

for any $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$. But this implies

$$\sum_{\boldsymbol{\varepsilon}(1),\ldots,\boldsymbol{\varepsilon}(n) \in \{*,1\}} \varphi\big(a_{\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots a_{\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\big) = \sum_{\boldsymbol{\varepsilon}(1),\ldots,\boldsymbol{\varepsilon}(n) \in \{*,1\}} \varphi\big(a_{\mathbf{j}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots a_{\mathbf{j}(n)}^{\boldsymbol{\varepsilon}(n)}\big),$$

which is just the expansion on both sides of the equation

$$\varphi\big(x_{\mathbf{i}(1)} \cdots x_{\mathbf{i}(n)}\big) = \varphi\big(x_{\mathbf{j}(1)} \cdots x_{\mathbf{j}(n)}\big).$$

Thus $\mathbf{x}$ is spreadable.
(ii) Let $y, z \in {}^*\text{-alg}\{x_i \mid i \in \mathbb{N}\backslash\{\ell\}\} \subset {}^*\text{-alg}\{a_i \mid i \in \mathbb{N}\backslash\{\ell\}\}$ for some $\ell \in \mathbb{N}$. Thus

$$\varphi(yx_\ell z) = \varphi(ya_\ell z) + \varphi(ya_\ell^* z) = 0$$

by the SVP of the sequence $\mathbf{a}$ from Theorem 4.2.9(ii). Consequently $\mathbf{x}$ has the SVP.
(iii) Let $y, z$ be as stated in (ii). We compute, using the factorization property of $\mathbf{a}$ from Theorem 4.2.9(iii),

$$\varphi\big(yx_\ell x_\ell z\big) = \varphi\big(y(a_\ell + a_\ell^*)^2 z\big) = \varphi\big(ya_\ell a_\ell^* z\big) + \varphi\big(ya_\ell^* a_\ell z\big) = \varphi(yz)$$

and $\varphi(x_\ell x_\ell) = 1$.
(iv) Let $\mathbf{i}\colon [n] \to \mathbb{N}$ be given with $\pi := \ker(\mathbf{i}) \in \mathcal{P}_2(n)$. We compute

$$\varphi\big(x_{\mathbf{i}(1)} \cdots x_{\mathbf{i}(n)}\big) = \sum_{\boldsymbol{\varepsilon}\colon [n]\to\{*,1\}} \varphi\big(a_{\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots a_{\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\big) = \sum_{\substack{\boldsymbol{\varepsilon}\colon [n]\to\{*,1\} \\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} \varphi\big(a_{\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots a_{\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\big),$$

since $a_{\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots a_{\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)} = 0$ if $\boldsymbol{\varepsilon}$ is not $\pi$-balanced, by the defining relations (4.4) and (4.5) of an $\omega$-sequence of partial isometries. We infer from the $\pi$-balancedness of $\boldsymbol{\varepsilon}$ that $\ker(\mathbf{i}) \in \mathcal{P}_2(n, \boldsymbol{\varepsilon})$. Thus Theorem 4.2.9(iv) applies to each summand of the last equation such that

$$\varphi\big(a_{\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots a_{\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\big) = \frac{1}{2^k}\, \omega^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\, \overline{\omega}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)},$$

where the pair partition $\pi = \{V_1, V_2, \ldots, V_k\} \in \mathcal{P}_2(n, \boldsymbol{\varepsilon})$, the permutation $\sigma \in S_k$ and $k \in \mathbb{N}$ are uniquely given by $\ker(\mathbf{i}) = \pi$ and $\ker_{\mathcal{O}}(\mathbf{i}) = (V_{\sigma(1)}, \ldots, V_{\sigma(k)}) \in \mathcal{OP}_2(n, \boldsymbol{\varepsilon})$ with $n = 2k$. □

**Remark 4.2.11.** We have provided in Theorem 4.2.9(iv) an explicit formula for balanced pair distributions of an $\omega$-sequence $\mathbf{a}$ of partial isometries. Furthermore, an explicit formula is available for the general distribution of $\mathbf{a}$. For example, this explicit formula allows us also to determine moments of the form $\varphi(x_1 x_1^* x_1 x_1^*)$ or $\varphi(x_1 x_1^* x_2 x_1 x_2^* x_1^*)$ which are not supported by pair partitions. Here we are omitting this result, as knowing the balanced pair distribution suffices for establishing CLTs.

# 4.3 CLTs for $\omega$-Sequences of Partial Isometries

We have introduced $\omega$-sequences in Subsection 4.2 and shown that such sequences enjoy all properties as they are required for establishing CLTs. Here we focus first on proving a univariate CLT for $\omega$-sequences, before we turn our attention to certain multivariate versions of this CLT. In contrast to multivariate CLTs for exchangeable sequences, we will see that the mixed moments of a multivariate CLT for non-exchangeable sequences depend on the details of how one passes from a given single $\omega$-sequence to a tuple of jointly or separately spreadable sequences. Interesting on its own, this difference occurs starting a mixed moment of order 8 (see Example 4.3.13).

## 4.3.1 Univariate Version of the CLT

We have already established in Theorem 4.2.9 that an $\omega$-sequence of partial isometries $\mathbf{a}$ is spreadable and satisfies a SVP. These properties are inherited by the sequence $\mathbf{x} = \mathbf{a} + \mathbf{a}^*$, considered in Corollary 4.2.10, and thus ensure the existence of a *-algebraic CLT for the sequence $\mathbf{x}$. Consequently, the CLT, Theorem 3.4.9, applies to the sequence $\mathbf{x}$ when the considered 'color' set $C$ is a singleton. Thus all 'color'-related labels can be dropped in the formulas of Theorem 3.4.9 such that we obtain for

$$S_N = \frac{x_1 + \cdots + x_N}{\sqrt{N}}$$

the $n$-moment formula

$$\lim_{N\to\infty} \varphi(S_N^n) = \frac{1}{(n/2)!} \sum_{\pi \in \mathcal{OP}_2(n)} \varphi_\pi^{\mathcal{O}}, \tag{4.9}$$

where $\varphi_\bullet^{\mathcal{O}} \colon \mathcal{OP}_2(n) \to \mathbb{C}$, the moment function on ordered pair partitions associated to $\mathbf{x}$, is given by

$$\varphi_\pi^{\mathcal{O}} = \varphi\big(x_{\mathbf{i}(1)} \cdots x_{\mathbf{i}(n)}\big)$$

for any $\mathbf{i}\colon [n] \to \mathbb{N}$ with $\ker_{\mathcal{O}}(\mathbf{i}) = \pi$ (see also Definition 3.4.7). In the following, as $\mathcal{OP}(n) = \emptyset$ for odd $n \in \mathbb{N}$, we focus on the case $n = 2k$. We recall from Lemma 2.1.26 that there is a bijective correspondence between ordered pair partitions $\pi \in \mathcal{OP}_2(2k)$ and pairs $(\overline{\pi}, \sigma) \in \mathcal{P}_2(2k) \times S_k$ such that $(V_{\sigma(1)}, \ldots, V_{\sigma(k)}) = \pi \mapsto \overline{\pi} = \{V_1, \ldots, V_k\}$, where $(V_1, \ldots, V_k)$ is the unique standard ordered partition assigned to $\pi$. Consequently, we can rewrite (4.9) as

$$\lim_{N\to\infty} \varphi(S_N^{2k}) = \frac{1}{k!} \sum_{\sigma \in S_k} \sum_{\overline{\pi} \in \mathcal{P}_2(2k)} \varphi_{\overline{\pi},\sigma}, \tag{4.10}$$

where

$$\varphi_{\overline{\pi},\sigma} := \varphi\big(x_{\sigma(\mathbf{i}(1))} \cdots x_{\sigma(\mathbf{i}(2k))}\big)$$

for any $\sigma \in S_k$ and $\mathbf{i}\colon [2k] \to [k]$ with $\ker(\mathbf{i}) = \overline{\pi} = \{V_1, \ldots, V_k\} \in \mathcal{P}_2(2k)$. Here $\{V_1, \ldots, V_k\}$ is in standard order, i.e. one has $\min V_i < \min V_j$ for $1 \le i < j \le k$.

We are ready to formulate a concrete version of the CLT associated to an $\omega$-sequence of partial isometries. Recall that $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$.

**Theorem 4.3.1.** *Let $(\mathcal{A}, \varphi)$ be a tracial *-algebraic probability space and $\omega \in \mathbb{T}$. Furthermore, let $\mathbf{a} \equiv (a_n)_{n=1}^\infty \subset \mathcal{A}$ be an $\omega$-sequence of partial isometries and let the sequence $\mathbf{x} \equiv (x_n)_{n=1}^\infty \subset \mathcal{A}$ be the sequence defined by $x_n := a_n + a_n^*$. Then one has for*

$$S_N = \frac{1}{\sqrt{N}}(x_1 + \ldots + x_N)$$

*that, for all $k \in \mathbb{N}$,*

$$\lim_{N\to\infty} \varphi(S_N^{2k-1}) = 0$$

*and*

$$\lim_{N\to\infty} \varphi(S_N^{2k}) = \frac{1}{k!}\frac{1}{2^k} \sum_{\sigma \in S_k} \sum_{\pi \in \mathcal{P}_2(2k)} \sum_{\substack{\boldsymbol{\varepsilon}\colon [2k]\to\{*,1\} \\ \boldsymbol{\varepsilon} \textit{ is } \pi\textit{-balanced}}} \omega^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)} \overline{\omega}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}.$$

As $\omega \in \mathbb{T} \subset \mathbb{C}$ is unimodular, one has $\overline{\omega} = \omega^{-1}$ and thus

$$\omega^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\overline{\omega}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)} = \omega^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)-\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}.$$

*Proof.* The sequence $\mathbf{x}$ is spreadable and satisfies the SVP by Corollary 4.2.10. Thus the CLT, as formulated in Theorem 3.4.9, applies for the 'color' set $C = \{c\}$. We will drop the reference to this 'color' set in the notation of Theorem 3.4.9. Clearly, all odd moments of order $2k-1$ vanish in the large $N$-limit. We are left to further specify in terms of oriented crossings the following formula for even moments:

$$\lim_{N\to\infty} \varphi(S_N^{2k}) = \frac{1}{k!} \sum_{\widetilde{\pi}\in\mathcal{OP}_2(2k)} \varphi_{\widetilde{\pi}}^{\mathcal{O}}, \tag{4.11}$$

where $\varphi_{\widetilde{\pi}} = \varphi(x_{\mathbf{i}(1)} \cdots x_{\mathbf{i}(n)})$ with $\widetilde{\pi} = \ker_{\mathcal{O}}(\mathbf{i}) \in \mathcal{OP}_2(2k)$. We recall from Lemma 2.1.26 that there is a bijective correspondence between ordered pair partitions $\widetilde{\pi} \in \mathcal{OP}_2(2k)$ and pairs $(\overline{\widetilde{\pi}}, \sigma) \in \mathcal{P}_2(2k) \times S_k$ such that $(V_{\sigma(1)}, \ldots, V_{\sigma(k)}) = \widetilde{\pi} \mapsto \pi := \overline{\widetilde{\pi}} = \{V_1, \ldots, V_k\}$, where $(V_1, \ldots, V_k)$ is the unique standard ordered partition assigned to $\widetilde{\pi}$. Thus the sum over ordered pair partitions can also be written as

$$\lim_{N\to\infty} \varphi(S_N^{2k}) = \frac{1}{k!} \sum_{\sigma\in S_k} \sum_{\pi\in\mathcal{P}_2(2k)} \varphi\big(x_{\sigma(\mathbf{i}(1))} \cdots x_{\sigma(\mathbf{i}(2k))}\big),$$

where the pair partition $\pi = \ker(\mathbf{i})$ is in standard order (see Definition 2.1.24). Thus we can apply Corollary 4.2.10(iv) such that

$$\varphi\big(x_{\sigma(\mathbf{i}(1))} \cdots x_{\sigma(\mathbf{i}(2k))}\big) = \frac{1}{2^k} \sum_{\substack{\boldsymbol{\varepsilon}\colon [2k]\to\{*,1\} \\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} \omega^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\, \overline{\omega}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}.$$

Consequently, we arrive at the formula

$$\lim_{N\to\infty} \varphi(S_N^{2k}) = \frac{1}{k!}\frac{1}{2^k} \sum_{\sigma\in S_k} \sum_{\pi\in\mathcal{P}_2(2k)} \sum_{\substack{\boldsymbol{\varepsilon}\colon [2k]\to\{*,1\} \\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} \omega^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\, \overline{\omega}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}.$$

□

**Remark 4.3.2.** (i) If $\omega = \overline{\omega}$ (and thus $\omega = \pm 1$) in Theorem 4.3.1 then one has

$$\omega^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\, \overline{\omega}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)} = \omega^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)+\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)} = \omega^{\mathrm{cr}(\pi)}$$

such that the formula of the large $N$-limit for even moments simplifies to

$$\lim_{N\to\infty} \varphi(S_N^{2k}) = \sum_{\pi\in\mathcal{P}_2(2k)} \omega^{\mathrm{cr}(\pi)} = \begin{cases} (2k-1)!! & \text{if } \omega = 1, \\ \sum_{\pi\in\mathcal{P}_2(2k)} (-1)^{\mathrm{cr}(\pi)} & \text{if } \omega = -1. \end{cases}$$

These are the formulas for the $2k$-th moment of a centred Gaussian random variable with variance 1 in the case $\omega = 1$, and that of a centred Bernoulli random variable with variance 1 in the case $\omega = -1$.
(ii) The CLT associated to an $\omega$-sequence of partial isometries in the cases $\omega = \pm 1$ coincides with the CLT of a $q$-semicircular system (see Section 5.2) in the cases $q = \pm 1$, respectively.
(iii) Let $z \in \mathbb{C}$ and put

$$M_{2k}^{\mathrm{SCS}}(z) := \frac{1}{k!}\frac{1}{2^k}\sum_{\sigma\in S_k}\sum_{\pi\in\mathcal{P}_2(2k)}\sum_{\substack{\boldsymbol{\varepsilon}\colon [2k]\to\{*,1\}\\ \boldsymbol{\varepsilon}\text{ is }\pi\text{-balanced}}} z^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\overline{z}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}.$$

This formula generalizes the formula for the even moments of the CLT for $\omega \in \mathbb{T} \subset \mathbb{C}$. We will see in Section 5.4 that $M_{2k}^{\mathrm{SCS}}(z)$ are the even moments of a $z$-semicircular operator. In the special case $z = r$ for some $r \in \mathbb{R}$, one has $r^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}r^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)} = r^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)+\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)} = r^{\mathrm{cr}(\pi)}$ and thus the moment formula simplifies to

$$M_{2k}^{\mathrm{SCS}}(r) = \sum_{\pi\in\mathcal{P}_2(2k)} r^{\mathrm{cr}(\pi)}.$$

This is the even moment formula of a $q$-Gaussian random variable, also called a $q$-semicircular operator, as we will meet it again in Section 5.2.

In the following, we denote by $M_n(\omega)$ the $n$-th moment of the central limit law associated to the $\omega$-sequence of partial isometries as given in Theorem 4.3.1. More explicitly, for $k = 0, 1, 2, \ldots$, we have $M_{2k+1} = 0$ and

$$M_{2k}(\omega) := \lim_{N\to\infty}\varphi(S_N^{2k}) = \frac{1}{k!}\frac{1}{2^k}\sum_{\sigma\in S_k}\sum_{\pi\in\mathcal{P}_2(2k)}\sum_{\substack{\boldsymbol{\varepsilon}\colon [2k]\to\{*,1\}\\ \boldsymbol{\varepsilon}\text{ is }\pi\text{-balanced}}} \omega^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\overline{\omega}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}.$$

**Corollary 4.3.3.** *For each $k = 0, 1, 2, \ldots$, there exists a polynomial $P_{2k} \in \mathbb{Q}[q]$ such that*

$$P_{2k}(\Re\omega) = M_{2k}(\omega)$$

*for all $\omega \in \mathbb{T}$. Here $\Re\omega$ denotes the real part of $\omega$.*

*Proof.* Since the operator $S_N$ (as stated in Theorem 4.3.1) is self-adjoint, we know that

$$\overline{\varphi\big(S_N^{2k}\big)} = \varphi\big((S_N^{2k})^*\big) = \varphi(S_N^{2k})$$

for all $k, N \in \mathbb{N}$. We conclude from this in the large $N$-limit that $M_{2k}(\omega) \in \mathbb{R}$ for any $\omega \in \mathbb{T}$. We know from the moment formula in Theorem 4.3.1 that $M_{2k}(\omega)$ is a polynomial in $\omega$ and $\overline{\omega}$ of the form

$$M_{2k}(\omega) = \sum_{\ell=-L}^{L} c_\ell\omega^\ell$$

where $L := \frac{1}{2}(k-1)k$ is the maximal number of crossings of a pair partition in $\mathcal{P}_2(2k)$ and $c_\ell \in \mathbb{Q}$ for all $\ell = 0, \pm 1, \dots, \pm L_k$. Thus we know that

$$\sum_{\ell=-L}^{L} c_\ell \omega^{-\ell} = \overline{M_{2k}(\omega)} = M_{2k}(\omega) = \sum_{\ell=-L}^{L} c_\ell \omega^{\ell}$$

for all $\omega \in \mathbb{T}$. But this implies $c_\ell = c_{-\ell}$ for all $\ell = 0, 1, \dots, L$ by the fundamental theorem of algebra. Consequently, we can write

$$M_{2k}(\omega) = c_0 + \sum_{\ell=1}^{L} c_\ell(\omega^\ell + \overline{\omega}^\ell).$$

Now consider the symmetric polynomial $Q \in \mathbb{Q}[\omega, \overline{\omega}]$ with $Q(\omega, \overline{\omega}) = \sum_{\ell=0}^{n} a_\ell(\omega^\ell + \overline{\omega}^\ell)$ of degree $n$. We claim that there exists a symmetric polynomial $\widetilde{Q} \in \mathbb{Q}[\omega, \overline{\omega}]$ of the form $\widetilde{Q}(\omega, \overline{\omega}) = \sum_{\ell=0}^{n-1} \widetilde{a}_\ell(\omega^\ell + \overline{\omega}^\ell)$ with degree $n-1$ such that

$$Q(\omega, \overline{\omega}) = \widetilde{Q}(\omega, \overline{\omega}) + a_n(\omega + \overline{\omega})^n. \tag{4.12}$$

Indeed, both $\mathbb{T}^2 \ni (\omega, \overline{\omega}) \mapsto \omega^n + \overline{\omega}^n \in \mathbb{C}$ and $\mathbb{T}^2 \ni (\omega, \overline{\omega}) \mapsto (\omega + \overline{\omega})^n \in \mathbb{C}$ are symmetric polynomials. Thus their difference is a symmetric polynomial. More explicitly, by the binomial expansion and reducing products of the form $\omega\overline{\omega} = 1 = \overline{\omega}\omega$, one has

$$\begin{aligned}
\omega^n + \overline{\omega}^n &= (\omega + \overline{\omega})^n - \sum_{\ell=1}^{n-1} \binom{n}{\ell} \omega^\ell \overline{\omega}^{L-\ell} \\
&= (\omega + \overline{\omega})^n - \sum_{\ell=1}^{n-1} \binom{n}{\ell} \omega^{2\ell-n} \\
&= (\omega + \overline{\omega})^n - \sum_{\ell=1}^{n-1} d_\ell(\omega^\ell + \overline{\omega}^\ell)
\end{aligned}$$

for some coefficients $d_1, d_2, \dots, d_{L-1} \in \mathbb{Q}$. This establishes the existence of a symmetric polynomial $\widetilde{Q}$ of order $n-1$ satisfying (4.12) (as claimed above). Now a finite induction argument on the order $n$ establishes that there exists a polynomial $P \in \mathbb{Q}[q]$ of order $n$ such that $P\big(\frac{1}{2}(\omega+\overline{\omega})\big) = Q(\omega, \overline{\omega})$. Of course, these arguments apply to the symmetric polynomial $\mathbb{T}^2 \ni (\omega, \overline{\omega}) \mapsto M_{2k}(\omega) \in \mathbb{C}$. Thus there exists a polynomial $P_{2k} \in \mathbb{Q}[q]$ of order $2k$ such that $P_{2k}\big(\frac{1}{2}(\omega + \overline{\omega})\big) = Q(\omega, \overline{\omega})$. $\square$

Next we explicitly compute the even moments $M_{2k}(\omega)$ for $k = 1, 2, 3, 4$. Throughout these computations, we put

$$q := \Re\omega = \frac{1}{2}(\omega + \overline{\omega}).$$

Originally, these moments were computed by brute force, manually or with computer support, as the $n$-th moment can be realized as the normalized trace of a self-adjoint $2^n \times 2^n$ matrix. Here we present calculations of these moments, which are based on factorization rules and algebraic relations. Actually, these more structured computations revealed that the CLTs associated to $\omega$-sequences of partial isometries are counting oriented crossings.

