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Constant mean curvature slices in the extended Schwarzschild solution and the collapse of the lapse

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We give a detailed description of the constant mean curvature foliations in Schwarzschild spacetime, show that the lapse collapses exponentially, and compute the exponent.

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I. INTRODUCTION

In the analysis of general relativity as a Hamiltonian system [1], one chooses a time function and considers the foliation of the spacetime by the slices of constant time. Two natural geometrical quantities arise on such spacelike three-slices. One is the intrinsic three-metric, usually g_{ab}, and the other is the extrinsic curvature K^{ab}, the derivative of g_{ab} along the normal to the slice. They are related by the constraints, which in a vacuum spacetime read

$$\mathcal{R}^{(3)}-K^{ab}K_{ab}+(\mathrm{tr}\,K)^2=0, \tag{1}$$

$$\nabla_a K^{ab}-g^{ab}\nabla_a \mathrm{tr}\,K=0, \tag{2}$$

where $\mathcal{R}^{(3)}$ is the three-scalar curvature. Given the initial data, one chooses, essentially arbitrarily, the lapse N and the shift N^i, which determine the magnitude and direction of the unit time vector relative to the normal to the slice.

One can now write the evolution equations for the intrinsic metric and extrinsic curvature in vacuum: e.g. [2] (the reader should be warned that we follow Wald [3] in our definition of the extrinsic curvature, not [1]; positive K means increasing volume to the future),

$$\partial_t g_{ab}=2NK_{ab}+N_{a;b}+N_{b;a}, \tag{3}$$

$$\partial_t K_{ab}=N_{;ab}-N(R_{ab}-2K_a^dK_{bd}+K_{ab}\mathrm{tr}\,K)+K_{ab;c}N^c+K_{ac}N^c_{;b}+K_{cb}N^c_{;a}. \tag{4}$$

Let us stress that we are using the convention of signs that gives $\mathrm{tr}\,K=+n^\alpha_{;\alpha}$, where n^α is the timelike unit normal to the slice and $\partial_t\sqrt{g}=\sqrt{g}(N\,\mathrm{tr}\,K+N^a_{;a})$. In other words, a positive $\mathrm{tr}\,K$ means expansion. It is often useful to specify the foliation, and thus the time, by placing a condition on the extrinsic curvature. A very popular choice is to demand that the trace of the extrinsic curvature be constant on each slice ("CMC slicing") [4].

In this paper, we investigate the CMC slices of the extended Schwarzschild solution. The manifold consists of four segments, each of which can be covered by the standard Schwarzschild metric

$$ds^2=-\left(1-\frac{2m}{r}\right)dt^2+\frac{dr^2}{1-\frac{2m}{r}}+r^2(d\theta^2+\sin^2\theta d\phi^2), \tag{5}$$

where t is the "static" Killing vector and r is the "areal" or "Schwarzschild" radius. In the left and right zones, t is timelike and $r>2m$ is spacelike. In the bottom zone, r is timelike and runs forward from the past singularity at $r=0$ to $r=2m$. In the top zone, r is also timelike and runs forward from $r=2m$ to the future singularity, also at $r=0$. We do not seek the most general CMC slices. We are looking for those CMC slices which inherit the underlying spherical symmetry of the given spacetime.

There are two complementary ways of analyzing this problem. One way is to assume that one is given initial data (the intrinsic metric and extrinsic curvature), both parts of which have the desired symmetry. It turns out that one can explicitly solve the constraints. From the momentum constraint, Eq. (2), it is clear that the extrinsic curvature must be just a sum of the trace term and a part which is both trace- and divergence-free [transverse–trace-free (TT)]. There exists a unique spherically symmetric TT tensor. Therefore, the extrinsic curvature can be written down with just two free parameters. On substituting into the Hamiltonian constraint, Eq. (1), one discovers—see Sec. VII—that this also can be solved explicitly.

The alternative approach is to take the given spacetime and make a coordinate transformation in the (t,r) plane only, given by $t=h(r)$, leaving the rest untouched. $h(r)$ is called the height function. One now imposes the condition that the $t'=0$ slice be CMC. This gives a second-order equation for the height function which can be integrated explicitly once. This is enough to evaluate the intrinsic metric and extrinsic curvature of the slice, and, of course, they agree with the expressions obtained using the first approach.

One then can work out how these slices fit into the given spacetime and construct interesting CMC foliations. One ends up with a first-order equation for the height function which cannot be integrated explicitly. Nevertheless, one can

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make qualitative statements about the location of the slices. In this paper, we focus on a particular class of slicings in which we fix the value of $\mathrm{tr}\,K$ and vary the parameter defining the amount of the TT component in the extrinsic curvature. For small values of the parameter we have two foliations, one which runs from one infinity to the other, and one which emerges and returns to one of the singularities. As the parameter increases, the leaves of the foliations approach one another, and at a critical value of the parameter, they touch. For values of the parameter greater than the critical one, the nature of the CMC slices changes. They all now run from infinity into the singularity.

In this paper, we focus on the behavior of the slices as they approach the critical value. We find the classic ''collapse of the lapse'' phenomenon. Further, by looking carefully at the first-order equation for the height function, we obtain an explicit expression for how the lapse decays near criticality. The first part of this paper (Secs. III–VI) draws very heavily on the analysis given in [6] of the collapse of the lapse for maximal slices of the Schwarzschild solution.

II. EXTRINSIC AND INTRINSIC GEOMETRY OF CMC SLICES

We can generate an essentially general spherically symmetric slicing of the Schwarzschild solution by making a coordinate transformation $t'=t'(t,r), r'=r'(t,r)$. This will give us a spacetime metric of the form

$$ds^2=-g_{t't'}dt'^2+N_{r'}dt'dr'+g_{r'r'}dr'^2 + r^2[d\theta^2+\sin^2(\theta)d\phi^2]. \tag{6}$$

We have that $g_{t't'}$, $N_{r'}$, and $g_{r'r'}$ are functions of t' and r'. The coefficient r^2 in front of the two-metric is the original, Schwarzschild, coordinate r^2 but can be viewed as a function of t' and r' as well. We can make a further coordinate transformation of the form $r''=r''(t',r')$, leaving t' unchanged. This has the effect of changing the r' coordinate within each slice but leaving the slicing unchanged. This kind of transformation can be used to arrange that $\boldsymbol{\nabla} t'\cdot\boldsymbol{\nabla} r''=\mathbf{0}$. This is equivalent to dragging the r' coordinate along the normal to the slicing and thus sets the shift to zero. This will give us a spacetime metric,

$$ds^2=-N^2dt'^2+g_{r''r''}dr''^2+r^2[d\theta^2+\sin^2(\theta)d\phi^2]. \tag{7}$$

On any one given slice, we can arrange that $r''=r$, the original Schwarzschild coordinate. However, when one tries to propagate this condition, one discovers that in general it is not compatible with vanishing shift. Therefore, one can have a metric of the form (6) with $r'=r$ or a metric of the form (7) with zero shift, but not both. Another choice would be to set $g_{r'r'}=1$, i.e., to choose the r' coordinate as the proper distance along the slice. Again, this is not compatible with vanishing shift. One advantage that the ''proper distance'' coordinate choice has over the ''$r'=r$'' gauge is that, so long as the slice remains spacelike, the proper distance gauge always remains regular while the ''$r'=r$'' choice may well have coordinate singularities. However, in this paper we will largely stick to the metric form (7) and ignore questions such as the ''best'' choice of spatial coordinate.

To simplify the notation, we will write the line element as

$$ds^2=-N^2dt^2+a\,dr^2+R^2[d\theta^2+\sin^2(\theta)d\phi^2], \tag{8}$$

where the written (t,r) are *not* the original (t,r) while R *is* the original r. The geometry of $t=$const slices is encoded in two places. One is the dependence of a on r and the other is the relationship between R and r. This second piece is contained in the mean curvature of the surfaces of constant r as embedded two-surfaces in the spatial three-geometry,

$$p=\frac{2}{\sqrt{a}R}\frac{dR}{dr}.$$

The only nonzero (three) extrinsic curvature components with mixed-case indices are

$$K_r^r=\frac{\partial_t a}{2aN},\quad K_\theta^\theta=K_\phi^\phi=\frac{\partial_t R}{RN}=\frac{1}{2}(\mathrm{tr}\,K-K_r^r). \tag{9}$$

These can be viewed as evolution equations for a and R. The evolution equations for the extrinsic curvature can most compactly be written as

$$\partial_t \mathrm{tr}\,K=\nabla_i\nabla^i N-K_j^iK_i^jN, \tag{10}$$

$$\partial_t(\mathrm{tr}\,K-K_r^r)=NR_r^{(3)r}-NK_i^jK_j^i+NK_r^r\mathrm{tr}\,K+\frac{p\,\partial_r N}{\sqrt{a}}. \tag{11}$$

The form of the three-dimensional Ricci curvature component $R_r^{(3)r}$ is

$$R_r^{(3)r}=-\frac{\partial_r(pR)}{\sqrt{a}R}, \tag{12}$$

while the three-dimensional scalar curvature $R^{(3)}$ is

$$R^{(3)}=-\frac{2\partial_r(pR)}{R\sqrt{a}}-\frac{(pR)^2}{2R^2}+\frac{2}{R^2}. \tag{13}$$

It turns out that the momentum constraint can be written as

$$\partial_r(K_r^r-\mathrm{tr}\,K)=-\frac{3}{2}pK_r^r+\frac{1}{2}p\;\mathrm{tr}\,K, \tag{14}$$

and the Hamiltonian constraint is

$$\frac{1}{\sqrt{a}R}\partial_r(pR)=-\frac{3}{4}(K_r^r)^2-\frac{1}{4}p^2+\frac{1}{R^2} +\frac{1}{2}\mathrm{tr}\,KK_r^r+\frac{1}{4}(\mathrm{tr}\,K)^2. \tag{15}$$

We are interested in finding surfaces which have $\mathrm{tr}\,K=$const, where $\mathrm{tr}\,K=(1/N)\partial_t\ln(\sqrt{a}R^2)$ is the fractional rate of change of a coordinate volume during the temporal evolution. Assume that $K \equiv \mathrm{tr}\, K$ is constant on a fixed Cauchy hypersurface. Then the momentum constraint (14) is easily solved by

$$K^r_r = \frac{K}{3} + \frac{2C}{R^3}, \tag{16}$$

where C is again a constant on the chosen Cauchy slice. The other components of the extrinsic curvature are

$$K^\theta_\theta = K^\phi_\phi = \frac{K}{3} - \frac{C}{R^3}. \tag{17}$$

This can be recognized as a combination of the trace term plus the unique spherically symmetric TT tensor, the terms with coefficient C. Therefore, CMC slices of the Schwarzschild solution are completely defined by the two parameters K and C. The only residual freedom is the ability to drag any surface along the Killing vector without disturbing either the intrinsic or extrinsic geometry.

The insertion of Eqs. (16),(17) into the Hamiltonian constraint leads, after some minor manipulation, to the equation

$$\partial_r \left[\frac{R}{4}(pR)^2 - R - \frac{C^2}{R^3} - \frac{K^2}{9}R^3 \right] = 0. \tag{18}$$

Equation (18) is solved by

$$(pR)^2 = 4\left[1 - \frac{\beta}{R} + \left(\frac{KR}{3} - \frac{C}{R^2} \right)^2 \right]. \tag{19}$$

Here β is essentially the integration constant, modified by completing the square of K and C related terms. It is easy to show that $\beta = 2m$. If "r" is replaced by the "areal radius R," then one finds $a = 4/(pR)^2$. Notice also that the three-dimensional line elements in such a case read

$$ds^2_{(3)} = \frac{4}{(pR)^2} dR^2 + R^2 [d\theta^2 + \sin^2(\theta) d\phi^2]. \tag{20}$$

III. THE CYLINDRICAL CMC SLICES OF SCHWARZSCHILD SPACETIME

In the upper and lower quadrants of Schwarzschild spacetime, the Killing vector is spacelike and runs along the $r = \mathrm{const}$ surfaces. Since everything is constant along the Killing vector, the trace of the extrinsic curvature is preserved along these cylindrical surfaces. Therefore, each $r = \mathrm{const}$ surface is a CMC slice.