**Computation of the 2-th moment:**

$$M_2(\omega) = 1.$$

This is immediate from the moment formula (4.9), using the relations (4.5):

$$M_2(\omega) = \frac{1}{1!} \sum_{\pi \in \mathcal{OP}_2(2)} \varphi_\pi^{\mathcal{O}} = \varphi(x_1 x_1) = \varphi(a_1^* a_1) + \varphi(a_1^* a_1) = 1.$$

**Computation of the 4-th moment:**

$$M_4(\omega) = \frac{1}{2!}(4 + \omega + \overline{\omega}) = 2 + q.$$

Similar as done for the 2-nd moment, we compute for the moment formula (4.9) that

$$\begin{aligned} M_4(\omega) &= \frac{1}{2!} \sum_{\pi \in \mathcal{OP}_2(4)} \varphi_\pi^{\mathcal{O}} \\ &= \frac{1}{2} \sum_{\sigma \in S_2} \big( \varphi(x_{\sigma(1)} x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(2)}) + \varphi(x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(2)} x_{\sigma(1)}) \\ &\qquad\qquad + \varphi(x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(1)} x_{\sigma(2)}) \big) \\ &= \frac{1}{2} \big( 4 + \varphi(x_1 x_2 x_1 x_2) + \varphi(x_2 x_1 x_2 x_1) \big). \end{aligned}$$

Here we used for the terms corresponding to non-crossing pair partitions the factorization rule from Corollary 4.2.10(iii), to obtain

$$\varphi(x_{\sigma(1)} x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(2)}) = \varphi(x_{\sigma(1)} x_{\sigma(1)})\ \varphi(x_{\sigma(2)} x_{\sigma(2)}) = 1$$

and

$$\varphi(x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(2)} x_{\sigma(1)}) = \varphi(x_{\sigma(1)} x_{\sigma(1)})\ \varphi(x_{\sigma(2)} x_{\sigma(2)}) = 1,$$

as $\varphi(x_i x_i) = \varphi(x_1 x_1)$ for any $i \geq 1$ by spreadability. We are left with the computation of two terms corresponding to the crossing pair partitions. Using the quantum decomposition $x_i = a_i + a_i^*$, the relations (4.4) and (4.5), we find

$$\begin{aligned} \varphi(x_1 x_2 x_1 x_2) &= \varphi(a_1^* a_2^* a_1 a_2) + \varphi(a_1^* a_2 a_1 a_2^*) + \varphi(a_1 a_2^* a_1^* a_2) + \varphi(a_1 a_2 a_1^* a_2^*) \\ &= \frac{1}{4}(\omega + \overline{\omega} + \overline{\omega} + \omega) = \frac{1}{2}(\omega + \overline{\omega}) = q. \end{aligned}$$

Here we have used again that terms corresponding to non-crossing pair partitions are evaluated to 1. A similar calculation (or traciality of the state $\varphi$) yields

$$\varphi(x_2x_1x_2x_1) = \frac{1}{2}(\omega + \overline{\omega}) = q.$$

Altogether, we obtain

$$M_4(\omega) = \frac{1}{2}(4 + 2q) = 2 + q.$$

It is instructive to alternatively compute the 4-th moment based on the formula with counting oriented crossings from Theorem 4.3.1:

$$M_4(\omega) = \frac{1}{2!}\frac{1}{2^2}\sum_{\sigma\in S_2}\sum_{\pi\in\mathcal{P}_2(4)}\sum_{\substack{\boldsymbol{\varepsilon}\colon [4]\to\{*,1\}\\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} \omega^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\overline{\omega}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}.$$

There are 2 non-crossing pair partitions and 1 crossing pair partition in $\mathcal{P}_2(4)$. Each one of these two non-crossing pair partition contributes with

$$\frac{1}{2!}\frac{1}{2^2}\sum_{\sigma\in S_2}\sum_{\substack{\boldsymbol{\varepsilon}\colon [4]\to\{*,1\}\\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} \omega^0\overline{\omega}^0 = \frac{1}{2!}\frac{1}{2^2}2^2 2! = 1.$$

We are left to determine the contribution of the single crossing pair partition $\pi = \{\{1,3\},\{2,4\}\} \in \mathcal{P}_2(4)$ coming from

$$\frac{1}{2!}\frac{1}{2^2}\sum_{\sigma\in S_2}\sum_{\substack{\boldsymbol{\varepsilon}\colon [4]\to\{*,1\}\\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} \omega^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\overline{\omega}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)},$$

using the following arguments. Clearly, each of the $2!2^2 = 8$ summands is a scalar multiple of either $\omega$ or $\overline{\omega}$. Denote by $\sigma_0$ the neutral element of $S_2$. We note that $\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma_0) = 0$ if and only if $\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma_1) = 1$. Thus it holds

$$\sum_{\sigma\in S_2} \omega^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\overline{\omega}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)} = \omega + \overline{\omega},$$

as $\pi$ has a single crossing. This symmetry argument yields

$$\begin{aligned}\frac{1}{2!}\frac{1}{2^2}\sum_{\sigma\in S_2}\sum_{\substack{\boldsymbol{\varepsilon}\colon [4]\to\{*,1\}\\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} \omega^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\overline{\omega}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)} &= \frac{1}{2!}\frac{1}{2^2}\sum_{\substack{\boldsymbol{\varepsilon}\colon [4]\to\{*,1\}\\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} (\omega + \overline{\omega})\\ &= \frac{1}{2}(\omega + \overline{\omega}) = q.\end{aligned}$$

Note that the orientation of a crossing is best determined by using a graphical representation of the directed ordered pair partition (see Figure 4.1 for $\sigma = \sigma_1$ and $\boldsymbol{\varepsilon}(1) = \boldsymbol{\varepsilon}(4) = 1$, $\boldsymbol{\varepsilon}(2) = \boldsymbol{\varepsilon}(3) = *$).

**Computation of the 6-th moment:** Counting oriented crossings according to the moment formula from Theorem 4.3.1, one finds

$$\begin{aligned} M_6(\omega) &= \frac{1}{3!2^3}\big(312 + 162(\omega^1 + \varpi^1) + 36(\omega^2 + \varpi^2) + 6(\omega^3 + \varpi^3)\big) \\ &= \frac{1}{2^3}\big(52 + 27(\omega^1 + \varpi^1) + 6(\omega^2 + \varpi^2) + 1(\omega^3 + \varpi^3)\big) \\ &= 5 + 6q + 3q^2 + q^3. \end{aligned}$$

The coefficients inside of the parenthesis of this polynomial have the following combinatorial interpretation. There are total of $3! \cdot 5!! = 6 \cdot 15 = 90$ ordered pair partitions. Each block of a pair partition is equipped with a direction which gives $2^3 = 8$ additional choices. Consequently, there are a total of 720 summands which are grouped as follows:

$$\begin{aligned} &312 \text{ summands with } \mathrm{cr}_+(\pi, \boldsymbol{\varepsilon}, \sigma) - \mathrm{cr}_-(\pi, \boldsymbol{\varepsilon}, \sigma) = 0, \\ &162 \text{ summands with } \mathrm{cr}_+(\pi, \boldsymbol{\varepsilon}, \sigma) - \mathrm{cr}_-(\pi, \boldsymbol{\varepsilon}, \sigma) = 1, \\ &162 \text{ summands with } \mathrm{cr}_+(\pi, \boldsymbol{\varepsilon}, \sigma) - \mathrm{cr}_-(\pi, \boldsymbol{\varepsilon}, \sigma) = -1, \\ &36 \text{ summands with } \mathrm{cr}_+(\pi, \boldsymbol{\varepsilon}, \sigma) - \mathrm{cr}_-(\pi, \boldsymbol{\varepsilon}, \sigma) = 2, \\ &36 \text{ summands with } \mathrm{cr}_+(\pi, \boldsymbol{\varepsilon}, \sigma) - \mathrm{cr}_-(\pi, \boldsymbol{\varepsilon}, \sigma) = -2, \\ &6 \text{ summands with } \mathrm{cr}_+(\pi, \boldsymbol{\varepsilon}, \sigma) - \mathrm{cr}_-(\pi, \boldsymbol{\varepsilon}, \sigma) = 3, \\ &6 \text{ summands with } \mathrm{cr}_+(\pi, \boldsymbol{\varepsilon}, \sigma) - \mathrm{cr}_-(\pi, \boldsymbol{\varepsilon}, \sigma) = -3. \end{aligned}$$

The moment formula $M_6(\omega)$ can be expressed in terms of $q = \Re\omega$ along the following computations which also underlie the inductive proof of Corollary 4.3.3.

$$\omega^3 + \varpi^3 = (\omega + \varpi)^3 - \sum_{\ell=1}^{2} \binom{3}{\ell} \omega^\ell \varpi^{3-\ell} = (\omega + \varpi)^3 - 3(\varpi^1 + \omega^1).$$

Thus we obtain

$$\begin{aligned} &312 + 162(\omega^1 + \varpi^1) + 36(\omega^2 + \varpi^2) + 6(\omega^3 + \varpi^3) \\ &= 312 + 144(\omega^1 + \varpi^1) + 36(\omega^2 + \varpi^2) + 6(\omega + \varpi)^3 \\ &= 240 + 144(\omega^1 + \varpi^1) + 36(\omega + \varpi)^2 + 6(\omega + \varpi)^3. \end{aligned}$$

Here we have used $\omega^2 + \varpi^2 = (\omega + \varpi)^2 - 2$ for the last equality. Consequently, we obtain

$$\begin{aligned} M_6(\omega) &= \frac{1}{3!2^3}\Big(240 + 144(\omega^1 + \varpi^1) + 36(\omega + \varpi)^2 + 6(\omega + \varpi)^3\Big) \\ &= \frac{1}{3!2^3}\Big(240 + 288q + 144q^2 + 48q^3\Big) \\ &= 5 + 6q + 3q^2 + q^3. \end{aligned}$$

An alternative, more algebraic way to compute the 6-th moment starts with that there is a bijective correspondence between ordered pair partitions in $\mathcal{OP}_2(6)$ and pairs in $\mathcal{P}_2(6) \times S_3$. Thus we can expand the 6-th moment $M_6(\omega)$ as

$$\begin{aligned}
\frac{1}{3!}\sum_{\sigma\in S_3}\Big(&\underbrace{\varphi\big(x_{\sigma(1)}x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(3)}\big)}_{\text{non-crossing}}+\underbrace{\varphi\big(x_{\sigma(1)}x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(3)}x_{\sigma(2)}\big)}_{\text{non-crossing}}\\
&+\underbrace{\varphi\big(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(2)}x_{\sigma(1)}x_{\sigma(3)}x_{\sigma(3)}\big)}_{\text{non-crossing}}+\underbrace{\varphi\big(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(3)}x_{\sigma(1)}\big)}_{\text{non-crossing}}\\
&+\underbrace{\varphi\big(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(3)}x_{\sigma(2)}x_{\sigma(1)}\big)}_{\text{non-crossing}}+\underbrace{\varphi\big(x_{\sigma(1)}x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(2)}x_{\sigma(3)}\big)}_{\text{1 crossing}}\\
&+\underbrace{\varphi\big(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(3)}\big)}_{\text{1 crossing}}+\underbrace{\varphi\big(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(1)}x_{\sigma(3)}x_{\sigma(3)}x_{\sigma(2)}\big)}_{\text{1 crossing}}\\
&+\underbrace{\varphi\big(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(1)}x_{\sigma(3)}\big)}_{\text{1 crossing}}+\underbrace{\varphi\big(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(3)}x_{\sigma(1)}x_{\sigma(2)}\big)}_{\text{1 crossing}}\\
&+\underbrace{\varphi\big(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(1)}\big)}_{\text{1 crossing}}+\underbrace{\varphi\big(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(1)}x_{\sigma(3)}x_{\sigma(2)}x_{\sigma(3)}\big)}_{\text{2 crossings}}\\
&+\underbrace{\varphi\big(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(1)}x_{\sigma(3)}x_{\sigma(2)}\big)}_{\text{2 crossings}}+\underbrace{\varphi\big(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(2)}x_{\sigma(1)}x_{\sigma(3)}\big)}_{\text{2 crossings}}\\
&+\underbrace{\varphi\big(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}\big)}_{\text{3 crossings}}\Big).
\end{aligned}$$

Each of these 15 terms can be evaluated with the help of the relations (4.4) and (4.5). Before taking into account the summation over permutations from $S_3$, each of the 5 terms corresponding to a non-crossing partition gives 1. And each of the 6 terms corresponding to partition with one crossing contributes with $\frac{1}{2}(\omega+\overline{\omega})$, for example

$$\varphi\big(x_{\sigma(1)}x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(2)}x_{\sigma(3)}\big) = \frac{1}{2}(\omega+\overline{\omega}) = q.$$

There are 3 terms corresponding to pair partitions with two crossings and each of them contributes with $\frac{1}{4}(\omega+\overline{\omega})^2$, for example

$$\varphi\big(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(1)}x_{\sigma(3)}x_{\sigma(2)}x_{\sigma(3)}\big) = \frac{1}{4}(\omega+\overline{\omega})^2 = q^2.$$

Finally, there is the single term

$$\varphi\big(x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}\big)$$

corresponding to a pair partition with 3 crossings which needs to be computed separately for each permutation $\sigma \in S_3$. Using the cyclicity of the trace, it suffices to consider the two permutations $\sigma = e$ and $\sigma = \sigma_1\sigma_2\sigma_1$. One computes for these two permutations that

$$\varphi(x_1x_2x_3x_1x_2x_3) = \frac{1}{8}(2\omega^3 + 6\overline{\omega})$$

and

$$\varphi(x_3x_2x_1x_3x_2x_1) = \frac{1}{8}(2\overline{\omega}^3 + 6\omega).$$

Furthermore, the sum of these two terms can be rewritten in terms of the real part of $\omega$ as

$$\varphi(x_1x_2x_3x_1x_2x_3) + \varphi(x_3x_2x_1x_3x_2x_1) = \frac{2}{8}(\omega^3 + 3\overline{\omega} + \overline{\omega}^3 + 3\omega) = 2q^3.$$

Altogether, we arrive at

$$M_6(\omega) = \frac{1}{3!}\sum_{\sigma\in S_3}\Big(5 + 6q + 3q^2\Big) + \frac{3}{3!}2q^3 = 5 + 6q + 3q^2 + q^3.$$

**Computation of the 8-th moment:** Counting oriented crossings yields along the formula in Theorem 4.3.1 that

$$\begin{aligned} M_8(\omega) = \frac{1}{4!2^4}\Big(&12416 + 8768(\omega^1 + \overline{\omega}^1) + 3672(\omega^2 + \overline{\omega}^2) + 1152(\omega^3 + \overline{\omega}^3) \\ &+ 288(\omega^4 + \overline{\omega}^4) + 64(\omega^5 + \overline{\omega}^5) + 8(\omega^6 + \overline{\omega}^6)\Big) \\ = \frac{1}{3}&(44 + 88q + 81q^2 + 52q^3 + 30q^4 + 16q^5 + 4q^6). \end{aligned}$$

Similar to the 6-th moment, we observe that there are a total of $4! \cdot 7!! \cdot 2^4 = 40\,320$ summands which can again be grouped according to the value of the difference $\mathrm{cr}_+(\pi, \boldsymbol{\varepsilon}, \sigma) - \mathrm{cr}_-(\pi, \boldsymbol{\varepsilon}, \sigma)$. For example, there are 288 summands with $\mathrm{cr}_+(\pi, \boldsymbol{\varepsilon}, \sigma) - \mathrm{cr}_-(\pi, \boldsymbol{\varepsilon}, \sigma) = 4$ and 288 summands with $\mathrm{cr}_+(\pi, \boldsymbol{\varepsilon}, \sigma) - \mathrm{cr}_-(\pi, \boldsymbol{\varepsilon}, \sigma) = -4$. One has again that the 8-th moment only depends on $q$, the real part of $\omega$, as it is shown in above formula.

Similar to the 6-th moment, this 8-th moment can be calculated by averaging over all permutations $\sigma \in S_4$ and now summing over 105 pair partitions. As related calculations are quite lengthy but straightforward, we omit presenting them here.

Notably, the 8-th moment is the first moment which differs from the 8-th moment of a $q$-Gaussian random variable, see also Remark 4.3.5 below.

**Remark 4.3.4.** Recall from Corollary 4.3.3 that $P_n(q)$ is the $n$-th moment in the CLT associated to an $\omega$-sequence of partial isometries with $q = \Re\omega$, and that $M_n^{\mathrm{SCS}}(q)$ denotes the $n$-th moment of a $q$-Gaussian random variable (or a

$q$-semicircular operator). Directly comparing the computed moments, one verifies for $-1 < q < 1$ that

$$P_{2k}(q) = M_{2k}^{\mathrm{SCS}}(q) \qquad (k = 0, 1, 2, 3),$$

but

$$P_8(q) \neq M_8^{\mathrm{SCS}}(q).$$

It is an interesting question to ask why the equality of even moments breaks down starting the 8-th moment, and not earlier. We speculate that this phenomenon is connected to the fact that, for $k = 3$, any permutation in $S_3$ can be written as a power of the cycle $(1, 2, 3)$ or the 'reversed' cycle $(3, 2, 1)$. This is no longer true for $k = 4$, as there exist permutations in $S_4$ which can not be written as power of the cycle $(1, 2, 3, 4)$ or the 'reversed' cycle $(4, 3, 2, 1)$.

**Remark 4.3.5.** We obtain the following moments from the CLT associated to an $\omega$-sequence of partial isometries in the special case $q = \Re\omega = 0$:

$$M_2(\pm\mathrm{i}) = 1, \quad M_4(\pm\mathrm{i}) = 2, \quad M_6(\pm\mathrm{i}) = 5, \quad M_8(\pm\mathrm{i}) = 44/3.$$

It is known that the $2k$-moment of a centred $q$-Gaussian random variable with variance 1 is described in the special case $q = 0$ by the Catalan numbers $C_k = \frac{1}{k+1}\binom{2k}{k}$:

$$C_1 = 1, \quad C_2 = 2, \quad C_3 = 5, \quad C_4 = 14.$$

We observe that $M_{2k(\pm\mathrm{i})} = C_k$ for $k = 1, 2, 3$ but $M_8(\pm\mathrm{i}) = 14.\overline{6} \neq 14 = C_4$.

We close this subsection by providing explicit formulas for the moments of order 10, 12, 14 and 16. We are grateful to Andreas Amann who carried out these calculations with PYTHON and made his results available to us [Am19].

**Computation of the 10-th moment:** There are a total of $9!! \cdot 5! = 945 \cdot 120 = 113\,400$ ordered pair partitions and $2^5$ direction maps. Thus a brute force computation of the 10-th moment amounts to sum $9!! \cdot 5! \cdot 2^5 = 3\,628\,800$ terms.

The polynomials $M_{10}(\omega)$ and $P_{10}(q)$ are displayed in the following with reduced fractions of the coefficients, somewhat hiding the combinatorial interpretation of all coefficients. For example, it can be seen that the integer coefficients of the polynomial $5! \cdot 2^5 \cdot M_{10}(\omega) = 3840 \cdot M_{10}(\omega)$ actually count the number of directed ordered pair partitions with a fixed difference of the number of positively and negatively oriented crossings.

$$\begin{aligned} M_{10}(\omega) = &\frac{1}{384}(\omega^{10} + \overline{\omega}^{10}) + \frac{5}{192}(\omega^9 + \overline{\omega}^9) + \frac{55}{384}(\omega^8 + \overline{\omega}^8) + \frac{55}{96}(\omega^7 + \overline{\omega}^7) \\ &+ \frac{255}{128}(\omega^6 + \overline{\omega}^6) + \frac{1207}{192}(\omega^5 + \overline{\omega}^5) + \frac{575}{32}(\omega^4 + \overline{\omega}^4) + \frac{8965}{192}(\omega^3 + \overline{\omega}^3) \\ &+ \frac{19985}{192}(\omega^2 + \overline{\omega}^2) + \frac{11675}{64}(\omega^1 + \overline{\omega}^1) + \frac{14375}{64} \end{aligned}$$

or, with $q = \Re\omega$,

$$\begin{aligned} P_{10}(q) = {} & \frac{8}{3}q^{10} + \frac{40}{3}q^9 + 30q^8 + \frac{130}{3}q^7 + 60q^6 + \frac{286}{3}q^5 + 140q^4 + 180q^3 \\ & + \frac{575}{3}q^2 + 140q^1 + \frac{146}{3}. \end{aligned}$$

The highest order coefficient of the polynomials $M_{10}(\omega)$ and $P_{10}(q)$ are obtained by the following combinatorial arguments. There is a single pair partition in $\mathcal{P}_2(10)$ with $\frac{1}{2} \cdot \frac{10}{2} \cdot (\frac{10}{2} - 1) = 10$ crossings, the maximal possible number of crossings. Due to the cyclicity of the trace, there are 10 directed ordered pair partitions in $\mathcal{OP}_2(10)$ with 10 positive crossings, and another 10 directed ordered pair partitions with 10 negative crossings. Thus the coefficient of the term $(\omega^{10} + \overline{\omega}^{10})$ in $M_{10}(\omega)$ is $\frac{1}{5!} \cdot \frac{1}{2^5} \cdot 10 = \frac{1}{384}$. Since $q = \Re\omega$, one has to further re-scale by the factor $2^{10}$ to obtain the leading coefficient of $P_{10}(q)$, such that $10 \cdot \frac{1}{5!}\frac{1}{2^5} \cdot 2^{10} = \frac{8}{3}$.

**Computation of the 12-th moment:** There are a total of $11!! \cdot 6! = 10\,395 \cdot 720 = 7\,484\,400$ ordered pair partitions and $2^6$ direction maps. Thus a brute force computation of the 12-th moment amounts to sum $11!! \cdot 6! \cdot 2^6 = 479\,001\,600$ terms.

$$\begin{aligned} M_{12}(\omega) = {} & \frac{1}{3840}(\omega^{15} + \overline{\omega}^{15}) + \frac{1}{320}(\omega^{14} + \overline{\omega}^{14}) + \frac{13}{640}(\omega^{13} + \overline{\omega}^{13}) + \frac{91}{960}(\omega^{12} + \overline{\omega}^{12}) \\ & + \frac{91}{256}(\omega^{11} + \overline{\omega}^{11}) + \frac{371}{320}(\omega^{10} + \overline{\omega}^{10}) + \frac{3313}{960}(\omega^9 + \overline{\omega}^9) + \frac{3067}{320}(\omega^8 + \overline{\omega}^8) \\ & + \frac{3207}{128}(\omega^7 + \overline{\omega}^7) + \frac{3985}{64}(\omega^6 + \overline{\omega}^6) + \frac{188483}{1280}(\omega^5 + \overline{\omega}^5) \\ & + \frac{104779}{320}(\omega^4 + \overline{\omega}^4) + \frac{640813}{960}(\omega^3 + \overline{\omega}^3) + \frac{381271}{320}(\omega^2 + \overline{\omega}^2) \\ & + \frac{2246211}{1280}(\omega^1 + \overline{\omega}^1) + \frac{966707}{480} \end{aligned}$$

or, with $q = \Re\omega$,

$$\begin{aligned} P_{12}(q) = {} & \frac{128}{15}q^{15} + \frac{256}{5}q^{14} + \frac{672}{5}q^{13} + \frac{3136}{15}q^{12} + \frac{1176}{5}q^{11} + \frac{1344}{5}q^{10} + \frac{6064}{15}q^9 \\ & + 628q^8 + \frac{4214}{5}q^7 + \frac{15812}{15}q^6 + \frac{6604}{5}q^5 + \frac{7752}{5}q^4 + \frac{7849}{5}q^3 + 1258q^2 \\ & + \frac{3406}{5}q^1 + \frac{892}{5}. \end{aligned}$$

We note that the highest order coefficient of $P_{12}(q)$ is again obtained as

$$\frac{1}{6!} \cdot \frac{1}{2^6} \cdot 12 \cdot 2^{15} = \frac{128}{15}.$$

Here 12 is the number of cyclic permutations and $15 = \frac{1}{2} \cdot \frac{12}{2} \cdot (\frac{12}{2} - 1)$ is the maximal number of crossings which a pair partition in $\mathcal{P}_2(12)$ can have.

**Computation of the 14-th moment:** There are a total of $13!! \cdot 7! = 135\,135 \cdot 5\,040 = 681\,080\,400$ ordered pair partitions and $2^7$ direction maps. Thus a brute force computation of the 14-th moment amounts to sum $13!! \cdot 7! \cdot 2^7 = 87\,178\,291\,200$ terms.

$$\begin{aligned} M_{14}(\omega) = {} & \frac{1}{46080}(\omega^{21}+\overline{\omega}^{21}) + \frac{7}{23040}(\omega^{20}+\overline{\omega}^{20}) + \frac{7}{3072}(\omega^{19}+\overline{\omega}^{19}) \\ & + \frac{7}{576}(\omega^{18}+\overline{\omega}^{18}) + \frac{119}{2304}(\omega^{17}+\overline{\omega}^{17}) + \frac{119}{640}(\omega^{16}+\overline{\omega}^{16}) \\ & + \frac{4543}{7680}(\omega^{15}+\overline{\omega}^{15}) + \frac{39551}{23040}(\omega^{14}+\overline{\omega}^{14}) + \frac{53543}{11520}(\omega^{13}+\overline{\omega}^{13}) \\ & + \frac{91483}{7680}(\omega^{12}+\overline{\omega}^{12}) + \frac{149219}{5120}(\omega^{11}+\overline{\omega}^{11}) + \frac{197323}{2880}(\omega^{10}+\overline{\omega}^{10}) \\ & + \frac{796817}{5120}(\omega^{9}+\overline{\omega}^{9}) + \frac{3948623}{11520}(\omega^{8}+\overline{\omega}^{8}) + \frac{11251147}{15360}(\omega^{7}+\overline{\omega}^{7}) \\ & + \frac{5819093}{3840}(\omega^{6}+\overline{\omega}^{6}) + \frac{69380591}{23040}(\omega^{5}+\overline{\omega}^{5}) + \frac{2036713}{360}(\omega^{4}+\overline{\omega}^{4}) \\ & + \frac{451584497}{46080}(\omega^{3}+\overline{\omega}^{3}) + \frac{69712951}{4608}(\omega^{2}+\overline{\omega}^{2}) + \frac{11548579}{576}(\omega^{1}+\overline{\omega}^{1}) \\ & + \frac{127380007}{5760} \end{aligned}$$

or, with $q = \Re\omega$,

$$\begin{aligned} P_{14}(q) = {} & \frac{2048}{45}q^{21} + \frac{14336}{45}q^{20} + \frac{14336}{15}q^{19} + \frac{14336}{9}q^{18} + \frac{73472}{45}q^{17} + \frac{55552}{45}q^{16} \\ & + \frac{19264}{15}q^{15} + \frac{34208}{15}q^{14} + \frac{173096}{45}q^{13} + \frac{78176}{15}q^{12} + \frac{90538}{15}q^{11} \\ & + \frac{309974}{45}q^{10} + \frac{385462}{45}q^{9} + \frac{492226}{45}q^{8} + \frac{65904}{5}q^{7} + \frac{670936}{45}q^{6} \\ & + \frac{717199}{45}q^{5} + \frac{231637}{15}q^{4} + \frac{189896}{15}q^{3} + \frac{40201}{5}q^{2} + \frac{30800}{9}q^{1} + \frac{31984}{45}. \end{aligned}$$

**Computation of the 16-th moment:** There are a total of $15!! \cdot 8! = 2\,027\,025 \cdot 40\,320 = 81\,729\,648\,000$ ordered pair partitions and $2^8$ direction maps. Thus a brute force computation of the 16-th moment amounts to sum $15!! \cdot 8! \cdot 2^8 = 20\,922\,789\,888\,000$ terms.

$$\begin{aligned} M_{16}(\omega) = {} & \frac{1}{645120}(\omega^{28}+\overline{\omega}^{28}) + \frac{1}{40320}(\omega^{27}+\overline{\omega}^{27}) + \frac{17}{80640}(\omega^{26}+\overline{\omega}^{26}) \\ & + \frac{17}{13440}(\omega^{25}+\overline{\omega}^{25}) + \frac{323}{53760}(\omega^{24}+\overline{\omega}^{24}) + \frac{323}{13440}(\omega^{23}+\overline{\omega}^{23}) \\ & + \frac{323}{3840}(\omega^{22}+\overline{\omega}^{22}) + \frac{1067}{4032}(\omega^{21}+\overline{\omega}^{21}) + \frac{49289}{64512}(\omega^{20}+\overline{\omega}^{20}) \\ & + \frac{16595}{8064}(\omega^{19}+\overline{\omega}^{19}) + \frac{422467}{80640}(\omega^{18}+\overline{\omega}^{18}) + \frac{128321}{10080}(\omega^{17}+\overline{\omega}^{17}) \\ & + \frac{4796041}{161280}(\omega^{16}+\overline{\omega}^{16}) + \frac{60145}{896}(\omega^{15}+\overline{\omega}^{15}) + \frac{790403}{5376}(\omega^{14}+\overline{\omega}^{14}) \end{aligned}$$

$$
\begin{aligned}
&+ \frac{6271}{20}(\omega^{13}+\overline{\omega}^{13}) + \frac{15044311}{23040}(\omega^{12}+\overline{\omega}^{12}) + \frac{53653127}{40320}(\omega^{11}+\overline{\omega}^{11})\\
&+ \frac{214318043}{80640}(\omega^{10}+\overline{\omega}^{10}) + \frac{14993087}{2880}(\omega^{9}+\overline{\omega}^{9}) + \frac{1612001207}{161280}(\omega^{8}+\overline{\omega}^{8})\\
&+ \frac{75633779}{4032}(\omega^{7}+\overline{\omega}^{7}) + \frac{2759152349}{80640}(\omega^{6}+\overline{\omega}^{6}) + \frac{172938161}{2880}(\omega^{5}+\overline{\omega}^{5})\\
&+ \frac{64416303137}{645120}(\omega^{4}+\overline{\omega}^{4}) + \frac{177554335}{1152}(\omega^{3}+\overline{\omega}^{3})\\
&+ \frac{2173169297}{10080}(\omega^{2}+\overline{\omega}^{2}) + \frac{2152202999}{8064}(\omega^{1}+\overline{\omega}^{1}) + \frac{23161160759}{80640}
\end{aligned}
$$

or, with $q = \Re\omega$,

$$
\begin{aligned}
P_{16}(q) = {} & \frac{131072}{315}q^{28} + \frac{1048576}{315}q^{27} + \frac{393216}{35}q^{26} + \frac{2097152}{105}q^{25} + \frac{376832}{21}q^{24}\\
&+ \frac{131072}{35}q^{23} - \frac{65536}{15}q^{22} + \frac{2146304}{315}q^{21} + \frac{8171008}{315}q^{20} + \frac{2392064}{63}q^{19}\\
&+ \frac{13612544}{315}q^{18} + \frac{14424064}{315}q^{17} + \frac{2941760}{63}q^{16} + \frac{2303744}{45}q^{15}\\
&+ \frac{2935424}{45}q^{14} + \frac{27143552}{315}q^{13} + \frac{33394288}{315}q^{12} + \frac{38228608}{315}q^{11}\\
&+ \frac{1226912}{9}q^{10} + \frac{5437136}{35}q^{9} + \frac{55306868}{315}q^{8} + \frac{19875376}{105}q^{7}\\
&+ \frac{60087476}{315}q^{6} + \frac{55595608}{315}q^{5} + \frac{45453083}{315}q^{4}\\
&+ \frac{30939616}{315}q^{3} + \frac{3225680}{63}q^{2} + \frac{623552}{35}q^{1} + \frac{191600}{63}.
\end{aligned}
$$

We note that $P_{16}(q)$ is the first polynomial where a new feature emerges: the coefficient of the power $q^{22}$ is negative.

We finally compare the even moments of $q$-Gaussian random variables and the even moments $M_{2k}(\omega) = P_{2k}(q)$ in the special case $q = \Re\omega = 0$. Recall that the $2k$-th moment of a centred 0-Gaussian random variable with variance 1 is given by the Catalan number $C_k = \frac{1}{k+1}\binom{2k}{k}$.

| $2k$ | $C_k$ | $P_{2k}(0)$ | $P_{2k}(0)/C_k$ | $(P_{2k}(0)/C_k)^{1/2k}$ |
|---|---|---|---|---|
| 2 | 1 | 1 | 1 | 1 |
| 4 | 2 | 2 | 1 | 1 |
| 6 | 5 | 5 | 1 | 1 |
| 8 | 14 | 44/3 | 1.047619... | 1.005831... |
| 10 | 42 | 146/3 | 1.158730... | 1.014841... |
| 12 | 132 | 892/5 | 1.351515... | 1.025419... |
| 14 | 429 | 31984/45 | 1.656772... | 1.036720... |
| 16 | 1430 | 191600/63 | 2.126762... | 1.048292... |

At the time of writing we could not resolve if the ratio $(P_{2k}(0)/C_k)^{1/2k}$ is bounded for $k \to \infty$.

We close this section with a strengthened version of the univariate CLT for $\omega$-sequences.

**Theorem 4.3.6.** *Suppose $(\mathcal{A},\varphi)$ is a tracial *-algebraic probability space, $\mathbf{a}\equiv(a_n)_{n=1}^{\infty}\subset\mathcal{A}$ is an $\omega$-sequence for some fixed $\omega\in\mathbb{T}$, and let*

$$S_N=\frac{1}{\sqrt{N}}\big(a_1+a_1^*+\ldots+a_N+a_N^*\big).$$

*Then there exists a unique probability measure $\mu_\omega$ on $\mathbb{R}$ such that, for any $n\in\mathbb{N}$,*

$$M_n(\omega)=\lim_{N\to\infty}\varphi(S_N^n)=\int_{\mathbb{R}}t^n\mu_\omega(dt).$$

*Proof.* The $\omega$-sequence $\mathbf{a}\subset\mathcal{A}$ has the same distribution as the concrete $\omega$-sequence $\mathbf{b}\subset\mathcal{B}_\infty$, as considered in Remark 4.1.10 for $\kappa=1$. Thus, letting

$$\widetilde{S}_N=\frac{1}{\sqrt{N}}\big(b_1+b_1^*+\ldots+b_N+b_N^*\big),$$

we have

$$\varphi(S_N^n)=\psi_\infty(\widetilde{S}_N^n)\qquad(N,n\in\mathbb{N}).$$

By the construction of the concrete sequence $\mathbf{b}\subset\mathcal{B}_\infty$, one has $\widetilde{S}_N\in\mathcal{B}_N$, where $\mathcal{B}_N$ is an amplification of the matrix algebra $\mathbb{M}_2(\mathbb{C})^{\otimes N}$. Since the latter, and thus its amplification, is a finite dimensional unital C*-algebra, Theorem 3.6.3 applies. This ensures the existence of a probability measure $\mu_\omega$ on $\mathbb{R}$ such that

$$\lim_{N\to\infty}\varphi(S_N^n)=\lim_{N\to\infty}\varphi(\widetilde{S}_N^n)=\int_{\mathbb{R}}t^n\mu_\omega(dt)$$

for all $n\in\mathbb{N}$. It is known for the Hamburger Moment Problem that the probability measure $\mu_\omega$ is unique if there exist positive constants $C,D$ such that

$$|M_n(\omega)|\le CD^n n!$$

for all $n\in\mathbb{N}$ (see [RS75, Example 4 on p205]). This inequality is clearly satisfied for odd $n$. Let $n=2k$ for $k\in\mathbb{N}$. Then the explicit moment formula from Theorem 4.3.1 has the estimate

$$\begin{aligned}
|M_{2k}(\omega)| &= \left|\frac{1}{k!}\frac{1}{2^k}\sum_{\sigma\in S_k}\sum_{\pi\in\mathcal{P}_2(2k)}\sum_{\substack{\boldsymbol{\varepsilon}\colon[2k]\to\{*,1\}\\ \boldsymbol{\varepsilon}\text{ is }\pi\text{-balanced}}}\omega^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\overline{\omega}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}\right|\\
&\le \frac{1}{k!}\frac{1}{2^k}\sum_{\sigma\in S_k}\sum_{\pi\in\mathcal{P}_2(2k)}\sum_{\substack{\boldsymbol{\varepsilon}\colon[2k]\to\{*,1\}\\ \boldsymbol{\varepsilon}\text{ is }\pi\text{-balanced}}}\left|\omega^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\right|\cdot\left|\overline{\omega}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}\right|\\
&= \frac{1}{k!}\frac{1}{2^k}\sum_{\sigma\in S_k}\sum_{\pi\in\mathcal{P}_2(2k)}\sum_{\substack{\boldsymbol{\varepsilon}\colon[2k]\to\{*,1\}\\ \boldsymbol{\varepsilon}\text{ is }\pi\text{-balanced}}}1\\
&= (2k-1)!!=M_{2k}(1).
\end{aligned}$$

Altogether, this estimate ensures that $|M_n(\omega)|\le CD^n n!$ with $C=D=1$. ∎

### 4.3.2 Multivariate Versions of the CLT

We will generalize the CLT from Theorem 4.3.1 to multivariate settings. Recall the two versions of the singleton vanishing property (SVP) from Definition 3.2.3.

**Proposition 4.3.7.** *Let $(\mathcal{A}, \varphi)$ be a tracial *-algebraic probability space and $\mathbf{a} \equiv (a_n)_{n=1}^{\infty} \subset \mathcal{A}$ an $\omega$-sequence of partial isometries. Furthermore, suppose that the injective function $J\colon [d] \times \mathbb{N} \to \mathbb{N}$ is given by*

$$J(c,n) := (n-1)d + c.$$

*Then the family $\{(a_{c,n})_{n=1}^{\infty} \mid c \in [d]\}$, defined by*

$$a_{c,n} := a_{J(c,n)},$$

*is $[d]$-jointly spreadable and has the $[d]$-separate SVP (and thus also the $[d]$-joint SVP). Furthermore, $\{(a_{c,n})_{n=1}^{\infty} \mid c \in [d]\}$ has the factorization property*

$$\varphi\big(yP(a_{\ell,\bullet}, a_{\ell,\bullet}^*)z\big) = \varphi\big(P(a_{\ell,\bullet}, a_{\ell,\bullet}^*)\big)\varphi\big(yz\big)$$

*for $y, z \in$ *-alg$\{a_{k,c} \mid c \in [d], k \in \mathbb{N}, k \neq \ell\}$ and $\ell \in \mathbb{N}$, and $P(X_\bullet, Y_\bullet) \in \mathbb{C}\langle X_c, Y_c \mid c \in [d]\rangle$.*

*Proof.* The $[d]$-joint spreadability and $[d]$-separate (and thus $[d]$-joint) SVP is immediate from Lemma 3.4.15 and Example 3.4.16. We are left to show the claimed factorization property which is equivalent to the following factorization property by Corollary 3.5.6:

$$\varphi\Big(P(a_{1,\bullet}, a_{1,\bullet}^*)^* P(a_{2,\bullet}, a_{2,\bullet}^*)\Big) = \varphi\Big(P(a_{1,\bullet}, a_{1,\bullet}^*)^*\Big)\varphi\Big(P(a_{2,\bullet}, a_{2,\bullet}^*)\Big)$$

for $P(X_\bullet, Y_\bullet) \in \mathbb{C}\langle X_c, Y_c \mid c \in [d]\rangle$. Using the relations of an $\omega$-sequence of partial isometries, in particular (4.6) and (4.7), a term of the polynomial $P(a_{1,\bullet}, a_{1,\bullet}^*)$ is of the general form

$$M_1 M_2 \cdots M_d,$$

where each factor $M_i$ equals $a_i^{\epsilon_i}$ or $a_i a_i^*$ or $a_i^* a_i$ for some $\epsilon_i \in \{*, 1, 0\}$ and $i = 1, 2, \ldots, d$. Similarly, a term of the polynomial $P(a_{2,\bullet}, a_{2,\bullet}^*)$ is of the general form

$$N_1 N_2 \cdots N_d$$

where each factor $N_i$ equals $a_{i+d}^{\epsilon_i}$ or $a_{i+d} a_{i+d}^*$ or $a_{i+d}^* a_{i+d}$, corresponding to the form of $P(a_{1,\bullet}, a_{1,\bullet}^*)$. We are left to prove the factorization

$$\varphi((M_1 M_2 \cdots M_d)^*\, N_1 N_2 \cdots N_d) = \varphi((M_1 M_2 \cdots M_d)^*)\, \varphi(N_1 N_2 \cdots N_d).$$

But this factorization is established by a repeated application of the SVP and factorization property as stated in Theorem 4.2.9 (ii) and (iii). $\square$

We state next the multivariate version of a CLT as it is valid for jointly spreadable families obtained from a single $\omega$-sequence of partial isometries.