The trace of the extrinsic curvature (in the upper quadrant) is given by

$$K = \frac{2r - 3m}{\sqrt{2mr^3 - r^4}}. \tag{21}$$

This is large and positive near $r = 2m$, zero at $r = 3m/2$, and becomes large and negative as r becomes small. This transforms into

$$r^4 - 2mr^3 + \frac{(2r-3m)^2}{K^2} = 0. \tag{22}$$

This is a quartic equation with two real roots. One lies between $r = 2m$ and $r = 3m/2$ and the other between $r = 3m/2$ and $r = 0$. This is clear by looking for the extrema of the quartic. To find these, we just differentiate to get

$$4r^3 - 6mr^2 + 4\frac{2r-3m}{K^2} = 2(2r-3m)\left(r + \frac{2}{K^2} \right) = 0. \tag{23}$$

Therefore, it has only one minima (at $r = 3m/2$) and the quartic is negative there. Hence it has two real roots, one on each side of $r = 3m/2$. On substituting back into Eq. (21) it is clear that the solution of Eq. (22) with $r > 3m/2$ has $K > 0$ and the solution with $r < 3m/2$ has $K < 0$.

In the lower quadrant, things are somewhat different. The trace of the extrinsic curvature is now given by

$$K = \frac{3m - 2r}{\sqrt{2mr^3 - r^4}}. \tag{24}$$

This is because, in the upper quadrant, the future is in the direction of decreasing r, while in the lower quadrant the future is in the direction of increasing r. This is now large and positive near $r = 0$, zero at $r = 3m/2$, and becomes large and negative as r approaches $2m$. We get the same quartic, Eq. (22), with the same roots, but now with the order reversed. The root which is less than $3m/2$ corresponds to $K > 0$, while the other root has $K < 0$.

Given r, we can work out, from Eq. (21), the value of the trace of the extrinsic curvature. We can, in fact, work out the entire extrinsic curvature and evaluate the constant C associated with these cylindrical CMC slices. In the upper quadrant we get

$$C = \frac{3mr^3 - r^4}{3\sqrt{2mr^3 - r^4}}. \tag{25}$$

Therefore, $C > 0$ on each of these slices.

In the bottom quadrant, the extrinsic curvature picks up a minus sign. Therefore, for the cylindrical CMC slices we get

$$K = \frac{2r - 3m}{\sqrt{2mr^3 - r^4}}, \quad C = \frac{r^4 - 3mr^3}{3\sqrt{2mr^3 - r^4}}, \tag{26}$$

and so C is negative in the lower quadrant.

IV. THE EMBEDDING OF CMC SLICES IN SCHWARZSCHILD SPACETIME

In addition to the cylindrical CMC slices described in Sec. III, there are many other spherically symmetric ones. In this section, we will discuss how they run through the spacetime.

The first comprehensive analysis of CMC slices in Schwarzschild spacetime appeared in [5]. The analysis given here closely follows the analysis of the related problem of maximal slices ($\mathrm{tr}\,K=0$) in the Schwarzschild solution given in [6]. Let us start off with the Schwarzschild metric

$$ds^2=-\left(1-\frac{2m}{r}\right)dt^2+\frac{dr^2}{1-\frac{2m}{r}}+r^2(d\theta^2+\sin^2\theta d\phi^2), \tag{27}$$

and look at the slice given by $t=h(r)$, where $h(r)$, for obvious reasons, is called the height function. One way of understanding the geometry of this slice is to make the following coordinate transformation:

$$\overline{t}=t-h(r),\quad t=\overline{t}+h(\overline{r}),$$

$$\overline{r}=r,\quad r=\overline{r},$$

$$\overline{\theta}=\theta,\quad \theta=\overline{\theta},$$

$$\overline{\phi}=\phi,\quad \phi=\overline{\phi},$$

where the $\overline{t}=0$ surface is the slice in which we are interested. The transformed metric becomes

$$\overline{g}_{\mu\nu}=\begin{pmatrix} -\left(1-\frac{2m}{r}\right), & -h'\left(1-\frac{2m}{r}\right) & 0 & 0\\ -h'\left(1-\frac{2m}{r}\right), & \left(1-\frac{2m}{r}\right)^{-1}-h'^2\left(1-\frac{2m}{r}\right) & 0 & 0\\ 0 & 0 & r^2 & 0\\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix},$$

$$\overline{g}^{\mu\nu}=\begin{pmatrix} -\left(1-\frac{2m}{r}\right)^{-1}+h'^2\left(1-\frac{2m}{r}\right), & -h'\left(1-\frac{2m}{r}\right) & 0 & 0\\ -h'\left(1-\frac{2m}{r}\right), & \left(1-\frac{2m}{r}\right) & 0 & 0\\ 0 & 0 & \frac{1}{r^2} & 0\\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix},$$

where $h'=\partial h/\partial r$. The intrinsic metric is given by

$$ds^2=\left[\left(1-\frac{2m}{r}\right)^{-1}-h'^2\left(1-\frac{2m}{r}\right)\right]dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{28}$$

The lapse N of this slicing is given by

$$N=\left[\left(1-\frac{2m}{r}\right)^{-1}-h'^2\left(1-\frac{2m}{r}\right)\right]^{-1/2}, \tag{29}$$

the shift N_a by

$$N_a=\left[-h'\left(1-\frac{2m}{r}\right),0,0\right], \tag{30}$$

and the future-pointing unit normal by

$$n^\mu=\frac{\left[\left(1-\frac{2m}{r}\right)^{-1}-h'^2\left(1-\frac{2m}{r}\right),h'\left(1-\frac{2m}{r}\right),0,0\right]}{\sqrt{\left(1-\frac{2m}{r}\right)^{-1}-h'^2\left(1-\frac{2m}{r}\right)}}. \tag{31}$$

Given any three-slice in the four-manifold, we can drag it along by the Killing vector. This will give a slicing where the time translation is along the Killing vector. It is this slicing that is generated by the coordinate transformation above. Therefore, the N and N_a defined by Eqs. (29) and (30) are nothing more than the projections of the Killing vector perpendicular to and onto the given slice. Of course, the slicing given by dragging along the Killing vector cannot form a foliation because the Killing vector has a fixed point on the bifurcation sphere.

The mean curvature of the $\bar{t}=0$ slice is given by

$$K=n^{\mu}_{;\mu}=\frac{1}{\sqrt{-g}}(\sqrt{-g}n^{\mu})_{,\mu}$$
$$=\frac{1}{r^2}\partial_r\left[\frac{r^2h'\left(1-\frac{2m}{r}\right)}{\sqrt{\left(1-\frac{2m}{r}\right)^{-1}-h'^2\left(1-\frac{2m}{r}\right)}}\right]. \quad (32)$$

If K is a constant, this can be integrated to give

$$\frac{Kr}{3}-\frac{C}{r^2}=\frac{h'\left(1-\frac{2m}{r}\right)}{\sqrt{\left(1-\frac{2m}{r}\right)^{-1}-h'^2\left(1-\frac{2m}{r}\right)}}, \quad (33)$$

where C is a constant of integration. In turn, this can be manipulated to give

$$\left(1-\frac{2m}{r}\right)^{-1}-h'^2\left(1-\frac{2m}{r}\right)=\frac{1}{\left(1-\frac{2m}{r}\right)+\left(\frac{Kr}{3}-\frac{C}{r^2}\right)^2}, \quad (34)$$

and hence

$$h'=\frac{\frac{Kr}{3}-\frac{C}{r^2}}{\left(1-\frac{2m}{r}\right)\sqrt{\left(1-\frac{2m}{r}\right)+\left(\frac{Kr}{3}-\frac{C}{r^2}\right)^2}}. \quad (35)$$

If one could integrate this one more time and find $h(r)$ in closed form, we would have a complete description of the slices. We cannot do so. Nevertheless, we can extract a significant amount of information from Eq. (35) as it stands.

First, from the expressions Eq. (28), Eq. (29), and Eq. (30) it is clear that the intrinsic metric, the lapse, and the shift depend only on h'. Thus we get

$$N=\frac{pr}{2}=\sqrt{\left(1-\frac{2m}{r}\right)+\left(\frac{Kr}{3}-\frac{C}{r^2}\right)^2},$$

$$N_r=\frac{\frac{C}{r^2}-\frac{Kr}{3}}{\sqrt{\left(1-\frac{2m}{r}\right)+\left(\frac{Kr}{3}-\frac{C}{r^2}\right)^2}}. \quad (36)$$

Finally, we can find the extrinsic curvature of the slice by using

$$2NK_{ab}=\frac{\partial g_{ab}}{\partial \bar{t}}-N_{a;b}-N_{b;a}, \quad (37)$$

and we recover Eqs. (16)–(20).

When we look at Eq. (35), it is clear that the right-hand side does not decay for large r and thus the integral must diverge as we approach infinity. This is not surprising as we expect the CMC slices to go to null infinity.

This agrees with the behavior of the spherical CMC slices in Minkowski space. Consider the mass hyperboloid $t^2-r^2=9/K^2$ in flat spacetime. If we choose the one which goes to future null infinity, then the future-pointing timelike normal is $n^{\alpha}=(t,r)/\sqrt{t^2-r^2}$, where we have to choose the positive root of $\sqrt{t^2-r^2}$. We then find $\mathrm{tr}\,K=n^{\alpha}_{;\alpha}=3/\sqrt{t^2-r^2}=|K|>0$. To find out where on null infinity the slice ends up, we need to introduce null coordinates $v=(t+r)/2, u=(t-r)/2$. Using $t=r\sqrt{1+9/r^2K^2}\approx r+9/2rK^2$, it is clear that as $v\to\infty, u\approx 9/rK^2\to 0$. If we time-translate it to $(t-t_0)^2=r^2+9/K^2$, we find $u\to t_0$. Therefore, it slides up null infinity.

If we look at Eq. (35) for large r, we see that $h'\approx r/\sqrt{9/K^2+r^2}$. The integral of this is $h\approx\sqrt{9/K^2+r^2}$, which is in complete agreement with the flat space expression. Therefore, the slices remain spacelike but go null infinity as $r\to\infty$. Further, if $K>0$, the slices all go to future null infinity, whereas if $K<0$ the slices go to past null infinity.

One place we can find interesting information, without solving for h, is by looking at the expression for the mean curvature of the spherical two-surfaces. In particular, we know that

$$\frac{p^2r^2}{4}=\left(1-\frac{2m}{r}\right)+\left(\frac{Kr}{3}-\frac{C}{r^2}\right)^2\geqslant 0. \quad (38)$$

Therefore, the polynomial on the right-hand side of Eq. (38) must be non-negative. Further, we know that the zeros of the polynomial are the points where $p=0$ and therefore are the extrema of the area of the round two-spheres as embedded surfaces in the three-slice.

Let us first fix some $K>0$ and see what happens as we vary C. (The cases where $K<0$ are remarkably similar.) First consider the case where $C=0$. This is the so-called umbilical slice, where $K_{ab}\propto g_{ab}$. In this case the polynomial reduces to $1-2m/r+K^2r^2/9$. This is a cubic equation with only one real root, call it r_u. Outside $r=r_u$, the polynomial is positive; inside it, it is negative. It is clear that $r_u\leqslant 2m$ and that $r_u=2m$ if $K=0$. Therefore, we know that the umbilical slice with $K>0$ starts out at future null infinity, comes in to a minimum at $r=r_u$, and then passes out to the other future null infinity. The obvious question is whether this occurs above or below the bifurcation sphere.

To settle this, we need to look at the optical scalars [7–9]

$$\omega_{+}=2\sqrt{\left(1-\frac{2m}{r}\right)+\left(\frac{Kr}{3}-\frac{C}{r^2}\right)^2}+2\left(\frac{Kr}{3}-\frac{C}{r^2}\right) \quad (39)$$

and

$$\omega_{-}=2\sqrt{\left(1-\frac{2m}{r}\right)+\left(\frac{Kr}{3}-\frac{C}{r^2}\right)^2}-2\left(\frac{Kr}{3}-\frac{C}{r^2}\right). \quad (40)$$

These are essentially the null expansions in the outgoing-future and outgoing-past directions, respectively. They are both positive in Minkowski space and in the exterior regimes of the Schwarzschild solution. Since the product satisfies $\omega_{+}\omega_{-}/4=1-2m/r$, one or the other becomes negative in the interior quadrants of the Schwarzschild solution. It turns out that the upper quadrant satisfies $\omega_{+}<0,\omega_{-}>0$ while in the lower quadrant we have $\omega_{-}<0,\omega_{+}>0$. It is clear that at $r=2m$, when $K>0$ and $C=0$, ω_{-} goes negative while ω_{+} remains positive. Therefore, the umbilical slice (with $K>0$) must pass through the lower quadrant. Therefore, it starts at future null infinity, comes down so as to cross the $t=0$ axis, passes through the Schwarzschild throat below the bifurcation point to some minimum radius r_u, and then rises up again to the other future null infinity.