**Theorem 4.3.8.** *Let $(\mathcal{A},\varphi)$ be a tracial *-algebraic probability space which is equipped with the $\omega$-sequence of partial isometries $\mathbf{a} \equiv (a_n)_{n=1}^{\infty} \subset \mathcal{A}$. Furthermore, for fixed $d \in \mathbb{N}$ and $x_n := a_n + a_n^*$, put*

$$\begin{aligned}
S_{1,N} &:= \frac{1}{\sqrt{N}}\big(x_1 + x_{d+1} + \ldots + x_{(N-1)d+1}\big),\\
S_{2,N} &:= \frac{1}{\sqrt{N}}\big(x_2 + x_{d+2} + \ldots + x_{(N-1)d+2}\big),\\
&\;\vdots\\
S_{d,N} &:= \frac{1}{\sqrt{N}}\big(x_d + x_{2d} + \ldots + x_{Nd}\big).
\end{aligned}$$

*Then one has that, for all $k \in \mathbb{N}$,*

$$\lim_{N\to\infty} \varphi\big(S_{\mathbf{t}(1),N}\cdots S_{\mathbf{t}(2k-1),N}\big) = 0$$

*for all $\mathbf{t}\colon [2k-1] \to [d]$ and*

$$\lim_{N\to\infty} \varphi\big(S_{\mathbf{t}(1),N}\cdots S_{\mathbf{t}(2k),N}\big) = \frac{1}{k!}\frac{1}{2^k}\sum_{\sigma\in S_k}\sum_{\substack{\pi\in\mathcal{P}_2(2k)\\ \pi\leqslant\ker(\mathbf{t})}}\sum_{\substack{\boldsymbol{\varepsilon}\colon [2k]\to\{*,1\}\\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} \omega^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\overline{\omega}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}$$

*for all $\mathbf{t}\colon [2k] \to [d]$.*

We will see in Section 5.4 that the distribution of this multivariate CLT resembles that of an $\omega$-semicircular system, as introduced in Definition 5.4.1.

*Proof.* We have from Theorem 3.4.9 that

$$\lim_{N\to\infty} \varphi\big(S_{\mathbf{t}(1),N}\cdots S_{\mathbf{t}(2k),N}\big) = \frac{1}{k!}\sum_{\substack{\pi\in\mathcal{OP}_2(2k)\\ \overline{\pi}\leqslant\ker(\mathbf{t})}} \varphi^{\mathcal{O}}_{\pi,\mathbf{t}},$$

where $\varphi^{\mathcal{O}}_{\pi,\mathbf{t}} = \varphi\big(x_{\mathbf{t}(1),\mathbf{i}(1)}\cdots x_{\mathbf{t}(2k),\mathbf{i}(2k)}\big)$ for $\mathbf{i}\colon [2k] \to [k]$ with $\ker_{\mathcal{O}}(\mathbf{i}) = \pi$ and

$$x_{c,\ell} = x_{(\ell-1)d+c}.$$

Using the bijection from Lemma 2.1.26, we address an ordered pair partition $\pi \in \mathcal{OP}_2(2k)$ through the pair $(\overline{\pi},\sigma) \in \mathcal{P}_2(2k)\times S_k$. This allows us to rewrite $\varphi^{\mathcal{O}}_{\pi,\mathbf{t}}$ such that

$$\frac{1}{k!}\sum_{\substack{\pi\in\mathcal{OP}_2(2k)\\ \overline{\pi}\leqslant\ker(\mathbf{t})}} \varphi^{\mathcal{O}}_{\pi,\mathbf{t}} = \frac{1}{k!}\sum_{\sigma\in S_k}\sum_{\substack{\overline{\pi}\in\mathcal{P}_2(2k)\\ \overline{\pi}\leqslant\ker(\mathbf{t})}} \varphi_{\overline{\pi},\mathbf{t},\sigma},$$

where

$$\varphi_{\overline{\pi},\mathbf{t},\sigma} = \varphi\big(x_{\mathbf{t}(1),\sigma(\mathbf{i}(1))} \cdots x_{\mathbf{t}(2k),\sigma(\mathbf{i}(2k))}\big)$$

with $\overline{\pi} = \ker(\mathbf{i})$ in standard order for $\mathbf{i}\colon [2k] \to [k]$. We infer from spreadability that

$$\begin{aligned}\varphi\big(x_{\mathbf{t}(1),\sigma(\mathbf{i}(1))} \cdots x_{\mathbf{t}(2k),\sigma(\mathbf{i}(2k))}\big) &= \varphi\big(x_{(\sigma(\mathbf{i}(1))-1)d+\mathbf{t}(1)} \cdots x_{(\sigma(\mathbf{i}(2k))-1)d+\mathbf{t}(2k)}\big)\\ &= \varphi\big(x_{(\sigma(\mathbf{i}(1))-1)d} \cdots x_{(\sigma(\mathbf{i}(2k))-1)d}\big)\\ &= \varphi\big(x_{\sigma(\mathbf{i}(1))} \cdots x_{\sigma(\mathbf{i}(2k))}\big).\end{aligned}$$

This last expression is determined by Corollary 4.2.10 (iv), and as we have already shown in the proof of Theorem 4.3.1, to be of the form

$$\varphi\big(x_{\sigma(\mathbf{i}(1))} \cdots x_{\sigma(\mathbf{i}(2k))}\big) = \frac{1}{2^k} \sum_{\substack{\boldsymbol{\varepsilon}\colon [2k]\to\{*,1\}\\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} \omega^{\mathrm{cr}_+(\overline{\pi},\boldsymbol{\varepsilon},\sigma)}\, \overline{\omega}^{\mathrm{cr}_-(\overline{\pi},\boldsymbol{\varepsilon},\sigma)}.$$

Altogether, we arrive at

$$\frac{1}{k!} \sum_{\substack{\pi\in\mathcal{OP}_2(2k)\\ \overline{\pi}\leqslant\ker(\mathbf{t})}} \varphi^{\mathcal{O}}_{\pi,\mathbf{t}} = \frac{1}{k!}\frac{1}{2^k} \sum_{\sigma\in S_k} \sum_{\substack{\overline{\pi}\in\mathcal{P}_2(2k)\\ \overline{\pi}\leqslant\ker(\mathbf{t})}} \sum_{\substack{\boldsymbol{\varepsilon}\colon [2k]\to\{*,1\}\\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} \omega^{\mathrm{cr}_+(\overline{\pi},\boldsymbol{\varepsilon},\sigma)}\, \overline{\omega}^{\mathrm{cr}_-(\overline{\pi},\boldsymbol{\varepsilon},\sigma)}.$$

As the right-hand side of this equations involves the summation over all pair partitions instead of ordered pair partitions, we can finally replace $\overline{\pi} \in \mathcal{P}_2(2k)$ by $\pi \in \mathcal{P}_2(2k)$ in the final formula. $\square$

We turn now our attention to a multivariate setting with separate spreadability of sequences. Note that we are using local versions of separate spreadability and separate SVPs in the next result, as the 'block coloring' of a single spreadable sequence can be done only for finite parts of the sequence, in contrast to the situation of 'interleaving colorings'.

**Proposition 4.3.9.** *Let $(\mathcal{A}, \varphi)$ be a tracial *-algebraic probability space and $\mathbf{a} \equiv (a_n)_{n=1}^\infty \subset \mathcal{A}$ an $\omega$-sequence of partial isometries. Furthermore, suppose that the injective function $\{J_M\}_M\colon [d] \times [M] \to [dM]$ is given by*

$$J_M(t,n) := (t-1)M + n \qquad (t \in [d], n \in [M]).$$

*Then the family $\{(a_{c,n})_{n=1}^M \mid c \in [d]\}$, defined by*

$$a_{c,n} := a_{J_M(c,n)},$$

*is locally $[d]$-separately spreadable, i.e., for any $n \in \mathbb{N}$, for every $\boldsymbol{\varepsilon}\colon [n] \to \{*,1\}$, $\mathbf{i},\mathbf{j}\colon [n] \to [M]$, and $\mathbf{t}\colon [n] \to [d]$,*

$$\varphi\big(a^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),\mathbf{i}(1)} \cdots a^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),\mathbf{i}(n)}\big) = \varphi\big(a^{\boldsymbol{\varepsilon}(1)}_{\mathbf{t}(1),\mathbf{j}(1)} \cdots a^{\boldsymbol{\varepsilon}(n)}_{\mathbf{t}(n),\mathbf{j}(n)}\big)$$

*whenever* $\mathbf{i}|_W \sim_{\mathcal{O}} \mathbf{j}|_W$ *for every block* $W \in \ker(\mathbf{t})$*. Moreover, one has the local* $[d]$*-separate SVP, i.e., for any* $n \in \mathbb{N}$*, for every* $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$*,* $\mathbf{i}\colon [n] \to [M]$ *and* $\mathbf{t}\colon [n] \to [d]$*,*

$$\varphi\big(a_{\mathbf{t}(1),\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots a_{\mathbf{t}(n),\mathbf{i}(n)}^{\boldsymbol{\varepsilon}(n)}\big) = 0,$$

*whenever there exists a singleton* $\{\ell\} \in \ker(\mathbf{i}|_W)$ *for some* $\ell \in W$ *and some block* $W \in \ker(\mathbf{t})$*. Furthermore,* $\{(a_{c,n})_{n=1}^M \mid c \in [d]\}$ *has the factorization property*

$$\varphi\big(yP(a_{\ell,c}, a_{\ell,c}^*)z\big) = \varphi\big(P(a_{\ell,c}, a_{\ell,c}^*)\big)\varphi\big(yz\big) \tag{4.13}$$

*for* $y, z \in$ *-alg$\{x_{k,t} \mid t \in [d], k \in [M], (k,t) \neq (\ell, c)\}$*,* $\ell \in [M]$*, and* $P(X_c, Y_c) \in \mathbb{C}\langle X_c, Y_c \mid c \in [d]\rangle$*.*

*Proof.* The $[d]$-separate spreadability and $[d]$-separate SVP is immediate from Lemma 3.4.19 and Example 3.4.20, after adapting therein results from the infinite set $\mathbb{N}$ to the finite set $M$. Thus we are left to prove the factorization property (4.13), which is again immediate upon adapting Corollary 3.5.7 from $n$-tuples $\mathbf{i}\colon [n] \to \mathbb{N}$ to $n$-tuples $\mathbf{i}\colon [n] \to [M]$. Indeed, as we have already established the (local) $[d]$-separate spreadability of $\{(a_{c,n})_{n=1}^M \mid c \in [d]\}$, it suffices to check for $P(X_c, Y_c) \in \mathbb{C}\langle X_c, Y_c\rangle$ with $c \in [d]$ that

$$\varphi\Big(P(x_{c,1}, x_{c,1}^*)^* P(x_{c,2}, x_{c,2}^*)\Big) = \varphi\Big(P(x_{c,1}, x_{c,1}^*)^*\Big)\varphi\Big(P(x_{c,2}, x_{c,2}^*)\Big).$$

But this is immediate from the definition of the finite sequences $\{(a_{c,n})_{n=1}^M \mid c \in [d]\}$, the spreadability, and factorization property of the underlying $\omega$-sequence of partial isometries $\mathbf{a}$ (see Theorem 4.2.9(iii) for example). □

**Theorem 4.3.10.** *Let* $(\mathcal{A}, \varphi)$ *be a tracial *-algebraic probability space which is equipped with the* $\omega$*-sequence of partial isometries* $\mathbf{a} \equiv (a_n)_{n=1}^\infty \subset \mathcal{A}$*. Furthermore, for fixed* $d \in \mathbb{N}$ *and* $x_n := a_n + a_n^*$*, put*

$$\begin{aligned}
\widetilde{S}_{1,N} &:= \frac{1}{\sqrt{N}}\big(x_1 + \ldots + x_N\big),\\
\widetilde{S}_{2,N} &:= \frac{1}{\sqrt{N}}\big(x_{N+1} + \ldots + x_{2N}\big),\\
&\ \vdots\\
\widetilde{S}_{d,N} &:= \frac{1}{\sqrt{N}}\big(x_{(d-1)N+1} + \ldots + x_{2dN}\big).
\end{aligned}$$

*Then*

$$\lim_{N\to\infty} \varphi(\widetilde{S}_{\mathbf{t}(1),N} \cdots \widetilde{S}_{\mathbf{t}(2k-1),N}) = 0$$

*for all $k \in \mathbb{N}$ and $\mathbf{t}\colon [2k-1] \to [d]$. Furthermore, for all $k \in \mathbb{N}$,*

$$\lim_{N\to\infty} \varphi(\widetilde{S}_{\mathbf{t}(1),N} \cdots \widetilde{S}_{\mathbf{t}(2k),N}) = \frac{1}{k_1!} \cdots \frac{1}{k_d!} \frac{1}{2^{k_1}} \cdots \frac{1}{2^{k_d}} \sum_{\substack{\sigma\in S_k \\ \text{with } \sigma(W)=W \\ \text{for all } W\in\ker(\mathbf{t})}} \sum_{\substack{\pi\in\mathcal{P}_2(2k) \\ \pi\leqslant\ker(\mathbf{t})}} \sum_{\substack{\boldsymbol{\varepsilon}\colon [2k]\to\{*,1\} \\ \boldsymbol{\varepsilon}|_W \text{ is } \pi|_W\text{-balanced} \\ \text{for all } W\in\ker(\mathbf{t})}} \omega^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\overline{\omega}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}.$$

*for all $\mathbf{t}\colon [2k] \to [d]$. Here one has $k = k_1 + k_2 + \ldots + k_d$, where $k_c = |\mathbf{t}^{-1}\{c\}|/2$ for $c \in [d]$.*

*Proof.* We restrict our considerations to moments of even order $n = 2k$ with $k \in \mathbb{N}$, as moments of odd order are easily seen to vanish. There are essentially two ways to prove the large $N$-limit moment formula stated in the theorem. One way is to immediately use

$$x_{c,n} = x_{(c-1)M+n}, \qquad (c \in [d], n \in [M]),$$

with the goal to directly verify the claimed formula for an $\omega$-sequence of partial isometries. Here we take an alternative way and use the CLT already provided earlier for $[d]$-separately spreadable sequences. We know from Theorem 3.4.13 that, for all $k \in \mathbb{N}$ and $\mathbf{t}\colon [2k] \to [d]$,

$$\lim_{N\to\infty} \varphi(\widetilde{S}_{\mathbf{t}(1),N} \cdots \widetilde{S}_{\mathbf{t}(2k),N}) = \prod_{W\in\ker(\mathbf{t})} \frac{1}{(|W|/2)!} \sum_{\pi_{|\bullet}\in\prod_{W\in\ker(\mathbf{t})}\mathcal{OP}_2(W)} \varphi^{\mathcal{O}}_{\pi_|,\mathbf{t}},$$

where

$$\varphi^{\mathcal{O}}_{\pi_|,\mathbf{t}} = \varphi(x_{\mathbf{t}(1),\mathbf{i}(1)} \cdots x_{\mathbf{t}(2k),\mathbf{i}(2k)})$$

for any $\mathbf{i}\colon [2k] \to [M]$ with $\pi_{|W} = \ker_{\mathcal{O}}(\mathbf{i}|_W)$ for all $W \in \ker(\mathbf{t})$ and

$$x_{c,n} := x_{(c-1)M+n}, \qquad (c \in [d], n \in [M]).$$

Since $\pi_{|\bullet} \in \prod_{W\in\ker(\mathbf{t})} \mathcal{OP}_2(W)$, we can assume that every block $W \in \ker(\mathbf{t})$ has an even cardinality such that

$$\operatorname{card}\{\sigma \in S_k \mid \sigma(W) = W \text{ for all } W \in \ker(\mathbf{t})\} = \prod_{W\in\ker(\mathbf{t})} (|W|/2)!.$$

Putting $W_c := \mathbf{t}^{-1}(\{c\})$ and $k_c := |W_c|/2$ for $c = 1, 2, \ldots, d$, one has

$$\prod_{W\in\ker(\mathbf{t})} (|W|/2)! = \prod_{c=1}^{d} k_c!.$$

Furthermore, the ordered pair partition $\pi_{|\bullet} \in \prod_{W\in\ker(\mathbf{t})} \mathcal{OP}_2(W)$ can be uniquely addressed by the pair

$$(\overline{\pi}, \sigma) \in \mathcal{P}_2(2k) \times S_k$$

which satisfies $\overline{\pi} \leq \ker(\mathbf{t})$ and $\sigma(W) = W$ for all $W \in \ker(\mathbf{t})$. Thus we have arrived at

$$\lim_{N\to\infty} \varphi(\widetilde{S}_{\mathbf{t}(1),N} \cdots \widetilde{S}_{\mathbf{t}(2k),N}) = \frac{1}{k_1!} \cdots \frac{1}{k_d!} \sum_{\substack{\sigma\in S_k \\ \text{with } \sigma(W)=W \\ \text{for all } W\in\ker(\mathbf{t})}} \sum_{\substack{\overline{\pi}\in\mathcal{P}_2(2k) \\ \overline{\pi}\leqslant\ker(\mathbf{t})}} \varphi_{\overline{\pi},\mathbf{t},\sigma}, \tag{4.14}$$

where

$$\varphi_{\overline{\pi},\mathbf{t},\sigma} := \varphi\big(x_{\mathbf{t}(1),\sigma(\mathbf{i}(1))} \cdots x_{\mathbf{t}(2k),\sigma(\mathbf{i}(2k))}\big)$$

for $\mathbf{i}\colon [2k] \to [k]$ with $\overline{\pi} = \ker(\mathbf{i})$. We apply next Corollary 4.2.10(iv) such that

$$\begin{aligned}
\varphi_{\overline{\pi},\mathbf{t},\sigma} &= \varphi\big(x_{\mathbf{t}(1),\sigma(\mathbf{i}(1))} \cdots x_{\mathbf{t}(2k),\sigma(\mathbf{i}(2k))}\big) \\
&= \varphi\big(x_{(\mathbf{t}(1)-1)M+\sigma(\mathbf{i}(1))} \cdots x_{(\mathbf{t}(2k)-1)M+\sigma(\mathbf{i}(2k))}\big) \\
&= \frac{1}{2^k} \sum_{\substack{\boldsymbol{\varepsilon}\colon [2k]\to\{*,1\} \\ \boldsymbol{\varepsilon} \text{ is } \overline{\pi}\text{-balanced}}} \omega^{\mathrm{cr}_+(\overline{\pi},\boldsymbol{\varepsilon},\sigma)}\overline{\omega}^{\mathrm{cr}_-(\overline{\pi},\boldsymbol{\varepsilon},\sigma)}.
\end{aligned}$$

Since $\overline{\pi} \in \mathcal{P}_2(2k)$ satisfies $\overline{\pi} \leq \ker(\mathbf{t})$, the two sets

$$\{\boldsymbol{\varepsilon}\colon [2k] \to \{*,1\} \mid \ \boldsymbol{\varepsilon}|_W \text{ is } \pi|_W\text{-balanced for all } W \in \ker(\mathbf{t})\}$$

and

$$\{\boldsymbol{\varepsilon}\colon [2k] \to \{*,1\} \mid \ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}\}$$

are the same. Moreover, we note that $2^k = 2^{k_1} \cdot 2^{k_2} \cdots 2^{k_d}$. Altogether, this shows that (4.14) can be rewritten as claimed by the theorem. □

There are many possible choices of how one can construct multivariate spreadable sequences from a single spreadable sequence. These choices may yield different multivariate CLTs, as we have seen in Theorem 4.3.8 and Theorem 4.3.10. This difference of mixed moments occurs starting the 8-th order, but it is absent for the 2-nd, 4-th and 6-th order. We illustrate this in the following examples.

**Example 4.3.11** (4-th mixed moments)**.** Let $d = 2$ and consider $S_{1,N}$ and $S_{2,N}$ as introduced in Theorem 4.3.8 for jointly spreadable sequences. Using the result of Theorem 4.3.8 for $k = 2$ and $\mathbf{t}(1) = \mathbf{t}(2) = 1$ and $\mathbf{t}(3) = \mathbf{t}(4) = 2$ , one has

$$\lim_{N\to\infty} \varphi(S_{1,N}S_{1,N}S_{2,N}S_{2,N}) = \frac{1}{2!}\frac{1}{2^2} \sum_{\sigma\in S_2} \sum_{\substack{\pi\in\mathcal{P}_2(4) \\ \pi\leqslant\ker(\mathbf{t})}} \sum_{\substack{\boldsymbol{\varepsilon}\colon [4]\to\{*,1\} \\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} \omega^0\overline{\omega}^0 = 1.$$

Taking instead $\mathbf{t}(1) = \mathbf{t}(3) = 1$ and $\mathbf{t}(2) = \mathbf{t}(4) = 2$, one obtains

$$\lim_{N\to\infty} \varphi(S_{1,N}S_{2,N}S_{1,N}S_{2,N}) = \frac{1}{2!}\frac{1}{2^2} \sum_{\substack{\boldsymbol{\varepsilon}\colon [4]\to\{*,1\} \\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} (\omega + \overline{\omega}) = \Re\omega,$$

due to the following arguments. One has only one pair partition $\pi$ satisfying $\pi \leq \ker(\mathbf{t})$. Moreover, one has either a single positive crossing or a single negative crossing for each triple $(\pi, \boldsymbol{\varepsilon}, \sigma)$. Since the permutation $\sigma \neq e$ changes the order, the summation over the symmetric group $S_2$ yields the factor $\omega + \overline{\omega}$.