Let us now hold $K>0$ fixed but change C so as to be slightly larger than zero. Now the polynomial becomes sixth order with two roots, which we call r_{ms} (ms=max.-small) and r_{ml} (ml=min.-large). Near $r=0$, the dominant term is the positive term C^2/r^4 so the polynomial starts off large and positive while the next term is the negative $-2m/r$, which pulls it negative at $r=r_{\mathrm{ms}}$ with $r_{\mathrm{ms}}\approx\sqrt[3]{C^2/2m}$. We know that the polynomial must become positive before $r=2m$ and the $K^2r^2/9$ term does just that at r_{ml} with $r_{\mathrm{ml}}\approx r_u$. If $C>0$, then the effect of the C term is to diminish the effectiveness of the K^2 term, so we get that $r_{\mathrm{ml}}>r_u$ while if $C<0$ we get $r_{\mathrm{ml}}<r_u$. Therefore, for $K>0$ and $C>0$ (but small) we have two different regimes in which the polynomial is positive. One is for small r, which represents a CMC slice that starts at $r=0$, expands out to r_{ms}, which is the maximum area, and then contracts again back to $r=0$.

When we look at the null expansions, it is clear that when $C>0$ for small r we have that $\omega_{+}<0,\omega_{-}>0$, so it must be in the upper quadrant. Hence when $C>0$ the small r slice comes out of and goes back to the future singularity while the other slice runs from future null infinity to future null infinity and passes through the center at a slightly larger radius than the umbilical one. Thus it runs to the future of the umbilical slice and crosses closer to the bifurcation sphere. As C increases away from zero we continue to have two CMC slices, one which comes from the future singularity out to some small radius $r_{\mathrm{ms}}\approx\sqrt[3]{C^2/2m}$ and the other which goes from future null infinity to future null infinity but will be slightly to the future of the umbilical slice. We find r_{ml} monotically increases as C does until $C=8Km^3/3$. For this value of C, it is easy to show that $r_{\mathrm{ml}}=2m$ so that this CMC slice will pass through the bifurcation point.

Increasing C acts like a time translation near infinity. From what happens in Minkowski space, we expect the slice to slide up along null infinity.

As C increases even further, r_{ml} will start to decrease again while the CMC slice continues to move forward in time and passes through the throat above the bifurcation point. As C increases, we find that r_{ms} increases so that the CMC slicing that begins and ends at the future singularity moves backwards in time. The minimum of the polynomial rises up and the two roots, r_{ml} and r_{ms}, will approach each other as C approaches the critical value $C=C_*^+$. For this value of C, the polynomial is everywhere positive except at one point. This will be at a radius we call R_*^+. This will satisfy $R_*^+>3m/2$. R_*^+ is nothing more than the larger of the two roots of Eq. (22) and C_*^+ is the value of C given by Eq. (25). This is because the cylindrical CMC slices act as barriers to the noncylindrical CMC slices.

As C approaches C_*^+, each of the two CMC slices will develop long cylindrical regions. The one from null infinity will run along, but just above, the surface with $r=R_*^+$ while the one from $r=0$ will run just below. The closer to the critical value, the longer the cylinders.

When $C=C_*^+$, we get a sudden change. Instead of having two solutions with long cylinders, we have five. Two come from the left and right null infinity, respectively, and asymptote (from above) to infinite cylinders of radius $r=R_*^+$. Two others come, left and right, from $r=0$ and asymptote from below to the same cylinders. The fifth solution is the $r=R_*^+$ cylinder itself.

When C exceeds the critical value, we get another change. The polynomial becomes everywhere positive. This means that the CMC slice cannot have any extremum. It must run all the way from $r=0$ to $r=\infty$. If $C>C_*^+$, we will have two CMC slices, one from the left future null infinity which runs into the future singularity and a mirror one from the right future null infinity.

Starting from the umbilical slice, holding K fixed, and letting C become negative, we get the opposite behavior. The slice from null infinity to null infinity moves backwards in time, while a new CMC slice emerges from the past singularity and goes back to it. As C approaches a negative critical value C_*^-, the two roots of the polynomial approach one another and coincide at a radius $R_*^-<3m/2$. This is the smaller root of Eq. (22) and C_*^- is the value of C given by Eq. (26).

We conjecture that the slicings we have described for fixed K and for C in the range $C_*^-<C<C_*^+$ form three separate foliations: one for $r<R_*^-$ near the past singularity, one for $r<R_*^+$ near the future singularity, and the third formed by the slices that run from one null infinity to the other. We further conjecture that these three foliations cover the entire extended spacetime.

V. DIFFERENTIATING THE HEIGHT FUNCTION

Let us consider the foliation that runs from null infinity to null infinity. Each slice has the same value of K but C spans an interval. We could use the value of C as a label on the slices, but we want to use some time coordinate as a label.

The obvious choice is the ''time at infinity.'' This is given by

$$\tau(C)=\int_{r_{\rm ml}}^{\infty}\frac{\left(\frac{Kr}{3}-\frac{C}{r^2}\right)dr}{\left(1-\frac{2m}{r}\right)\sqrt{1-\frac{2m}{r}+\left(\frac{Kr}{3}-\frac{C}{r^2}\right)^2}}$$
$$=\int_{r_{\rm ml}}^{\infty}\frac{\left(\frac{Kr^3}{3}-C\right)dr}{\left(1-\frac{2m}{r}\right)\sqrt{r^4-2mr^3+\left(\frac{Kr^3}{3}-C\right)^2}}. \tag{41}$$

This has three divergences. The first is due to the $(1-2m/r)$, which diverges at the horizon. This can be integrated through in the Cauchy principal value sense. A similar divergence arose in [6]. The second is due to the fact that the polynomial inside the square root vanishes at $r=r_{\rm ml}$. This is the definition of $r_{\rm ml}$. This is not a problem because the polynomial goes to zero linearly at $r_{\rm ml}$. Therefore, the integral is of the form $\int dx/\sqrt{x}$, which is regular at $x=0$.

The third divergence is due to the fact that the integral itself diverges as $r\rightarrow\infty$. This has to be because the slice goes to null infinity. To leading order the integral becomes

$$\tau\approx\int\frac{rdr}{\sqrt{\frac{9}{K^2}+r^2}}=\sqrt{r^2+\frac{9}{K^2}}, \tag{42}$$

which is just the flat spacetime mass hyperboloid. If we want a finite time label on the CMC slices, the obvious thing to do would be to subtract off the leading flat space divergent expression. Unfortunately, the difference still logarithmically diverges (like $2m\ln r$). If we want a finite expression, it is better to subtract off the height function of some favored slice of the foliation itself. One obvious choice is to pick the umbilical slice (the $C=0$ slice). Therefore, a natural time label is given by

$$\tau(C)=\int_{r_{\rm ml}}^{\infty}\frac{\left(\frac{Kr^3}{3}-C\right)dr}{\left(1-\frac{2m}{r}\right)\sqrt{r^4-2mr^3+\left(\frac{Kr^3}{3}-C\right)^2}}$$
$$-\int_{r_u}^{\infty}\frac{\frac{Kr^3}{3}dr}{\left(1-\frac{2m}{r}\right)\sqrt{r^4-2mr^3+\left(\frac{Kr^3}{3}\right)^2}}. \tag{43}$$

This, from the argument given above, is finite for all $C<C_*$. As $C\rightarrow C_*$, we have that $r_{\rm ml}\rightarrow R_*$. At this point, the two roots of the polynomial coincide, and the slope of the tangent to the polynomial at $r=r_{\rm ml}$ goes to zero. The integral close to $r_{\rm ml}$ approximates $\int a\,dx/\sqrt{sx}$, where a is some constant and s is the slope. Integrating this over some small but finite fixed interval $(0,\Delta x)$, we get a contribution to τ of $a\sqrt{\Delta x}/2\sqrt{s}$. As $s\rightarrow 0$, this contribution becomes unboundedly large. Therefore, we get ''collapse of the lapse'' in the interior. The foliation moves only a finite distance at the center to reach $r=R_*$ while the passage of ''time at infinity'' becomes unboundedly large.

At the critical point, both $1/h'$ and the first derivative of $1/h'$ vanish at the throat. The coefficient in the exponential decay is nothing more than the second derivative of $1/h'$ at the critical point. This is the dominant term in any expansion of the time function near the critical point. The rest of this paper is devoted to demonstrating this.

We wish to investigate the behavior of the central lapse. In [6] we discussed the situation where we had a foliation defined by some time function τ with lapse α. Say we are given a vector field ξ^μ. The projection of ξ normal to the time slice (call it N) is given by

$$N=\alpha\xi^\mu\nabla_\mu\tau\Rightarrow\alpha=N(\xi^\mu\nabla_\mu\tau)^{-1}. \tag{44}$$

If we choose ξ to be the Killing vector, we know what N is from Eq. (29) and we can also write

$$(\xi^\mu\nabla_\mu\tau)^{-1}=\left(\frac{d\tau}{dC}\right)^{-1}\frac{dh}{dC}\bigg|_r\Rightarrow\alpha$$
$$=\left(\frac{d\tau}{dC}\right)^{-1}N\frac{dh}{dC}\bigg|_r. \tag{45}$$

To evaluate expression (45) we need to differentiate the height function with respect to C. This looks to be highly unpleasant. The square root in the denominator is promoted to 3/2 power so the integral has a term $dx/x^{3/2}$ which diverges at the origin. Further, $r_{\rm ml}$ depends on C so there will also be an end-point variation. This will take the integrand (which is infinite) outside the integral sign. We know $d\tau/dC$ must be finite so these infinities must cancel. A very similar problem arose in [6] and a way was found around it. This essentially involved an integration by parts before differentiating and much more malleable expressions were found. We can repeat this trick.

We begin by defining the following function:

$$J=-\int\frac{\left[r^4-2mr^3+\left(\frac{Kr^3}{3}-C\right)^2\right]^{1/2}dr}{\left(1-\frac{2m}{r}\right)}. \tag{46}$$

This is constructed so that $dJ/dC=h$. Now rewrite J as

$$J=-\frac{2}{3}\int\frac{\frac{d}{dr}\left[r^4-2mr^3+\left(\frac{Kr^3}{3}-C\right)^2\right]^{3/2}dr}{\left(1-\frac{2m}{r}\right)\left(4r^3-6mr^2+2Kr^2\left[\frac{Kr^3}{3}-C\right]\right)}. \tag{47}$$

This now can be integrated by parts to give

$$J=-\frac{2}{3}\frac{\left[r^4-2mr^3+\left(\frac{Kr^3}{3}-C\right)^2\right]^{3/2}}{\left(1-\frac{2m}{r}\right)\left(4r^3-6mr^2+2Kr^2\left[\frac{Kr^3}{3}-C\right]\right)}+\frac{2}{3}\int\left[r^4-2mr^3+\left(\frac{Kr^3}{3}-C\right)^2\right]^{3/2}\frac{d}{dr}\left[\frac{1}{\left(1-\frac{2m}{r}\right)\left(4r^3-6mr^2+2Kr^2\left[\frac{Kr^3}{3}-C\right]\right)}\right]dr. \tag{48}$$

We need to differentiate this twice with respect to C to get dh/dC. We will do this in two parts. Let us call the not-integral part J_1 and the integral J_2.

We get

$$\frac{dJ_1}{dC}=2\frac{\left[r^4-2mr^3+\left(\frac{Kr^3}{3}-C\right)^2\right]^{1/2}\left(\frac{Kr^3}{3}-C\right)}{\left(1-\frac{2m}{r}\right)\left(4r^3-6mr^2+2Kr^2\left[\frac{Kr^3}{3}-C\right]\right)}-\frac{4}{3}\frac{\left[r^4-2mr^3+\left(\frac{Kr^3}{3}-C\right)^2\right]^{3/2}Kr^2}{\left(1-\frac{2m}{r}\right)\left(4r^3-6mr^2+2Kr^2\left[\frac{Kr^3}{3}-C\right]\right)^2}, \tag{49}$$

$$\begin{aligned}\frac{d^2J_1}{dC^2}={}&-2\frac{\left[r^4-2mr^3+\left(\frac{Kr^3}{3}-C\right)^2\right]^{-1/2}\left(\frac{Kr^3}{3}-C\right)^2}{\left(1-\frac{2m}{r}\right)\left(4r^3-6mr^2+2Kr^2\left[\frac{Kr^3}{3}-C\right]\right)}-2\frac{\left[r^4-2mr^3+\left(\frac{Kr^3}{3}-C\right)^2\right]^{1/2}}{\left(1-\frac{2m}{r}\right)\left(4r^3-6mr^2+2Kr^2\left[\frac{Kr^3}{3}-C\right]\right)}\\&+8\frac{\left[r^4-2mr^3+\left(\frac{Kr^3}{3}-C\right)^2\right]^{1/2}Kr^2\left(\frac{Kr^3}{3}-C\right)}{\left(1-\frac{2m}{r}\right)\left(4r^3-6mr^2+2Kr^2\left[\frac{Kr^3}{3}-C\right]\right)^2}-\frac{16}{3}\frac{\left[r^4-2mr^3+\left(\frac{Kr^3}{3}-C\right)^2\right]^{3/2}K^2r^4}{\left(1-\frac{2m}{r}\right)\left(4r^3-6mr^2+2Kr^2\left[\frac{Kr^3}{3}-C\right]\right)^3}.\end{aligned} \tag{50}$$

One interesting property of d^2J_1/dC^2 is that it vanishes for large r. This means that it does not contribute to $d\tau/dC$. Note also that the first term in d^2J_1/dC^2 diverges as $r\to r_{\mathrm{ml}}$. However, we must remember that to compute α we multiply by N, which goes to zero in the matching fashion so that everything is regular. Further, only the first term is finite at the throat; all the other ones vanish.