Alternatively, consider $\widetilde{S}_{1,N}$ and $\widetilde{S}_{2,N}$ as introduced in Theorem 4.3.10. Then one has

$$\lim_{N\to\infty} \varphi(\widetilde{S}_{1,N}\widetilde{S}_{1,N}\widetilde{S}_{2,N}\widetilde{S}_{2,N}) = \frac{1}{1!}\frac{1}{1!}\frac{1}{2^1}\frac{1}{2^1}\sum_{\substack{\sigma\in S_2\\ \text{with } \sigma(W)=W\\ \text{for all } W\in\ker(\mathbf{t})}} \sum_{\substack{\pi\in\mathcal{P}_2(4)\\ \pi\leqslant\ker(\mathbf{t})}} \sum_{\substack{\boldsymbol{\varepsilon}\colon [4]\to\{*,1\}\\ \boldsymbol{\varepsilon}|_W \text{ is } \pi|_W\text{-balanced}\\ \text{for all } W\in\ker(\mathbf{t})}} \omega^0\overline{\omega}^0 = 1$$

and

$$\lim_{N\to\infty} \varphi(\widetilde{S}_{1,N}\widetilde{S}_{2,N}\widetilde{S}_{1,N}\widetilde{S}_{2,N}) = \Re\omega.$$

**Example 4.3.12** (6-th mixed moments)**.** Explicit computations for $d = 2$ show that all mixed moments of 6-th order are the same for the joint spreadable sequences and the locally separately spreadable sequences considered in Theorem 4.3.8 and Theorem 4.3.10. For example, one has

$$\lim_{N\to\infty} \varphi(S_{1,N}S_{1,N}S_{1,N}S_{1,N}S_{2,N}S_{2,N}) = \Re\omega + 2$$

and

$$\lim_{N\to\infty} \varphi(\widetilde{S}_{1,N}\widetilde{S}_{1,N}\widetilde{S}_{1,N}\widetilde{S}_{1,N}\widetilde{S}_{2,N}\widetilde{S}_{2,N}) = \Re\omega + 2.$$

**Example 4.3.13** (8-th mixed moments)**.** In contrast to the situation for mixed moments of 6-th order, differences occur for some mixed moments of 8-th order. For example, one obtains

$$\lim_{N\to\infty} \varphi(S_{1,N}S_{1,N}S_{2,N}S_{2,N}S_{1,N}S_{1,N}S_{2,N}S_{2,N}) = \frac{1}{48}(\omega^6 + \overline{\omega}^6) + P(\omega, \overline{\omega})$$

in the jointly spreadable setting, but

$$\lim_{N\to\infty} \varphi(\widetilde{S}_{1,N}\widetilde{S}_{1,N}\widetilde{S}_{2,N}\widetilde{S}_{2,N}\widetilde{S}_{1,N}\widetilde{S}_{1,N}\widetilde{S}_{2,N}\widetilde{S}_{2,N}) = \frac{1}{32}(\omega^6 + \overline{\omega}^6) + \widetilde{P}(\omega, \overline{\omega})$$

in the locally separately spreadable setting. Here $P$ and $\widetilde{P}$ are polynomials in the commuting variables of $\omega$ and $\overline{\omega}$ of degree 5 at most.

# Chapter 5

# Circular and Semicircular Systems

This chapter starts with reviewing multivariate versions of CLTs for $q$-circular and $q$-semicircular systems. We show that such systems are exchangeable and thus yield CLTs. In particular, we show that certain multivariate CLTs associated to $q$-circular systems and $q$-semicircular systems have moment formulas which reproduce those of $q$-circular systems and $q$-semicircular systems, respectively.

Inspired by the notion of a '*$z$-circular system*', defined and studied by Mingo and Nica in [MN01], we introduce the notion of a '*$z$-semicircular system*'. These generalize the corresponding notions of $q$-circular and $q$-semicircular systems from the parameter $q \in [-1, 1]$ to the parameter $z \in \mathbb{C}$ with $|z| \leq 1$. We show that such systems are spreadable and satisfy SVPs, as we have met them earlier in the context of $\omega$-sequences. Thus $z$-circular systems and $z$-semicircular systems yield CLTs such that their moment formulas generalize those moment formulas obtained from CLTs associated to $\omega$-sequences of partial isometries. In particular, we show that certain multivariate CLTs for $z$-(semi)circular systems yield $z$-(semi)circular systems in the large $N$-limit.

## 5.1 $q$-Circular Systems

In this section we review the notion of so-called $q$-circular systems. Such systems are defined in [MN01] for the open interval $q \in (-1, 1)$ in the framework of C*-algebraic probability spaces. Our approach adapts that of [MN01] to the framework of *-algebraic probability spaces for $q \in [-1, 1]$.

**Definition 5.1.1.** ([MN01]) Let $(\mathcal{A}, \psi)$ be a *-algebraic probability space and let $q \in [-1, 1]$. The family $\{c_r\}_{r=1}^{s} \subseteq \mathcal{A}$ ($s \geq 1$) is said to form a *$q$-circular system* in $(\mathcal{A}, \psi)$ if, for every $n \geq 1$, $\mathbf{r}\colon [n] \to [s]$, and $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$,

$$\psi(c_{\mathbf{r}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots c_{\mathbf{r}(n)}^{\boldsymbol{\varepsilon}(n)}) = \sum_{\pi \in \mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon})} q^{\mathrm{cr}(\pi)}. \tag{5.1}$$

Here $\mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon})$ denotes the set of all pair partitions

$$\pi = \{\{a_1, b_1\}, \dots, \{a_k, b_k\}\}$$

of $\{1, \dots, n\}$ which have the property $a_i < b_i$, $\mathbf{r}(a_i) = \mathbf{r}(b_i)$, and $\boldsymbol{\varepsilon}(a_i) \neq \boldsymbol{\varepsilon}(b_i)$ for all $i \in [k]$.

**Notation 5.1.2.** Given $\mathbf{r}\colon [n] \to [s]$ and $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$, the set $\mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon})$ will occasionally be written as $\mathcal{P}_2(\mathbf{r}(1), \dots, \mathbf{r}(n); \boldsymbol{\varepsilon}(1), \dots, \boldsymbol{\varepsilon}(n))$.

**Remark 5.1.3.** (i) If n odd, then $\mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon}) \subset \mathcal{P}_2(n) = \emptyset$. Thus

$$\psi(c_{\mathbf{r}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots c_{\mathbf{r}(n)}^{\boldsymbol{\varepsilon}(n)}) = 0.$$

(ii) More generally, suppose $\mathbf{r}\colon [n] \to [s]$ is such that its kernel set partition $\pi = \ker(\mathbf{r}) \in \mathcal{P}(n)$ contains a block $V \in \pi$ with odd cardinality $|V|$. Then one obtains $\mathcal{P}_2(\mathbf{r}; \boldsymbol{\varepsilon}) = \emptyset$ and thus

$$\psi(c_{\mathbf{r}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots c_{\mathbf{r}(n)}^{\boldsymbol{\varepsilon}(n)}) = 0. \tag{5.2}$$

(iii) If $n$ even and $|\boldsymbol{\varepsilon}^{-1}(\{1\})| \neq |\boldsymbol{\varepsilon}^{-1}(\{*\})|$, then $\mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon}) = \emptyset$ and thus

$$\psi(c_{\mathbf{r}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots c_{\mathbf{r}(n)}^{\boldsymbol{\varepsilon}(n)}) = 0.$$

(iv) If $n$ even and $|\boldsymbol{\varepsilon}^{-1}(\{1\})| = |\boldsymbol{\varepsilon}^{-1}(\{*\})|$, then $\mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon})$ may still be the empty set. One has $\mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon}) \neq \emptyset$ if and only if $n$ is even and $\mathbf{r}(a) = \mathbf{r}(b)$ implies $\boldsymbol{\varepsilon}(a) \neq \boldsymbol{\varepsilon}(b)$ for any $a < b$. For example, if $\mathbf{r}(1) = \mathbf{r}(2) = \ldots = \mathbf{r}(n)$ and $n = 2k$, then we arrive at

$$|\mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon})| = k!.$$

To further specify this example, consider $n = 4$ and assume $\mathbf{r}(1) = \mathbf{r}(2) = \mathbf{r}(3) = \mathbf{r}(4) = 1$, as well as $\boldsymbol{\varepsilon}(1) = \boldsymbol{\varepsilon}(2) = *$, $\boldsymbol{\varepsilon}(3) = \boldsymbol{\varepsilon}(4) = 1$. Then one has

$$\mathcal{P}_2(1, 1, 1, 1; *, *, 1, 1) = \big\{\{\{1, 3\}, \{2, 4\}\}, \{\{1, 4\}, \{2, 3\}\}\big\}.$$

These two pair partitions are visualized in the following diagrams, respectively.


Then, by Definition 5.1.1, one has

$$\psi(c_1^* c_1^* c_1 c_1) = \sum_{\pi \in \mathcal{P}_2(1,1,1,1;*,*,1,1)} q^{\mathrm{cr}(\pi)} = q + 1.$$

Note also that, if the tuple $(\mathbf{r}(1), \mathbf{r}(2), \dots, \mathbf{r}(n))$ defines a pair partition of the set $[n]$, then one has that either $\mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon}) = \emptyset$ or

$$|\mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon})| = 1,$$

depending on the choice of the direction map $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$. Let us say, for example, $n = 4$, $\mathbf{r}(1) = \mathbf{r}(3)$, $\mathbf{r}(2) = \mathbf{r}(4)$, and $\boldsymbol{\varepsilon}(1) = \boldsymbol{\varepsilon}(4) = *$, $\boldsymbol{\varepsilon}(2) = \boldsymbol{\varepsilon}(3) = 1$. Then the right-hand side of the moment formula (5.1) is

$$\sum_{\pi \in \mathcal{P}_2(r_1, r_2, r_3, r_4; *, 1, 1, *)} q^{\mathrm{cr}(\pi)} = q.$$

**Remark 5.1.4.** We further discuss some properties of this set of pair partitions $\mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon})$.

(i) Consider the case $s = \{1\}$ and suppose that the direction map $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$ satisfies $|\boldsymbol{\varepsilon}^{-1}(\{*\})| = |\boldsymbol{\varepsilon}^{-1}(\{1\})|$ for $n = 2k$ even. Then the set $\mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon})$ contains $k!$ pair partitions.

(ii) We consider next the case $s > 1$. Given the 'color' map $\mathbf{r}\colon [n] \to [s]$ and the direction map $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$ for some $n \in \mathbb{N}$, we formulate conditions which ensure that $\mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon})$ is not the empty set. Clearly, the cardinality of each pre-image $2k_m := |\mathbf{r}^{-1}(\{m\})|$ needs to be an even number, for some $k_m \in \mathbb{N}$ and $m = 1, \dots, s$. Thus, putting $k := k_1 + k_2 + \ldots + k_s$, one has $n = 2k$. Furthermore, one needs

$$|\{i \in [n] \mid \mathbf{r}(i) = m, \boldsymbol{\varepsilon}(i) = *\}| = |\{i \in [n] \mid \mathbf{r}(i) = m, \boldsymbol{\varepsilon}(i) = 1\}|$$

for all $m = 1, 2, \dots, s$. Then the set $\mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon})$ contains $k_1! \cdots k_s!$ pair partitions.

Note also that, having chosen the color map $\mathbf{r}\colon [n] \to [s]$ such that all its level sets contain an even number of elements, there are

$$\binom{2k_1}{k_1} \cdot \binom{2k_2}{k_2} \cdots \binom{2k_s}{k_s}$$

choices of $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$ such that the set $\mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon})$ is non-empty.

We show next that $q$-circular systems provide an interesting class of (finite) exchangeable sequences. As we have introduced exchangeability and various other properties only for infinite sequences, we will concentrate in the following on dealing with $q$-circular systems containing infinitely many elements, corresponding to the case $s = \infty$. Note that most of our results easily transfer to the finite case $1 \leq s < \infty$.

**Proposition 5.1.5.** *Let $(\mathcal{A}, \psi)$ be a *-algebraic probability space and suppose $\mathcal{Y} \equiv \{c_r\}_{r=1}^{\infty} \subseteq \mathcal{A}$ is a q-circular system in $(\mathcal{A}, \psi)$. Then $\mathcal{Y}$ is exchangeable and satisfies the SVP.*

*Proof.* Let $\mathbf{r}, \widehat{\mathbf{r}} \colon [n] \to \mathbb{N}$ for $n \in \mathbb{N}$ such that $\widehat{\mathbf{r}} = \sigma \circ \mathbf{r}$ for some permutation $\sigma \in S_\infty$. We need to show that

$$\psi(c_{\mathbf{r}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots c_{\mathbf{r}(n)}^{\boldsymbol{\varepsilon}(n)}) = \psi(c_{\widehat{\mathbf{r}}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots c_{\widehat{\mathbf{r}}(n)}^{\boldsymbol{\varepsilon}(n)}).$$

For this purpose, it is sufficient to show that $\mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon}) = \mathcal{P}_2(\widehat{\mathbf{r}}, \boldsymbol{\varepsilon})$. Indeed, it follows from the assumptions on the two maps $\mathbf{r}$ and $\widehat{\mathbf{r}}$ that

$$\pi \in \mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon}) \Leftrightarrow \pi \in \mathcal{P}_2(\widehat{\mathbf{r}}, \boldsymbol{\varepsilon}).$$

Since we are summing over the same set of pair partitions, we deduce from the mixed moment formula (5.1) that

$$\psi(c_{\mathbf{r}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots c_{\mathbf{r}(n)}^{\boldsymbol{\varepsilon}(n)}) = \sum_{\pi \in \mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon})} q^{\mathrm{cr}(\pi)} = \sum_{\pi \in \mathcal{P}_2(\widehat{\mathbf{r}}, \boldsymbol{\varepsilon})} q^{\mathrm{cr}(\pi)} = \psi(c_{\widehat{\mathbf{r}}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots c_{\widehat{\mathbf{r}}(n)}^{\boldsymbol{\varepsilon}(n)}).$$

We are left to show that $\mathcal{Y}$ satisfies the SVP. Suppose the partition $\pi := \ker(\mathbf{r}) \in \mathcal{P}(n)$ has the singleton $\{\ell\}$ for $1 \leq \ell \leq n$. This implies $\mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon}) = \emptyset$ and thus $\psi(c_{\mathbf{r}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots c_{\mathbf{r}(\ell)}^{\boldsymbol{\varepsilon}(\ell)} \cdots c_{\mathbf{r}(n)}^{\boldsymbol{\varepsilon}(n)}) = 0$ by (5.2). □

**Theorem 5.1.6.** *Suppose the family $\mathcal{Y} \equiv \{c_r\}_{r=1}^{\infty} \subseteq \mathcal{A}$ forms a q-circular system in $(\mathcal{A}, \psi)$. Let*

$$S_{1,N} := \frac{1}{\sqrt{N}}\big(c_1 + c_{s+1} + \ldots + c_{(N-1)s+1}\big),$$
$$S_{2,N} := \frac{1}{\sqrt{N}}\big(c_2 + c_{s+2} + \ldots + c_{(N-1)s+2}\big),$$
$$\vdots$$
$$S_{s,N} := \frac{1}{\sqrt{N}}\big(c_s + c_{2s} + \ldots + c_{Ns}\big).$$

*Then one has, for all $k \in \mathbb{N}$, $\mathbf{r} \colon [2k] \to \mathbb{N}$, and $\boldsymbol{\varepsilon} \colon [2k] \to \{*, 1\}$,*

$$\lim_{N \to \infty} \psi(S_{\mathbf{r}(1),N}^{\boldsymbol{\varepsilon}(1)} \cdots S_{\mathbf{r}(2k),N}^{\boldsymbol{\varepsilon}(2k)}) = \sum_{\pi \in P_2(\mathbf{r}, \boldsymbol{\varepsilon})} q^{\mathrm{cr}(\pi)}.$$

*Proof.* We conclude from Proposition 5.1.5 that $\mathcal{Y}$ is exchangeable and satisfies the SVP. Therefore, the CLT applies as formulated in Theorem 3.3.10 (or Theorem 3.3.13). Clearly, all odd moments of order $2k-1$ vanish in the large $N$-limit.

We are left to verify that the even moment formulas of a $q$-circular system are reproduced in the large $N$-limit of the considered multivariate CLT.

$$\lim_{N\to\infty} \psi(S^{\boldsymbol{\varepsilon}(1)}_{\mathbf{r}(1),N} \cdots S^{\boldsymbol{\varepsilon}(2k)}_{\mathbf{r}(2k),N}) = \sum_{\substack{\pi\in\mathcal{P}_2(2k)\\ \pi\leq\ker(\mathbf{r})}} \psi_{\pi,\mathbf{r},\boldsymbol{\varepsilon}}$$

where $\psi_{\pi,\mathbf{r},\boldsymbol{\varepsilon}} := \psi(c^{\boldsymbol{\varepsilon}(1)}_{\mathbf{r}(1)} \cdots c^{\boldsymbol{\varepsilon}(n)}_{\mathbf{r}(n)})$ for $\mathbf{r}\colon [2k] \to \mathbb{N}$ with $\pi = \ker(\mathbf{r})$. One obtains

$$\begin{aligned}\sum_{\substack{\pi\in\mathcal{P}_2(2k)\\ \pi\leq\ker(\mathbf{r})}} \psi_{\pi,\mathbf{r},\boldsymbol{\varepsilon}} &= \sum_{\pi\in\mathcal{P}_2(\mathbf{r},\boldsymbol{\varepsilon})} \psi(c^{\boldsymbol{\varepsilon}(1)}_{\mathbf{r}(1)} \cdots c^{\boldsymbol{\varepsilon}(2k)}_{\mathbf{r}(2k)})\\ &= \sum_{\pi\in\mathcal{P}_2(\mathbf{r},\boldsymbol{\varepsilon})} q^{\mathrm{cr}(\pi)}.\end{aligned}$$

Here we have used for the first equality that a summand $\psi_{\pi,\mathbf{r},\boldsymbol{\varepsilon}}$ may be non-zero for a pair partition $\pi \in \mathcal{P}_2(2k)$ with $\pi \leq \ker(\mathbf{r})$ only if $\boldsymbol{\varepsilon}(i) \neq \boldsymbol{\varepsilon}(j)$ for all blocks $\{i,j\} \in \pi$. Thus we can restrict the summation from the set of pair partitions $\{\pi \in \mathcal{P}_2(2k) \mid \pi \leq \ker(\mathbf{r})\}$ to its subset $\mathcal{P}_2(\mathbf{r},\boldsymbol{\varepsilon})$. □

A limit model (as it is the subject of Theorem 3.6.1) for this multivariate CLT can be identified again in terms of $q$-circular systems.

**Corollary 5.1.7.** *Let the $q$-circular system $\mathcal{Y}$ in $(\mathcal{A},\psi)$ and $S_{1,N},\ldots,S_{s,N}$ be given as in Theorem 5.1.6. Then there exists a *-algebraic probability space $(\widehat{\mathcal{A}},\widehat{\psi})$ and a $q$-circular system $\widehat{\mathcal{Y}} \equiv \{\widehat{c}_r\}_{r=1}^s$ in $(\widehat{\mathcal{A}},\widehat{\psi})$ such that*

$$\lim_{N\to\infty} \psi\big(S_{\mathbf{r}(1),N} \cdots S_{\mathbf{r}(n),N}\big) = \widetilde{\psi}\big(\widetilde{c}^{\boldsymbol{\varepsilon}(1)}_{\mathbf{r}(1)} \cdots \widetilde{c}^{\boldsymbol{\varepsilon}(n)}_{\mathbf{r}(n)}\big)$$

*for all $n \in \mathbb{N}$, $\mathbf{r}\colon [n] \to [s]$, and $\boldsymbol{\varepsilon}\colon [n] \to \{*,1\}$.*

*Proof.* This is immediate from the mixed moment formula of the CLT in Theorem 5.1.6, as its right-hand side is the defining moment formula for the $q$-circular system $\{c_r\}_{r=1}^s$ in $(\mathcal{A},\psi)$. □

## 5.2 $q$-Semicircular Systems

In this section we discuss *$q$-semicircular systems* which are also known in the published literature under the name of *(systems of) $q$-Gaussian random variables* (see also [BS92]). Here we introduce $q$-semicircular systems in the framework of *-algebraic probability spaces through their moments in a combinatorial way. Afterwards we discuss how one can obtain a $q$-semicircular system from a $q$-circular system. Furthermore, we show that a $q$-semicircular system is exchangeable and satisfies the SVP. We close this section with a CLT for a $q$-semicircular system, Theorem 5.2.7, and Corollary 5.2.8, where the latter shows that multivariate CLTs of $q$-semicircular systems yield in the large $N$-limit again $q$-semicircular systems.

**Definition 5.2.1.** Let $(\mathcal{A}, \psi)$ be a *-algebraic probability space and let $q \in [-1, 1]$. The family $\{\widehat{s}_r\}_{r=1}^{s} \subseteq \mathcal{A}$ $(s \geq 1)$ is said to form a *$q$-semicircular system* in $(\mathcal{A}, \psi)$ if

- $\widehat{s}_r = \widehat{s}_r^*$ for $r = 1, 2, \ldots, s$;

- for every $n \geq 1$ and $\mathbf{r}\colon [n] \to [s]$,

$$\psi(\widehat{s}_{\mathbf{r}(1)} \cdots \widehat{s}_{\mathbf{r}(n)}) = \sum_{\substack{\pi \in \mathcal{P}_2(n) \\ \pi \leq \ker(\mathbf{r})}} q^{\operatorname{cr}(\pi)}.$$

**Example 5.2.2.** The even moments of a single $q$-semicircular operator $\widehat{s}$ (for example obtained for $s = 1$ and writing $\widehat{s}_1$ just as $\widehat{s}$) are given by

$$\psi(\widehat{s}^{2k}) = \sum_{\pi \in \mathcal{P}_2(2k)} q^{\operatorname{cr}(\pi)}.$$

Thus the first few even moments are given by

$$\psi(\widehat{s}^2) = 1$$

$$\psi(\widehat{s}^4) = 1 + q$$

$$\psi(\widehat{s}^6) = 5 + 6q + 3q^2 + q^3$$

$$\psi(\widehat{s}^8) = 14 + 28q + 28q^2 + 20q^3 + 10q^4 + 4q^5 + q^6$$

In the following, we prove that one can obtain a $q$-semicircular system from a $q$-circular system. We will make use of the following lemma.

**Lemma 5.2.3.** *The two sets*

$$\Big\{(\pi, \boldsymbol{\varepsilon}) \in \mathcal{P}_2(2k) \times \big\{f\colon [2k] \to \{*, 1\}\big\} \Big| \pi \in \mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon})\Big\}$$

*and*

$$\Big\{(\pi, \boldsymbol{\varepsilon}) \in \mathcal{P}_2(2k) \times \big\{f\colon [2k] \to \{*, 1\}\big\} \Big| \pi \leq \ker(\mathbf{r}), \boldsymbol{\varepsilon} \textit{ is } \pi\textit{-balanced}\Big\}$$

*are the same.*

*Proof.* This is immediate from Definition 2.1.33 where both $\mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon})$ and the $\pi$-balancedness of a direction map $\boldsymbol{\varepsilon}$ are introduced. $\square$

**Proposition 5.2.4.** *Let $(\mathcal{A}, \psi)$ be a *-algebraic probability space. Given the $q$-circular system $\{c_r\}_{r=1}^{s} \subseteq \mathcal{A}$ $(s \geq 1)$, let $\widehat{s}_m := \dfrac{1}{\sqrt{2}}(c_m + c_m^*)$ for $m \in [s]$. Then $\{\widehat{s}_1, \ldots, \widehat{s}_s\}$ is a $q$-semicircular system.*

*Proof.* Let $n \in \mathbb{N}$ and $\mathbf{r}\colon [n] \to [s]$ be fixed. We compute that

$$\begin{aligned}
\psi(\widehat{s}_{\mathbf{r}(1)} \cdots \widehat{s}_{\mathbf{r}(n)}) &= \frac{1}{2^{n/2}} \sum_{\boldsymbol{\varepsilon}\colon [n]\to\{*,1\}} \psi(c_{\mathbf{r}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots c_{\mathbf{r}(n)}^{\boldsymbol{\varepsilon}(n)}) \\
&= \frac{1}{2^{n/2}} \sum_{\boldsymbol{\varepsilon}\colon [n]\to\{*,1\}} \sum_{\pi\in\mathcal{P}_2(\mathbf{r},\boldsymbol{\varepsilon})} q^{\mathrm{cr}(\pi)} \qquad \text{(by Definition 5.1.1)} \\
&= \frac{1}{2^{n/2}} \sum_{\substack{\pi\in\mathcal{P}_2(n) \\ \pi\leq\ker(\mathbf{r})}} \sum_{\substack{\boldsymbol{\varepsilon}\colon [n]\to\{*,1\} \\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} q^{\mathrm{cr}(\pi)} \\
&= \sum_{\substack{\pi\in\mathcal{P}_2(n) \\ \pi\leq\ker(\mathbf{r})}} q^{\mathrm{cr}(\pi)}.
\end{aligned}$$

Here we have used Lemma 5.2.3 for the third equality. □

**Conjecture 5.2.5.** *Let $(\mathcal{A}, \psi)$ be a *-algebraic probability space. Given the $q$-semicircular system $\{\widehat{s}_r\}_{r=1}^{2s} \subseteq \mathcal{A}$ $(s \geq 1)$, let $c_m := \frac{1}{\sqrt{2}}(\widehat{s}_{2m-1} + \mathrm{i}\,\widehat{s}_{2m})$ for $m \in [s]$. Then $\{c_1, \ldots, c_s\}$ is a $q$-circular system.*

We expect that this conjecture can be verified using a concrete realization of $q$-circular systems and $q$-semicircular systems in terms of creation and annihilation operators on the $q$-Fock space, similar to the approach of Mingo and Nica in [MN01].

We prove next that $q$-semicircular systems provide interesting classes of exchangeable sequences in the case $s = \infty$, similar to our results for $q$-circular systems (see Proposition 5.1.5).

**Proposition 5.2.6.** *Let $(\mathcal{A}, \psi)$ be *-algebraic probability space and suppose that the family $\widetilde{\mathcal{Y}} \equiv \{\widehat{s}_r\}_{r=1}^{\infty} \subseteq \mathcal{A}$ is a $q$-semicircular system in $(\mathcal{A}, \psi)$. Then $\widetilde{\mathcal{Y}}$ is exchangeable and satisfies the SVP.*

*Proof.* All arguments used in Proposition 5.1.5 transfer from $q$-circular systems to $q$-semicircular systems. □

**Theorem 5.2.7.** *Suppose the family $\widetilde{\mathcal{Y}} \equiv (\widehat{s}_r)_{r=1}^{\infty} \subseteq \mathcal{A}$ forms a $q$-semicircular system in $(\mathcal{A}, \psi)$. Let*

$$\begin{aligned}
\widetilde{S}_{1,N} &:= \frac{1}{\sqrt{N}}\big(\widehat{s}_1 + \widehat{s}_{s+1} + \ldots + \widehat{s}_{(N-1)s+1}\big), \\
\widetilde{S}_{2,N} &:= \frac{1}{\sqrt{N}}\big(\widehat{s}_2 + \widehat{s}_{s+2} + \ldots + \widehat{s}_{(N-1)s+2}\big), \\
&\vdots \\
\widetilde{S}_{s,N} &:= \frac{1}{\sqrt{N}}\big(\widehat{s}_s + \widehat{s}_{2s} + \ldots + \widehat{s}_{Ns}\big).
\end{aligned}$$

*Then one has for all $n \in \mathbb{N}$ and $\mathbf{r}\colon [n] \to \mathbb{N}$,*

$$\lim_{N\to\infty} \psi(\widetilde{S}_{\mathbf{r}(1),N} \dots \widetilde{S}_{\mathbf{r}(n),N}) = \sum_{\substack{\pi\in\mathcal{P}_2(n)\\ \pi\le\ker(\mathbf{r})}} q^{\mathrm{cr}(\pi)}.$$

*Proof.* We know from Proposition 5.2.6 that $\widetilde{\mathcal{Y}}$ is exchangeable and satisfies the SVP. Thus the CLT, as formulated in Theorem 3.3.10 (or Theorem 3.3.13) applies. Clearly, all odd moments of order $2k-1$ vanish in the large $N$-limit. We are left to verify the formulas for even moments of $q$-semicircular systems in the large $N$-limit. We know from Theorem 3.3.10 that

$$\lim_{N\to\infty} \psi(\widetilde{S}_{\mathbf{r}(1),N} \cdots \widetilde{S}_{\mathbf{r}(2k),N}) = \sum_{\substack{\pi\in\mathcal{P}_2(2k)\\ \pi\le\ker(\mathbf{r})}} \psi_{\pi,\mathbf{r}}$$

where $\psi_{\pi,\mathbf{r}} = \psi(\widehat{s}_{\mathbf{r}(1),\mathbf{i}(1)} \cdots \widehat{s}_{\mathbf{r}(2k),\mathbf{i}(2k)})$ for $\mathbf{i}\colon [2k] \to \mathbb{N}$ with $\ker(\mathbf{i}) = \pi$ and

$$\widehat{s}_{m,\ell} = \widehat{s}_{(\ell-1)s+m} \qquad \text{for } m \in [s].$$

It is elementary to see that

$$(\mathbf{i}(j)-1))s + \mathbf{r}(j) = (\mathbf{i}(j')-1)s + \mathbf{r}(j') \quad \Longleftrightarrow \quad \mathbf{i}(j) = \mathbf{i}(j')$$

for $\mathbf{i}\colon [2k] \to \mathbb{N}$ and $\mathbf{r}\colon [2k] \to [s]$ with $\ker(\mathbf{i}) \in \mathcal{P}_2(2k)$ and $\ker(\mathbf{i}) \le \ker(\mathbf{r})$. Thus we can infer from exchangeability that

$$\begin{aligned}\psi_{\pi,\mathbf{r}} = \psi(\widehat{s}_{\mathbf{r}(1),\mathbf{i}(1)} \cdots \widehat{s}_{\mathbf{r}(2k),\mathbf{i}(2k)}) &= \psi(\widehat{s}_{(\mathbf{i}(1)-1)s+\mathbf{r}(1)} \dots \widehat{s}_{(\mathbf{i}(2k)-1)s+\mathbf{r}(2k)})\\ &= \psi(\widehat{s}_{\mathbf{i}(1)} \dots \widehat{s}_{\mathbf{i}(2k)}).\end{aligned}$$

As we count the $q$-crossings of the pair partitions, we have that

$$\psi(\widehat{s}_{\mathbf{i}(1)} \dots \widehat{s}_{\mathbf{i}(2k)}) = q^{\mathrm{cr}(\pi)}$$

Altogether, we arrive at

$$\sum_{\substack{\pi\in\mathcal{P}_2(2k)\\ \pi\le\ker(\mathbf{r})}} \psi_{\pi,\mathbf{r}} = \sum_{\substack{\pi\in\mathcal{P}_2(2k)\\ \pi\le\ker(\mathbf{r})}} q^{\mathrm{cr}(\pi)}.$$

□

**Corollary 5.2.8.** *Let the $q$-semicircular system $\widetilde{\mathcal{Y}}$ in $(\mathcal{A}, \psi)$ and $\widetilde{S}_{1,N}, \dots, \widetilde{S}_{s,N}$ be given as in Theorem 5.2.7. Then there exists a *-algebraic probability space $(\widehat{\mathcal{A}}, \widehat{\psi})$ and a $q$-semicircular system $\widehat{\mathcal{Y}} \equiv \{\widetilde{s}_r\}_{r=1}^s$ in $(\widehat{\mathcal{A}}, \widehat{\psi})$ such that*

$$\lim_{N\to\infty} \psi\big(\widetilde{S}_{\mathbf{r}(1),N} \cdots \widetilde{S}_{\mathbf{r}(n),N}\big) = \widetilde{\psi}\big(\widetilde{s}_{\mathbf{r}(1)} \cdots \widetilde{s}_{\mathbf{r}(n)}\big)$$

*for all $n \in \mathbb{N}$ and $\mathbf{r}\colon [n] \to [s]$.*

*Proof.* This is immediate from the mixed moment formula of the CLT in Theorem 5.2.7, as its right-hand side is defining moment formula for the $q$-semicircular system $\{\widehat{s}_r\}_{r=1}^s$ in $(\mathcal{A}, \psi)$. □

## 5.3 $z$-Circular Systems

In this section we study the notion of $z$-circular systems which generalizes the notion of $q$-circular systems. Our approach again adapts that of [MN01] to the framework of *-algebraic probability spaces. We show that a $z$-circular system is spreadable and satisfies the SVP. Therefore, the central limit distribution of Theorem 5.1.6 for $q$-circular systems can be generalized to $z$-circular systems.

**Definition 5.3.1.** Let $(\mathcal{A}, \psi)$ be a *-algebraic probability space and fix $z \in \mathbb{C}$ with $|z| \leq 1$. The family $(c_r)_{r=1}^s \subseteq \mathcal{A}$ $(s \geq 1)$ is said to form a *$z$-circular system* in $(\mathcal{A}, \psi)$ if

- for every odd $n \geqslant 1$, $\mathbf{r}\colon [n] \to [s]$, and $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$,

$$\psi(c_{\mathbf{r}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots c_{\mathbf{r}(n)}^{\boldsymbol{\varepsilon}(n)}) = 0;$$

- for every even $n \geqslant 1$ with $n = 2k$, $\mathbf{r}\colon [n] \to [s]$, and $\boldsymbol{\varepsilon}\colon [n] \to \{*, 1\}$,

$$\psi(c_{\mathbf{r}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots c_{\mathbf{r}(n)}^{\boldsymbol{\varepsilon}(n)}) = \frac{1}{k!} \sum_{\sigma \in S_k} \sum_{\pi \in \mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon})} z^{\mathrm{cr}_+(\pi, \boldsymbol{\varepsilon}, \sigma)} \overline{z}^{\mathrm{cr}_-(\pi, \boldsymbol{\varepsilon}, \sigma)}. \tag{5.3}$$

  Here $S_k$ is the group of all permutations on the set $[k]$, and $\mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon})$ is as introduced in Definition 5.1.1.

**Remark 5.3.2.** This conditional set of pair partitions $\mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon})$ is denoted by $\mathcal{P}(\mathbf{r}(1), \ldots, \mathbf{r}(n); \boldsymbol{\varepsilon}(1), \ldots, \boldsymbol{\varepsilon}(n))$ in [MN01]. See also Remark 5.1.4 for a discussion of some properties of $\mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon})$.

Next we show that a $z$-circular system provides a spreadable sequence in the case $s = \infty$.

**Proposition 5.3.3.** *Let $(\mathcal{A}, \psi)$ be a *-algebraic probability space and suppose the family $\mathcal{Y} \equiv (c_r)_{r=1}^{\infty} \subseteq \mathcal{A}$ is a $z$-circular system in $(\mathcal{A}, \psi)$. Then $\mathcal{Y}$ is spreadable and satisfies the SVP.*

*Proof.* Let $n \in \mathbb{N}$ and $\mathbf{r}, \widehat{\mathbf{r}}\colon [n] \to \mathbb{N}$ such that $\mathbf{r} \sim_{\mathcal{O}} \widehat{\mathbf{r}}$. We need to show that

$$\psi(c_{\mathbf{r}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots c_{\mathbf{r}(n)}^{\boldsymbol{\varepsilon}(n)}) = \psi(c_{\widehat{\mathbf{r}}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots c_{\widehat{\mathbf{r}}(n)}^{\boldsymbol{\varepsilon}(n)}). \tag{5.4}$$

Due to the moment formula (5.3), it suffices to show that $\mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon}) = \mathcal{P}_2(\widehat{\mathbf{r}}, \boldsymbol{\varepsilon})$. It follows from the assumptions on the two maps $\mathbf{r}$ and $\widehat{\mathbf{r}}$ that

$$\pi \in \mathcal{P}_2(\mathbf{r}, \boldsymbol{\varepsilon}) \Leftrightarrow \pi \in \mathcal{P}_2(\widehat{\mathbf{r}}, \boldsymbol{\varepsilon}).$$

Since we are summing over the same set of ordered pair partitions, we conclude from the mixed moment formula (5.3) that

$$\begin{aligned}\psi(c_{\mathbf{r}(1)}^{\boldsymbol{\varepsilon}(1)}\cdots c_{\mathbf{r}(n)}^{\boldsymbol{\varepsilon}(n)}) &= \frac{1}{k!}\sum_{\sigma\in S_k}\sum_{\pi\in\mathcal{P}_2(\mathbf{r},\boldsymbol{\varepsilon})} z^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\overline{z}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\boldsymbol{\varepsilon})}\\ &= \frac{1}{k!}\sum_{\sigma\in S_k}\sum_{\pi\in\mathcal{P}_2(\widehat{\mathbf{r}},\boldsymbol{\varepsilon})} z^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\overline{z}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}\\ &= \psi(c_{\widehat{\mathbf{r}}(1)}^{\boldsymbol{\varepsilon}(1)}\cdots c_{\widehat{\mathbf{r}}(n)}^{\boldsymbol{\varepsilon}(n)}).\end{aligned}$$

The SVP of $\mathcal{Y}$ is immediate from the moment formula (5.3), as $\mathcal{P}_2(\mathbf{r},\boldsymbol{\varepsilon}) = \emptyset$ if $\ker(\mathbf{r})$ contains a singleton. □

**Theorem 5.3.4.** *Suppose the family $\mathcal{Y} \equiv (c_r)_{r=1}^{\infty} \subseteq \mathcal{A}$ forms a $z$-circular system in $(\mathcal{A},\psi)$. Let*

$$\begin{aligned}S_{1,N} &:= \frac{1}{\sqrt{N}}\big(c_1 + c_{s+1} + \ldots + c_{(N-1)s+1}\big),\\ S_{2,N} &:= \frac{1}{\sqrt{N}}\big(c_2 + c_{s+2} + \ldots + c_{(N-1)s+2}\big),\\ &\ \vdots\\ S_{s,N} &:= \frac{1}{\sqrt{N}}\big(c_s + c_{2s} + \ldots + c_{Ns}\big).\end{aligned}$$

*Then one has for all $k\in\mathbb{N}$, $\mathbf{r}\colon [2k]\to\mathbb{N}$, and $\boldsymbol{\varepsilon}\colon [2k]\to\{*,1\}$,*

$$\lim_{N\to\infty}\psi(S_{\mathbf{r}(1),N}^{\boldsymbol{\varepsilon}(1)}\cdots S_{\mathbf{r}(2k),N}^{\boldsymbol{\varepsilon}(2k)}) = \frac{1}{k!}\sum_{\sigma\in S_k}\sum_{\pi\in\mathcal{P}_2(\mathbf{r},\boldsymbol{\varepsilon})} z^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\overline{z}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}$$

*and, for all $k\in\mathbb{N}$, $\mathbf{r}\colon [2k-1]\to\mathbb{N}$, and $\boldsymbol{\varepsilon}\colon [2k-1]\to\{*,1\}$,*

$$\lim_{N\to\infty}\psi(S_{\mathbf{r}(1),N}^{\boldsymbol{\varepsilon}(1)}\cdots S_{\mathbf{r}(2k-1),N}^{\boldsymbol{\varepsilon}(2k-1)}) = 0.$$