Now we can work out

$$N\frac{dh}{dC}\bigg|_{r_{\mathrm{ml}}}=-2\frac{\left(\frac{Kr^3}{3}-C\right)^2}{r^2\left(1-\frac{2m}{r}\right)\left(4r^3-6mr^2+2Kr^2\left[\frac{Kr^3}{3}-C\right]\right)}\Bigg|_{r_{\mathrm{ml}}}. \tag{51}$$

From the definition of r_{ml} as the zero of the polynomial, Eq. (38), it is clear that

$$\left(\frac{Kr^3}{3}-C\right)^2_{r_{\mathrm{ml}}}=(2mr^3-r^4)_{r_{\mathrm{ml}}},\quad \left(\frac{Kr^3}{3}-C\right)_{r_{\mathrm{ml}}}=-\sqrt{(2mr^3-r^4)}_{r_{\mathrm{ml}}}. \tag{52}$$

From Eq. (21) we have that $K=(2R_*-3m)/\sqrt{2mR_*^3-R_*^4}$. When these are substituted into Eq. (51), we get

$$N\frac{dh}{dC}\bigg|_{r_{\rm ml}}=\frac{1}{2r-3m-(2R_*-3m)\sqrt{\frac{2mr^3-r^4}{2mR_*^3-R_*^4}}}\Bigg|_{r_{\rm ml}}. \tag{53}$$

A natural variable to use (as in [6]) is $\delta=r_{\rm ml}-R_*$. We then get

$$N\frac{dh}{dC}\bigg|_{r_{\rm ml}}\approx\frac{1}{\left[2+\frac{(3m-2R_*)^2}{2mR_*-R_*^2}\right]\delta}=\frac{2mR_*-R_*^2}{(2R_*^2-8mR_*+9m^2)\,\delta}. \tag{54}$$

The polynomial

$$D=4r^3-6mr^2+2Kr^2\left[\frac{Kr^3}{3}-C\right] \tag{55}$$

in Eq. (51) is the first derivative of the sextic polynomial of Eq. (38). In general it does not vanish at $r_{\rm ml}$. However, we can see that

$$D\approx 2R_*^2\left[2+\frac{(3m-2R_*)^2}{2mR_*-R_*^2}\right]\delta \tag{56}$$

and thus, as expected, goes to zero as $C\to C_*$.

Now we need to look at the integral part of J as this is what gives us $d\tau/dC$,

$$J_2=\frac{2}{3}\int\left[r^4-2mr^3+\left(\frac{Kr^3}{3}-C\right)^2\right]^{3/2}\frac{-3r^2+7mr-\frac{5K^2r^4}{6}-3m^2+\frac{4mK^2r^3}{3}+(Kr-mK)C}{\left[2r^3-7mr^2+\frac{K^2r^5}{3}+6m^2r-\frac{2mKr^4}{3}+(2mKr-Kr^2)C\right]^2}dr, \tag{57}$$

$$\begin{aligned}\frac{dJ_2}{dC}=&-2\int\left[r^4-2mr^3+\left(\frac{Kr^3}{3}-C\right)^2\right]^{1/2}\left(\frac{Kr^3}{3}-C\right)\frac{-3r^2+7mr-\frac{5K^2r^4}{6}-3m^2+\frac{4mK^2r^3}{3}+(Kr-mK)C}{\left[2r^3-7mr^2+\frac{Kr^5}{3}+6m^2r-\frac{2mKr^4}{3}+(2mKr-Kr^2)C\right]^2}dr\\&+\frac{2}{3}\int\left[r^4-2mr^3+\left(\frac{Kr^3}{3}-C\right)^2\right]^{3/2}\frac{(Kr-mK)}{\left[2r^3-7mr^2+\frac{K^2r^5}{3}+6m^2r-\frac{2mK^2r^4}{3}+(2mKr-Kr^2)C\right]^2}dr\\&-\frac{4}{3}\int\left[r^4-2mr^3+\left(\frac{Kr^3}{3}-C\right)^2\right]^{3/2}\frac{\left(-3r^2+7mr-\frac{5K^2r^4}{6}-3m^2+\frac{4mK^2r^3}{3}+(Kr-mK)C\right)(2mKr-Kr^2)}{\left[2r^3-7mr^2+\frac{K^2r^5}{3}+6m^2r-\frac{2mK^2r^4}{3}+(2mKr-Kr^2)C\right]^3}dr,\end{aligned} \tag{58}$$

$$\begin{aligned}\frac{d^2J_2}{dC^2}=\frac{d\tau}{dC}=&+2\int_{r_{\rm ml}}^{\infty}\left[r^4-2mr^3+\left(\frac{Kr^3}{3}-C\right)^2\right]^{-1/2}\left(\frac{Kr^3}{3}-C\right)^2\\&\times\frac{-3r^2+7mr-\frac{5K^2r^4}{6}-3m^2+\frac{4mK^2r^3}{3}+(Kr-mK)C}{\left[2r^3-7mr^2+\frac{Kr^5}{3}+6m^2r-\frac{2mKr^4}{3}+(2mKr-Kr^2)C\right]^2}dr+\text{eight other terms.}\end{aligned} \tag{59}$$

All the nine terms in Eq. (59) fall off like $1/r^3$. Therefore, each of these terms is finite. We also know that $d\tau/dC\to\infty$ as $C\to C_*$. The term we have isolated is the term which generates this behavior. All the other terms remain finite. To estimate the blowup, we need to understand the behavior of it near $r_{\rm ml}$. It is useful to shift the origin of coordinates to $r=r_{\rm crit}=R_*$. Therefore, we define $y=r-R_*$. We know $K=(2R_*-3m)/\sqrt{2mR_*^3-R_*^4}$ and we also write $C=C_*-\epsilon$, where $C_*=(3mR_*^3-R_*^4)/3\sqrt{2mR_*^3-R_*^4}$.