*Proof.* Clearly, all odd moments of order $2k-1$ vanish in the large $N$-limit. Moreover, one can conclude from Theorem 3.4.9 that

$$\lim_{N\to\infty}\psi(S_{\mathbf{r}(1),N}^{\boldsymbol{\varepsilon}(1)}\cdots S_{\mathbf{r}(2k),N}^{\boldsymbol{\varepsilon}(2k)}) = \frac{1}{k!}\sum_{\substack{\pi\in\mathcal{OP}_2(2k)\\ \overline{\pi}\leqslant\ker(\mathbf{r})}}\psi^{\mathcal{O}}_{\pi,\mathbf{r},\boldsymbol{\varepsilon}},$$

where $\psi^{\mathcal{O}}_{\pi,\mathbf{r},\boldsymbol{\varepsilon}} = \psi(c_{\mathbf{r}(1),\mathbf{i}(1)}^{\boldsymbol{\varepsilon}(1)}\cdots c_{\mathbf{r}(2k),\mathbf{i}(2k)}^{\boldsymbol{\varepsilon}(2k)})$ for $\mathbf{i}\colon[2k]\to[k]$ with $\ker_{\mathcal{O}}(\mathbf{i}) = \pi$ and

$$c_{m,\ell} = c_{(\ell-1)s+m}.$$

Using the bijection from Lemma 2.1.26, we address an ordered pair partition $\pi \in \mathcal{OP}_2(2k)$ through the pair $(\overline{\pi}, \sigma) \in \mathcal{P}_2(2k) \times S_k$. This allows us to rewrite $\psi^{\mathcal{O}}_{\pi,\mathbf{r},\boldsymbol{\varepsilon}}$ such that we arrive at

$$\frac{1}{k!} \sum_{\substack{\pi \in \mathcal{OP}_2(2k) \\ \overline{\pi} \leqslant \ker(\mathbf{r})}} \psi^{\mathcal{O}}_{\pi,\mathbf{r},\boldsymbol{\varepsilon}} = \frac{1}{k!} \sum_{\sigma \in S_k} \sum_{\substack{\overline{\pi} \in \mathcal{P}_2(2k) \\ \overline{\pi} \leqslant \ker(\mathbf{r})}} \psi_{\overline{\pi},\mathbf{r},\boldsymbol{\varepsilon},\sigma}, \tag{5.5}$$

where

$$\begin{aligned} \psi_{\overline{\pi},\mathbf{r},\boldsymbol{\varepsilon},\sigma} &= \psi\big(c^{\boldsymbol{\varepsilon}(1)}_{\mathbf{r}(1),\sigma(\mathbf{i}(1))} \cdots x^{\boldsymbol{\varepsilon}(2k)}_{\mathbf{r}(2k),\sigma(\mathbf{i}(2k))}\big) \\ &= \psi\big(c^{\boldsymbol{\varepsilon}(1)}_{(\sigma(\mathbf{i}(1))-1)s+\mathbf{r}(1)} \cdots c^{\boldsymbol{\varepsilon}(2k)}_{(\sigma(\mathbf{i}(2k))-1)s+\mathbf{r}(2k)}\big) \\ &= \psi\big(c^{\boldsymbol{\varepsilon}(1)}_{\sigma(\mathbf{i}(1))} \cdots c^{\boldsymbol{\varepsilon}(2k)}_{\sigma(\mathbf{i}(2k))}\big), \end{aligned}$$

with $\overline{\pi} = \ker(\mathbf{i})$ in standard order for $\mathbf{i} \colon [2k] \to [k]$. Here we have used for the last equality that

$$\begin{aligned} &\Big(\big(\sigma(\mathbf{i}(1)) - 1\big)s + \mathbf{r}(1), \dots, \big(\sigma(\mathbf{i}(2k)) - 1\big)s + \mathbf{r}(2k)\Big) \\ &\qquad\qquad \sim_{\mathcal{O}} \Big(\big(\sigma(\mathbf{i}(1)) - 1\big)s, \dots, \big(\sigma(\mathbf{i}(2k)) - 1\big)s\Big) \\ &\qquad\qquad \sim_{\mathcal{O}} \Big(\sigma(\mathbf{i}(1)), \dots, \sigma(\mathbf{i}(2k))\Big). \end{aligned}$$

Altogether, we can write (5.5) as

$$\frac{1}{k!} \sum_{\substack{\pi \in \mathcal{OP}_2(2k) \\ \overline{\pi} \leqslant \ker(\mathbf{r})}} \psi^{\mathcal{O}}_{\pi,\mathbf{r},\boldsymbol{\varepsilon}} = \frac{1}{k!} \sum_{\sigma \in S_k} \sum_{\substack{\mathbf{i} \colon [2k] \to [k] \\ \ker(\mathbf{i}) \in \mathcal{P}_2(2k) \\ \ker(\mathbf{i}) \leqslant \ker(\mathbf{r})}} \psi\big(c^{\boldsymbol{\varepsilon}(1)}_{\sigma(\mathbf{i}(1))} \cdots c^{\boldsymbol{\varepsilon}(2k)}_{\sigma(\mathbf{i}(2k))}\big)$$

We note that $\ker(\sigma \circ \mathbf{i}) = \ker(\mathbf{i})$. By the definition of a $z$-circular system,

$$\psi\big(c^{\boldsymbol{\varepsilon}(1)}_{\sigma(\mathbf{i}(1))} \cdots c^{\boldsymbol{\varepsilon}(2k)}_{\sigma(\mathbf{i}(2k))}\big) = \frac{1}{k!} \sum_{\tau \in S_k} \sum_{\pi \in \mathcal{P}_2(\mathbf{i},\boldsymbol{\varepsilon})} z^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\tau)} \overline{z}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\tau)}.$$

Thus we arrive at

$$\begin{aligned} \frac{1}{k!} \sum_{\substack{\pi \in \mathcal{OP}_2(2k) \\ \overline{\pi} \leqslant \ker(\mathbf{r})}} \psi^{\mathcal{O}}_{\pi,\mathbf{r},\boldsymbol{\varepsilon}} &= \frac{1}{k!} \sum_{\sigma \in S_k} \sum_{\substack{\mathbf{i} \colon [2k] \to [k] \\ \ker(\mathbf{i}) \in \mathcal{P}_2(2k) \\ \ker(\mathbf{i}) \leqslant \ker(\mathbf{r})}} \psi\big(c^{\boldsymbol{\varepsilon}(1)}_{\sigma(\mathbf{i}(1))} \cdots c^{\boldsymbol{\varepsilon}(2k)}_{\sigma(\mathbf{i}(2k))}\big) \\ &= \frac{1}{k!} \sum_{\sigma \in S_k} \sum_{\substack{\mathbf{i} \colon [2k] \to [k] \\ \ker(\mathbf{i}) \in \mathcal{P}_2(2k) \\ \ker(\mathbf{i}) \leqslant \ker(\mathbf{r})}} \left( \frac{1}{k!} \sum_{\tau \in S_k} \sum_{\pi \in \mathcal{P}_2(\mathbf{i},\boldsymbol{\varepsilon})} z^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\tau)} \overline{z}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\tau)} \right) \\ &= \frac{1}{k!} \sum_{\tau \in S_k} \sum_{\substack{\mathbf{i} \colon [2k] \to [k] \\ \ker(\mathbf{i}) \in \mathcal{P}_2(2k) \\ \ker(\mathbf{i}) \leqslant \ker(\mathbf{r})}} \sum_{\pi \in \mathcal{P}_2(\mathbf{i},\boldsymbol{\varepsilon})} z^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\tau)} \overline{z}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\tau)}. \end{aligned}$$

We are left to show that, in the last formula,

$$\cdots \sum_{\substack{\mathbf{i}\colon [2k]\to[k] \\ \ker(\mathbf{i})\in\mathcal{P}_2(2k) \\ \ker(\mathbf{i})\leqslant\ker(\mathbf{r})}} \sum_{\pi\in\mathcal{P}_2(\mathbf{i},\boldsymbol{\varepsilon})} \cdots = \cdots \sum_{\pi\in\mathcal{P}_2(\mathbf{r},\boldsymbol{\varepsilon})} \cdots . \tag{5.6}$$

Since $\operatorname{card}\{\mathbf{i}\colon [2k] \to [k] \mid \ker(\mathbf{i}) \in \mathcal{P}_2(2k)\} = \operatorname{card}\{\mathcal{P}_2(2k)\}$, we can rewrite the outer summation over the functions $\mathbf{i}$ such that

$$\cdots \sum_{\substack{\mathbf{i}\colon [2k]\to[k] \\ \ker(\mathbf{i})\in\mathcal{P}_2(2k) \\ \ker(\mathbf{i})\leqslant\ker(\mathbf{r})}} \sum_{\pi\in\mathcal{P}_2(\mathbf{i},\boldsymbol{\varepsilon})} \cdots = \cdots \sum_{\substack{\widetilde{\pi}\in\mathcal{P}_2(2k) \\ \widetilde{\pi}\leqslant\ker(\mathbf{r})}} \sum_{\pi\in\mathcal{P}_2(\mathbf{i},\boldsymbol{\varepsilon})} \cdots .$$

Note that both summations are the same due to that the summation over the set of pair partitions $\{\widetilde{\pi} \in \mathcal{P}_2(2k) \mid \widetilde{\pi} \le \ker(\mathbf{r})\}$ can be restricted to the subset $\mathcal{P}_2(\mathbf{i}, \boldsymbol{\varepsilon})$, since a summand $\psi_{\pi,\mathbf{r},\boldsymbol{\varepsilon}}$ may be non-zero for a pair partition $\widetilde{\pi} \in \mathcal{P}_2(2k)$ with $\widetilde{\pi} \le \ker(\mathbf{r})$ only if $\boldsymbol{\varepsilon}(i) \neq \boldsymbol{\varepsilon}(j)$ for all blocks $\{i,j\} \in \widetilde{\pi}$. This ensures the claimed equality in (5.6). Altogether we have verified that

$$\lim_{N\to\infty} \psi(S^{\boldsymbol{\varepsilon}(1)}_{\mathbf{r}(1),N} \cdots S^{\boldsymbol{\varepsilon}(2k)}_{\mathbf{r}(2k),N}) = \frac{1}{k!} \sum_{\tau\in S_k} \sum_{\pi\in\mathcal{P}_2(\mathbf{r},\boldsymbol{\varepsilon})} z^{\operatorname{cr}_+(\pi,\boldsymbol{\varepsilon},\tau)} \overline{z}^{\operatorname{cr}_-(\pi,\boldsymbol{\varepsilon},\tau)}.$$

□

**Corollary 5.3.5.** *Let the $z$-circular system $\mathcal{Y}$ in $(\mathcal{A},\psi)$ and $S_{1,N},\ldots,S_{s,N}$ be given as in Theorem 5.3.4. Then there exists a *-algebraic probability space $(\widehat{\mathcal{A}},\widehat{\psi})$ and a $z$-circular system $\widehat{\mathcal{Y}} \equiv (\widehat{c}_r)_{r=1}^s$ in $(\widehat{\mathcal{A}},\widehat{\psi})$ such that*

$$\lim_{N\to\infty} \psi\big(S^{\boldsymbol{\varepsilon}(1)}_{\mathbf{r}(1),N} \cdots S^{\boldsymbol{\varepsilon}(n)}_{\mathbf{r}(n),N}\big) = \widetilde{\psi}\big(\widetilde{c}^{\,\boldsymbol{\varepsilon}(1)}_{\mathbf{r}(1)} \cdots \widetilde{c}^{\,\boldsymbol{\varepsilon}(n)}_{\mathbf{r}(n)}\big) \tag{5.7}$$

*for all $n \in \mathbb{N}$, $\mathbf{r}\colon [n] \to [s]$, and $\boldsymbol{\varepsilon}\colon [n] \to \{*,1\}$.*

*Proof.* This is immediate from the mixed moment formula of the CLT in Theorem 5.3.4, the existence of the large $N$-limit system $\widehat{\mathcal{Y}}$ by Theorem 3.6.1, and the defining moment formula for a $z$-circular system with $s$ elements. □

## 5.4 $z$-Semicircular Systems

We introduce $z$-semicircular systems which relate to $z$-circular systems ([MN01]) as $q$-semicircular systems relate to $q$-circular systems. In particular, $z$-semicircular systems provide a generalization of $q$-semicircular systems from $q \in [-1,1]$ to $z \in \mathbb{C}$ with $|z| \le 1$. Moreover, we discuss how one can obtain a $z$-semicircular system form a $z$-circular system. Also, we show that a $z$-semicircular system is spreadable and satisfies the SVP. We close this section with CLTs for $z$-semicircular systems, Theorem 5.4.6 and Corollary 5.4.7. Here the latter shows that a certain multivariate CLT of a $z$-semicircular system provides again a $z$-semicircular system in the large $N$-limit.

**Definition 5.4.1.** Let $(\mathcal{A}, \psi)$ be a *-algebraic probability space and fix $z \in \mathbb{C}$ with $|z| \leq 1$. The family $\widetilde{\mathcal{Y}} \equiv (\widehat{s}_r)_{r=1}^{s} \subseteq \mathcal{A}$ with $(s \geq 1)$ is said to form a *z-semicircular system* in $(\mathcal{A}, \psi)$ if

- $\widehat{s}_r = \widehat{s}_r^*$ for all $r = 1, 2, \dots, s$;
- for every odd $n \geqslant 1$, $\mathbf{r} \colon [n] \to [s]$,
$$\psi(\widehat{s}_{\mathbf{r}(1)} \cdots \widehat{s}_{\mathbf{r}(n)}) = 0;$$
- for every even $n \geqslant 1$ with $n = 2k$, $\mathbf{r} \colon [n] \to [s]$,
$$\psi(\widehat{s}_{\mathbf{r}(1)} \cdots \widehat{s}_{\mathbf{r}(2k)}) = \frac{1}{k!}\frac{1}{2^k} \sum_{\sigma \in S_k} \sum_{\substack{\pi \in \mathcal{P}_2(2k) \\ \pi \leq \ker(\mathbf{r})}} \sum_{\substack{\boldsymbol{\varepsilon} \colon [2k] \to \{*,1\} \\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} z^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)} \overline{z}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}. \tag{5.8}$$

Here $S_k$ denotes the group of all permutations on $[k]$.

**Example 5.4.2.** The even moments of a single $z$-semicircular operator $\widehat{s}$ (obtained from a $z$-semicircular system for $s = 1$, and writing $\widehat{s}_1$ just as $\widehat{s}$) are given by
$$\psi(\widehat{s}^{2k}) = \frac{1}{k!}\frac{1}{2^k} \sum_{\sigma \in S_k} \sum_{\pi \in \mathcal{P}_2(2k)} \sum_{\substack{\boldsymbol{\varepsilon} \colon [2k] \to \{*,1\} \\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} z^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)} \overline{z}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}.$$
Using the polar form of complex numbers, one can write $z = r\omega$ with $0 \leq r \leq 1$ and $\omega \in \mathbb{T}$ such that
$$\psi(\widehat{s}^{2k}) = \frac{1}{k!}\frac{1}{2^k} \sum_{\sigma \in S_k} \sum_{\pi \in \mathcal{P}_2(2k)} \sum_{\substack{\boldsymbol{\varepsilon} \colon [2k] \to \{*,1\} \\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} r^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)+\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)} \omega^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)-\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}.$$
So the even moments of a $z$-semicircular operator are not only counting the difference of oriented crossings (as it is the case for $\omega$-semicircular systems), they also count the number of crossings, which is the sum of oriented crossings $\mathrm{cr}_+(\pi, \boldsymbol{\varepsilon}, \sigma) + \mathrm{cr}_-(\pi, \boldsymbol{\varepsilon}, \sigma)$.

Putting $q = \Re\omega$ and using $\Re z = rq$ in the polar form, the first few even moments are given by
$$\begin{aligned} \psi(\widehat{s}^2) &= 1, \\ \psi(\widehat{s}^4) &= rq + 2, \\ \psi(\widehat{s}^6) &= 5 + 6rq + 3r^2q^2 + r^3q^3. \end{aligned}$$
These moments of an $r\omega$-semicircular operator coincide with those of an $rq$-Gaussian random variable. This coincidence of moments fails for even moments of 8-th order and higher.

In the following, we show that one can obtain $z$-semicircular systems from $z$-circular systems. We recall that $\mathcal{P}_2(\mathbf{r},\boldsymbol{\varepsilon})$ denotes the set of all pair partitions $\pi \in \mathcal{P}(2k)$ such that $\pi \leq \ker(\mathbf{r})$ and $\boldsymbol{\varepsilon}(\min V) \neq \boldsymbol{\varepsilon}(\max V)$ for all $V \in \pi$. Furthermore, we recall from Lemma 5.2.3 that the two sets

$$\Big\{(\pi,\boldsymbol{\varepsilon}) \in \mathcal{P}_2(2k) \times \big\{f\colon [2k] \to \{*,1\}\big\} \Big| \pi \in \mathcal{P}_2(\mathbf{r},\boldsymbol{\varepsilon})\Big\}$$

and

$$\Big\{(\pi,\boldsymbol{\varepsilon}) \in \mathcal{P}_2(2k) \times \big\{f\colon [2k] \to \{*,1\}\big\} \Big| \pi \leq \ker(\mathbf{r}), \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced } \Big\}$$

are the same.

**Proposition 5.4.3.** *Let $(\mathcal{A},\psi)$ be a *-algebraic probability space. Given the $z$-circular systems $\widetilde{\mathcal{Y}} \equiv (c_r)_{r=1}^{s} \subseteq \mathcal{A}$ with $s \geq 1$, let $\widehat{s}_m = \frac{1}{\sqrt{2}}(c_m + c_m^*)$ for $m \in [s]$. Then $\widehat{s}_1, \ldots, \widehat{s}_s$ is a $z$-semicircular system.*

*Proof.* Clearly each operator $\widehat{s}_m$ is self-adjoint. Let $k \in \mathbb{N}$, $\mathbf{r}\colon [2k] \to [s]$, and $\sigma \in S_\infty$. We compute that

$$\begin{aligned}
\psi(\widehat{s}_{\mathbf{r}(1)} \cdots \widehat{s}_{\mathbf{r}(2k)}) &= \frac{1}{2^k} \sum_{\boldsymbol{\varepsilon}\colon [2k]\to\{*,1\}} \psi(c_{\mathbf{r}(1)}^{\boldsymbol{\varepsilon}(1)} \cdots c_{\mathbf{r}(2k)}^{\boldsymbol{\varepsilon}(2k)}) \\
&= \frac{1}{k!}\frac{1}{2^k} \sum_{\boldsymbol{\varepsilon}\colon [2k]\to\{*,1\}} \sum_{\sigma\in S_k} \sum_{\pi\in\mathcal{P}_2(\mathbf{r},\boldsymbol{\varepsilon})} z^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)} \overline{z}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)} \\
&= \frac{1}{k!}\frac{1}{2^k} \sum_{\sigma\in S_k} \sum_{\substack{\pi\in\mathcal{P}_2(2k) \\ \pi\leq\ker(\mathbf{r})}} \sum_{\substack{\boldsymbol{\varepsilon}\colon [2k]\to\{*,1\} \\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} z^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)} \overline{z}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}.
\end{aligned}$$

We have used Definition 5.3.1 for the second equality and Lemma 5.2.3 for the last equality. □

**Conjecture 5.4.4.** *Let $(\mathcal{A},\psi)$ be a *-algebraic probability space. Given the $z$-semicircular system $(\widehat{s}_r)_{r=1}^{2s} \subseteq \mathcal{A}$ with $(s \geq 1)$, let $c_m = \frac{1}{\sqrt{2}}(\widehat{s}_{2m-1} + \mathrm{i}\,\widehat{s}_{2m})$ for $m \in [s]$. Then $(c_1, \ldots, c_s)$ is a $z$-circular system.*

We investigate next the limit distribution of multivariate CLTs emerging from a $z$-semicircular system. To do so, we need first to verify that a $z$-semicircular system is spreadable and satisfies the SVP.

**Proposition 5.4.5.** *Let $(\mathcal{A},\psi)$ be a *-algebraic probability space and suppose the family $\widetilde{\mathcal{Y}} \equiv (\widehat{s}_r)_{r=1}^{\infty} \subseteq \mathcal{A}$ is a $z$-semicircular system in $(\mathcal{A},\psi)$. Then $\widetilde{\mathcal{Y}}$ is spreadable and satisfies the SVP.*

*Proof.* All arguments in the proof of Proposition 5.3.3 transfer from $z$-circular systems to $z$-semicircular systems. □

**Theorem 5.4.6.** *Suppose the family* $\widetilde{\mathcal{Y}} \equiv (\widehat{s}_r)_{r=1}^{\infty} \subseteq \mathcal{A}$ *forms a z-semicircular system in* $(\mathcal{A}, \psi)$. *Let*

$$\begin{aligned}\widetilde{S}_{1,N} &:= \frac{1}{\sqrt{N}}\big(\widehat{s}_1 + \widehat{s}_{s+1} + \ldots + \widehat{s}_{(N-1)s+1}\big),\\ \widetilde{S}_{2,N} &:= \frac{1}{\sqrt{N}}\big(\widehat{s}_2 + \widehat{s}_{s+2} + \ldots + \widehat{s}_{(N-1)s+2}\big),\\ &\vdots\\ \widetilde{S}_{s,N} &:= \frac{1}{\sqrt{N}}\big(\widehat{s}_s + \widehat{s}_{2s} + \ldots + \widehat{s}_{Ns}\big).\end{aligned}$$

*Then one has for all* $k \in \mathbb{N}$ *and* $\mathbf{r}\colon [2k] \to \mathbb{N}$,

$$\lim_{N\to\infty} \psi\big(\widetilde{S}_{\mathbf{r}(1),N} \cdots \widetilde{S}_{\mathbf{r}(2k),N}\big) = \frac{1}{k!}\frac{1}{2^k} \sum_{\sigma\in S_k} \sum_{\substack{\pi\in\mathcal{P}_2(2k)\\ \pi\le\ker(\mathbf{r})}} \sum_{\substack{\boldsymbol{\varepsilon}\colon [2k]\to\{*,1\}\\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} z^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\sigma)}\overline{z}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\sigma)}$$

*and, for all* $k \in \mathbb{N}$ *and* $\mathbf{r}\colon [2k-1] \to \mathbb{N}$,

$$\lim_{N\to\infty} \psi\big(\widetilde{S}_{\mathbf{r}(1),N} \cdots \widetilde{S}_{\mathbf{r}(2k-1),N}\big) = 0.$$

*Proof.* One concludes from Theorem 3.4.9 that

$$\lim_{N\to\infty} \psi\big(\widetilde{S}_{\mathbf{r}(1),N} \cdots \widetilde{S}_{\mathbf{r}(2k),N}\big) = \frac{1}{k!} \sum_{\substack{\pi\in\mathcal{OP}_2(2k)\\ \overline{\pi}\leqslant\ker(\mathbf{r})}} \psi^{\mathcal{O}}_{\pi,\mathbf{r}},$$

where $\psi^{\mathcal{O}}_{\pi,\mathbf{r}} = \psi\big(\widehat{s}_{\mathbf{r}(1),\mathbf{i}(1)} \cdots \widehat{s}_{\mathbf{r}(2k),\mathbf{i}(2k)}\big)$ for $\mathbf{i}\colon [2k] \to [k]$ with $\ker_{\mathcal{O}}(\mathbf{i}) = \pi$ and

$$\widehat{s}_{m,\ell} = \widehat{s}_{(\ell-1)s+m} \qquad \text{for } m \in [s]$$

or, more explicitly,

$$\psi^{\mathcal{O}}_{\pi,\mathbf{r}} = \psi\big(\widehat{s}_{(\mathbf{i}(1)-1)s+\mathbf{r}(1)} \cdots \widehat{s}_{(\mathbf{i}(2k)-1)s+\mathbf{r}(2k)}\big).$$

By using the bijection from Lemma 2.1.26, we address an ordered pair partition $\pi \in \mathcal{OP}_2(2k)$ through the pair $(\overline{\pi}, \sigma) \in \mathcal{P}_2(2k) \times S_k$. Thus, similar to the arguments in the proof of Theorem 4.3.8,

$$\frac{1}{k!} \sum_{\substack{\pi\in\mathcal{OP}_2(2k)\\ \overline{\pi}\leqslant\ker(\mathbf{r})}} \psi^{\mathcal{O}}_{\pi,\mathbf{r}} = \frac{1}{k!} \sum_{\sigma\in S_k} \sum_{\substack{\overline{\pi}\in\mathcal{P}_2(2k)\\ \overline{\pi}\leqslant\ker(\mathbf{r})}} \psi_{\overline{\pi},\mathbf{r},\sigma}, \tag{5.9}$$

where

$$\begin{aligned}\psi_{\overline{\pi},\mathbf{r},\sigma} &= \psi\big(\widehat{s}_{\mathbf{r}(1),\sigma(\mathbf{i}(1))} \cdots \widehat{s}_{\mathbf{r}(2k),\sigma(\mathbf{i}(2k))}\big)\\ &= \psi\big(\widehat{s}_{(\sigma(\mathbf{i}(1))-1)s+\mathbf{r}(1)} \cdots \widehat{s}_{(\sigma(\mathbf{i}(2k))-1)s+\mathbf{r}(2k)}\big)\\ &= \psi\big(\widehat{s}_{\sigma(\mathbf{i}(1))} \cdots \widehat{s}_{\sigma(\mathbf{i}(2k))}\big),\end{aligned}$$

with $\overline{\pi} = \ker(\mathbf{i})$ in standard order for $\mathbf{i}\colon [2k] \to [k]$. Here we have used for the last equality that

$$\begin{aligned}\Big(\big(\sigma(\mathbf{i}(1)) - 1\big)s + \mathbf{r}(1), \dots, \big(\sigma(\mathbf{i}(2k)) - 1\big)s + \mathbf{r}(2k)\Big)& \\ \sim_{\mathcal{O}} \Big(\big(\sigma(\mathbf{i}(1)) - 1\big)s, \dots, \big(\sigma(\mathbf{i}(2k)) - 1\big)s\Big)& \\ \sim_{\mathcal{O}} \Big(\sigma(\mathbf{i}(1)), \dots, \sigma(\mathbf{i}(2k))\Big).&\end{aligned}$$

Altogether, we can write (5.9) as

$$\frac{1}{k!} \sum_{\substack{\pi \in \mathcal{OP}_2(2k) \\ \overline{\pi} \leqslant \ker(\mathbf{r})}} \psi^{\mathcal{O}}_{\pi,\mathbf{r}} = \frac{1}{k!} \sum_{\sigma \in S_k} \sum_{\substack{\mathbf{i}\colon [2k]\to[k] \\ \ker(\mathbf{i}) \in \mathcal{P}_2(2k) \\ \ker(\mathbf{i}) \leqslant \ker(\mathbf{r})}} \psi\big(\widehat{s}_{\sigma(\mathbf{i}(1))} \cdots \widehat{s}_{\sigma(\mathbf{i}(2k))}\big)$$

We note that $\ker(\sigma \circ \mathbf{i}) = \ker(\mathbf{i})$. By the definition of a $z$-semicircular system,

$$\psi\big(\widehat{s}_{\sigma(\mathbf{i}(1))} \cdots \widehat{s}_{\sigma(\mathbf{i}(2k))}\big) = \frac{1}{k!}\frac{1}{2^k} \sum_{\tau \in S_k} \sum_{\substack{\pi \in \mathcal{P}_2(2k) \\ \pi \leq \ker(\mathbf{i})}} \sum_{\substack{\boldsymbol{\varepsilon}\colon [2k]\to\{*,1\} \\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} z^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\tau)} \overline{z}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\tau)}.$$

Thus we arrive at

$$\begin{aligned}\frac{1}{k!} \sum_{\substack{\pi \in \mathcal{OP}_2(2k) \\ \overline{\pi} \leqslant \ker(\mathbf{r})}} \psi^{\mathcal{O}}_{\pi,\mathbf{r}} &= \frac{1}{k!} \sum_{\sigma \in S_k} \sum_{\substack{\mathbf{i}\colon [2k]\to[k] \\ \ker(\mathbf{i}) \in \mathcal{P}_2(2k) \\ \ker(\mathbf{i}) \leqslant \ker(\mathbf{r})}} \psi\big(\widehat{s}_{\sigma(\mathbf{i}(1))} \cdots \widehat{s}_{\sigma(\mathbf{i}(2k))}\big) \\ &= \frac{1}{k!} \sum_{\sigma \in S_k} \sum_{\substack{\mathbf{i}\colon [2k]\to[k] \\ \ker(\mathbf{i}) \in \mathcal{P}_2(2k) \\ \ker(\mathbf{i}) \leqslant \ker(\mathbf{r})}} \left( \frac{1}{k!}\frac{1}{2^k} \sum_{\tau \in S_k} \sum_{\substack{\pi \in \mathcal{P}_2(2k) \\ \pi \leq \ker(\mathbf{i})}} \sum_{\substack{\boldsymbol{\varepsilon}\colon [2k]\to\{*,1\} \\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} z^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\tau)} \overline{z}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\tau)} \right) \\ &= \frac{1}{k!}\frac{1}{2^k} \sum_{\tau \in S_k} \sum_{\substack{\mathbf{i}\colon [2k]\to[k] \\ \ker(\mathbf{i}) \in \mathcal{P}_2(2k) \\ \ker(\mathbf{i}) \leqslant \ker(\mathbf{r})}} \sum_{\substack{\pi \in \mathcal{P}_2(2k) \\ \pi \leq \ker(\mathbf{i})}} \sum_{\substack{\boldsymbol{\varepsilon}\colon [2k]\to\{*,1\} \\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} z^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\tau)} \overline{z}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\tau)}.\end{aligned}$$

Here we have used that all summands are independent from the permutation $\sigma \in S_k$ and thus the corresponding average can be carried out. Otherwise, we had only rearranged the order of some factors and the order of summation. We are left to show that, in the last formula,

$$\cdots \sum_{\substack{\mathbf{i}\colon [2k]\to[k] \\ \ker(\mathbf{i}) \in \mathcal{P}_2(2k) \\ \ker(\mathbf{i}) \leqslant \ker(\mathbf{r})}} \sum_{\substack{\pi \in \mathcal{P}_2(2k) \\ \pi \leq \ker(\mathbf{i})}} \cdots = \cdots \sum_{\substack{\pi \in \mathcal{P}_2(2k) \\ \pi \leq \ker(\mathbf{r})}} \cdots . \tag{5.10}$$

Since $\operatorname{card}\{\mathbf{i}\colon [2k] \to [k] \mid \ker(\mathbf{i}) \in \mathcal{P}_2(2k)\} = \operatorname{card}\{\mathcal{P}_2(2k)\}$, we can rewrite the outer summation over the functions $\mathbf{i}$ such that

$$\cdots \sum_{\substack{\mathbf{i}\colon [2k]\to[k] \\ \ker(\mathbf{i})\in\mathcal{P}_2(2k) \\ \ker(\mathbf{i})\leqslant\ker(\mathbf{r})}} \sum_{\substack{\pi\in\mathcal{P}_2(2k) \\ \pi\leq\ker(\mathbf{i})}} \cdots = \cdots \sum_{\substack{\widetilde{\pi}\in\mathcal{P}_2(2k) \\ \widetilde{\pi}\leqslant\ker(\mathbf{r})}} \sum_{\substack{\pi\in\mathcal{P}_2(2k) \\ \pi\leq\widetilde{\pi}}} \cdots .$$

Note that the condition $\pi \leq \widetilde{\pi}$ forces $\pi = \widetilde{\pi}$ since both $\pi$ and $\widetilde{\pi}$ are pair partitions. This ensures the claimed equality in (5.10). Altogether, we have verified that

$$\lim_{N\to\infty} \psi(\widetilde{S}_{\mathbf{r}(1),N} \cdots \widetilde{S}_{\mathbf{r}(2k),N}) = \frac{1}{k!}\frac{1}{2^k} \sum_{\tau\in S_k} \sum_{\substack{\pi\in\mathcal{P}_2(2k) \\ \pi\leq\ker(\mathbf{r})}} \sum_{\substack{\boldsymbol{\varepsilon}\colon [2k]\to\{*,1\} \\ \boldsymbol{\varepsilon} \text{ is } \pi\text{-balanced}}} z^{\mathrm{cr}_+(\pi,\boldsymbol{\varepsilon},\tau)} \overline{z}^{\mathrm{cr}_-(\pi,\boldsymbol{\varepsilon},\tau)}.$$

□

**Corollary 5.4.7.** *Let the $z$-semicircular system $\widetilde{\mathcal{Y}}$ in $(\mathcal{A}, \psi)$ and the operators $\widetilde{S}_{1,N}, \ldots, \widetilde{S}_{s,N}$ be given as in Theorem 5.4.6. Then there exist a *-algebraic probability space $(\widehat{\mathcal{A}}, \widehat{\psi})$ and a $z$-semicircular system $\widehat{\mathcal{Y}} \equiv (\widetilde{s}_r)_{r=1}^s$ in $(\widehat{\mathcal{A}}, \widehat{\psi})$ such that*

$$\lim_{N\to\infty} \psi\big(\widetilde{S}_{\mathbf{r}(1),N} \cdots \widetilde{S}_{\mathbf{r}(n),N}\big) = \widetilde{\psi}\big(\widetilde{s}_{\mathbf{r}(1)} \cdots \widetilde{s}_{\mathbf{r}(n)}\big) \tag{5.11}$$

*for all $n \in \mathbb{N}$ and $\mathbf{r}\colon [n] \to [s]$.*

*Proof.* This is immediate from the mixed moment formula of the CLT in Theorem 5.4.6, as the right-hand side of (5.11) defines the moment formula for a $z$-semicircular system $(\widehat{s}_r)_{r=1}^s$ in $(\mathcal{A}, \psi)$. □

# Chapter 6

# Future Work

It is well-known that $q$-Gaussian random variables ($-1 \leq q \leq 1$) interpolate between the normal distribution ($q = 1$) and the symmetric Bernoulli distribution ($q = -1$). A breakthrough result of Bożejko and Speicher was that these $q$-Gaussian random variables can be realized as operators on so-called $q$-Fock spaces [BS91, BS92, BKS97]. We note that $q$-Gaussian random variables are also addressed as $q$-semicircular systems in a multivariate setting, see [MN01] for example.

Our investigations reveal that CLTs associated to $\omega$-sequences of partial isometries give rise to $\omega$-semicircular systems (for $\omega \in \mathbb{T}$). These $\omega$-semicircular systems provide another interpolation between 1-semicircular systems and $-1$-semicircular systems, where the latter can be realized on a $q$-Fock space for $q = 1$ and $q = -1$, respectively. Furthermore, inspired by the notion of $z$-circular systems in [MN01], the notion of $\omega$-semicircular systems can be further generalized to that of $z$-semicircular systems such that the latter also comprise $q$-semicircular systems.

The well-developed theory for $q$-Gaussian random variables (or $q$-semicircular systems) may serve as a blueprint for future investigations on $\omega$-(semi)circular systems. An affirmative answer to the following question would probably be pivotal for a future theory of $\omega$-semicircular systems:

> Can $\omega$-semicircular systems be realized as the sum of creation and annihilation operators on an "$\omega$-Fock space"?

In particular, the construction of such an "$\omega$-Fock space" would shed also new light on how to realize $z$-(semi)circular systems.

# Bibliography


[Am19] Amann, A. (2019). Personal communication.

[BKS97] Bożejko, M.; Kümmerer, B.; and Speicher, R. (1997). $q$-Gaussian processes: non-commutative and classical aspects. *Comm. Math. Phys.* 185, no. 1, 129–154.

[BS91] Bozejko, M.; and Speicher R. (1991). An example of a generalized Brownian motion. *Commun. Math. Phys.* 137, no.3, 519–531.

[BS92] Bozejko, M.; and Speicher R. (1992). An example of a generalized Brownian motion, II. *Quantum probability & Related Topics*, 67–77. QP & PQ VII, World Scientific Publications.

[BS96] Bozejko, M.; and Speicher, R. (1996). Interpolations between bosonic and fermionic relations given by generalized Brownian motions. *Math. Z.* 222, no. 1, 135–159.

[CGH77] Cockroft, A. M.; Gudder, S. P.; and Hudson, R. L.(1977). A quantum-mechanical functional central limit theorem. *J. Multivariate Anal.* 7, 125–148.

[CH71] Cushen, C.D.; and Hudson, R.L. (1971). A quantum mechanical central limit theorem. *J. Appl. Probability* 8, 454–469.

[CKN21] Campbell, J.; Köstler, C.; and Nica, A. (2021). Publication in preparation.

[D90] Voiculescu; D. (1990). Circular and semicircular systems and free product factors, In: *Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory* (A. Connes et al. editors). Progress in Mathematics Volume 92, Birkhauser, Boston pp. 45–60.

[EGK17] Evans, D. G.; Gohm, R.; and Köstler, C. (2017). Semi-cosimplicial objects and spreadability. *Rocky Mountain J. Math.* 47 (6), 1839–1873.

[GHJ89] Goodman, F. M.; de la Harpe, P.; and Jones, V. F. R. (1989). *Coxeter Graphs and Towers of Algebras.* Springer-Verlag.

[GK09] Gohm, R.; and Köstler, C. (2009). Noncommutative independence from the braid group $\mathbb{B}_\infty$. *Comm. Math. Phys.* 289, 435–482.

[GK10] Gohm, R.; and Köstler, C. (2010). Noncommutative independence from characters of the infinite symmetric group $\mathbb{S}_\infty$. *Eprint arXiv:1005.5726v1.*

[GK12] Gohm, R.; and Köstler, C. (2012). Noncommutative independence in the infinite braid and symmetric group. *Noncommutative harmonic analysis with applications to probability III*, Banach Center Publ., Polish Acad. Sci. Inst. Math., Warsaw 96, 193–206.

[GW78] Giri, N.; and von Waldenfels, W.(1978). An algebraic version of the central limit theorem. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 42, no. 2, 129–134.

[H73] Hudson, R. L. (1973). A quantum-mechanical central limit theorem for anti-commuting observables. *J. Appl. Probability* 10 , 502–509.

[HL19] Hasebe, T.; and Lehner, F. (2019). Cumulants, spreadability and the Campbell-Baker-Hausdorff Series. *Eprint arXiv:1711.00219.*

[JS97] Jones, V. F. R.; and Sunder, V. S. (1997). *Introduction to Subfactors.* London Mathematical Society Lecture Note Series (234), Cambridge University Press.

[Ka92] Kallenberg, O. (1992). Symmetries on random arrays and set-indexed processes. *J. Theoret. Probab.* 5, no. 4, 727–765.

[Ka05] Kallenberg, O. (2005). *Probabilistic Symmetries and Invariance Principles.* Springer-Verlag.

[KN20] Köstler, C; and Nica, A. (2020). A central limit theorem for star-generators of $S_\infty$, which relates to the law of a GUE matrix. *J. Theoret. Probab.* 34, no. 3, 1248–1278.

[Kö10] Köstler, C. (2010). A noncommutative extended de Finetti theorem. *J. Funct. Anal.* 258 (4), 1073–1120.

[Kö21] Köstler, C. (2021). Personal communication.

[KSV96] Krishnan, U.; Sunder, V. S.; and Varughese, C. (1996). On some subfactors of integer index arising from vertex models. *J. Funct. Anal.*, 140, 449–471.

[LM95] van Leeuwen, H.; and Maassen, H. (1995). A $q$-deformation of the Gauss distribution. *J. Math. Phys.*, 36, no. 9, 4743–4756.

[MN01] Mingo, J; and Nica, A. (2001). Random unitaries in non-commutative tori, and an asymptotic model for $q$-circular systems. *Indiana Univ. Math. J.* 50, no. 2, 953–987

[Mu11] Muraki, N. (2011). Monotonic independence, monotonic central limit theorem and monotonic law of small numbers. *Infinite Dimensional Analysis Quantum Probability and Related Topics* 04, no. 01, 39–58.

[NS06] Nica, A.; and Speicher, R. (2006). *Lectures on the Combinatorics of Free Probability*. London Mathematical Society Lecture Note Series, 335. Cambridge University Press, Cambridge.

[RS75] Reed, M.; Simon, B. (1975). *Fourier Analysis, Self-Adjointness. Methods of Modern Mathematical Physics*, 2, Academic Press.

[S91] Schürmann, M. (1991). Quantum $q$-white noise and a $q$-central limit theorem. *Comm. Math. Phys.* 140, no. 3, 589–615.

[Sp90] Speicher, R. (1990). A new example of “independence” and “white noise”. *Probab. Theory Related Fields* 84, no. 2, 141–159.

[Sp92] Speicher, R. (1992). A noncommutative central limit theorem. *Math. Z.* 209, no. 1, 55–66.

[Sp93] Speicher, R. (1993). Generalized statistics of macroscopic fields. *Lett. Math. Phys.* 27, no. 2, 97–104.

[Sp19] Speicher, R. (2019). Lecture Notes on “Free Probability Theory”. *Eprint arXiv:1908.08125.*

[SW88] Schürmann, M.; von Waldenfels, W. (1988). A central limit theorem on the free Lie group. *Quantum Probability and Applications*, III (Oberwolfach, 1987), 300–318, Lecture Notes in Math., 1303.

[SW94] Speicher, R.; and von Waldenfels, W. (1994). A general central limit theorem and invariance principle. *Institut für Angewandte Mathematik University Heidelberg.*

[Sz09] Szablowski, P. J. (2009). $q$-Gaussian distributions: simplifications and simulations. *Journal of Probability and Statistics*, Art. ID 752430, 18 pp.

[T04] Henk Tijms. (2004). *Understanding Probability: Chance Rules in Everyday Life.* Cambridge University Press.

[W78] von Waldenfels, W.(1978). An algebraic central limit theorem in the anti-commuting case. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 42 , no. 2, 135–140.

[W86] von Waldenfels, W.(1986). Noncommutative algebraic central limit theorems. *Probability Measures on Groups*, VIII (Oberwolfach, 1985), 174–202, Lecture Notes in Math., 1210.

[W87] von Waldenfels, W.(1987). Proof of an algebraic central limit theorem by moment generating functions. *Stochastic Processes – Mathematics and Physics*, II (Bielefeld, 1985), 342–347, Lecture Notes in Math., 1250.

[W90] von Waldenfels, W. (1990). Illustration of the quantum central limit theorem by independent addition of spins. *Séminaire de Probabilités*, XXIV, 1988/89, 349–356, Lecture Notes in Math., 1426.

[Wy08] Wysoczański J. (2008). bm-Central limit theorems for positive definite real symmetric matrices. *Infin. Dimens. Anal. Quantum Probab. Relat. Top. 11*, no. 1, 33–51.