We now write out the polynomial Eq. (38) in terms of (R_*,m,y,z) to give

$$r^4\cdots=\frac{2R_*^3-8mR_*^2+9m^2R_*}{2m-R_*}y^2+\frac{12R_*^2-56R_*m+54m^2}{3(2m-R_*)}y^3+\frac{17R_*^2-54mR_*+45m^2}{3a(2m-R_*)}y^4$$
$$+\frac{4R_*^2-12R_*m+9m^2}{9R_*^3(2m-R_*)}[6R_*y^5+y^6]+\frac{2R_*^3-4R_*^2m}{\sqrt{2mR_*-R_*^2}}\epsilon+\epsilon^2+\frac{4R_*-6m}{3\sqrt{2mR_*-R_*^2}}[3R_*^2y+3R_*y^2+y^3]\epsilon. \tag{60}$$

The polynomial begins at y^2 because we know that when $\epsilon=0$, both the polynomial itself and its first derivative vanish at $y=0$. More generally, we know that the polynomial vanishes when $r=r_{\rm ml}$, i.e., when $y=r_{\rm ml}-R_*=\delta$. If ϵ is small, and if we are close to the critical value, then

$$\frac{2R_*^3-8mR_*^2+9m^2R_*}{2m-R_*}\delta^2+\frac{2R_*^3-4R_*^2m}{\sqrt{2mR_*-R_*^2}}\epsilon\approx 0\Rightarrow\epsilon\approx\frac{2R_*^4-8mR_*^3+9m^2R_*^2}{(4R_*^2m-2R_*^3)\sqrt{2mR_*-R_*^2}}\delta^2. \tag{61}$$

Further, near $r=r_{\rm ml}$, we find that the polynomial approximates

$$r^4\cdots=\frac{2R_*^3-8mR_*^2+9m^2R_*}{2m-R_*}(y^2-\delta^2). \tag{62}$$

The polynomial D of Eq. (55) is the first derivative of Eq. (60). Thus we have

$$D\approx 2\frac{2R_*^3-8mR_*^2+9m^2R_*}{2m-R_*}y+O(y^2) \tag{63}$$

and

$$\frac{dD}{dr}=2\frac{2R_*^3-8mR_*^2+9m^2R_*}{2m-R_*}+O(y). \tag{64}$$

We also need to approximate the other terms in Eq. (59). The denominator equals $(1-2m/r)^2D^2/4$. When we write this out in terms of (R_*,m,y), we get

$$2r^3\cdots\approx\left(1-\frac{2m}{r}\right)^2\frac{2R_*^3-8mR_*^2+9m^2R_*}{2m-R_*}^2y^2+O(y^3). \tag{65}$$

The polynomial in the numerator of Eq. (59) is

$$-\frac{1}{2}\frac{d}{dr}\left[\left(1-\frac{2m}{r}\right)D\right]. \tag{66}$$

Therefore,

$$-3r^2+\cdots=-\frac{1}{2}\left(1-\frac{2m}{r}\right)\frac{2R_*^3-8mR_*^2+9m^2R_*}{2m-R_*}+O(y). \tag{67}$$

Therefore, the term in Eq. (59) which blows up as $\delta\to 0$ is dominated by

$$\frac{d^2J_2}{dC^2}=\frac{d\tau}{dC}\approx+\int_\delta^\infty\frac{\left(\frac{Kr^3}{3}-C\right)^2}{\frac{2m}{r}-1}\left(\frac{2m-R_*}{2R_*^3-8mR_*^2+9m^2R_*}\right)^{3/2}\frac{1}{y^2\sqrt{y^2-\delta^2}}dy. \tag{68}$$

We know that

$$\frac{\left(\frac{Kr^3}{3}-C\right)^2}{\frac{2m}{r}-1}\approx R_*^4 \tag{69}$$

and

$$\int \frac{1}{y^2\sqrt{y^2-\delta^2}}dy=\frac{\sqrt{y^2-\delta^2}}{y\delta^2}. \tag{70}$$

Therefore, we have

$$\frac{d^2J_2}{dC^2}=\frac{d\tau}{dC}\approx\left(\frac{2m-R_*}{2R_*^3-8mR_*^2+9m^2R_*}\right)^{3/2}\frac{R_*^4}{\delta^2}. \tag{71}$$

VI. COLLAPSE OF THE LAPSE

We now have calculated the various terms necessary to compute the central lapse. From Eq. (54), we have

$$N\frac{dh}{dC}\bigg|_{r_{\mathrm{ml}}}\approx\frac{2mR_*-R_*^2}{(2R_*^2-8mR_*+9m^2)\delta}. \tag{72}$$

From Eq. (61), we have

$$\epsilon=C_*-C\approx\frac{2R_*^2-8mR_*+9m^2}{(4m-2R_*)\sqrt{2mR_*-R_*^2}}\delta^2. \tag{73}$$

We also have, from Eq. (71),

$$\frac{d\tau}{dC}\approx\left(\frac{2m-R_*}{2R_*^3-8mR_*^2+9m^2R_*}\right)^{3/2}\frac{R_*^4}{\delta^2}. \tag{74}$$

We can differentiate Eq. (73) to get

$$\frac{dC}{d\delta}\approx-\frac{2R_*^2-8mR_*+9m^2}{(2m-R_*)\sqrt{2ma-R_*^2}}\delta. \tag{75}$$

We multiply Eq. (74) by Eq. (75) to get

$$\frac{d\tau}{dC}\frac{dC}{d\delta}=\frac{d\tau}{d\delta}\approx-\left(\frac{R_*^4}{2R_*^2-8mR_*+9m^2}\right)^{1/2}\frac{1}{\delta}. \tag{76}$$

Integrating Eq. (76) gives

$$\tau=-\left(\frac{R_*^4}{2R_*^2-8mR_*+9m^2}\right)^{1/2}\ln\delta+A, \tag{77}$$

where A is a constant, or

$$\delta=\exp\left[-\left(\frac{2R_*^2-8mR_*+9m^2}{R_*^4}\right)^{1/2}(\tau-A)\right]. \tag{78}$$

From Eq. (45), we have

$$\alpha=\left(\frac{d\tau}{dC}\right)^{-1}N\frac{dh}{dC}\bigg|_r. \tag{79}$$

Therefore, to compute the central lapse we need to divide Eq. (72) by Eq. (74) to get

$$\alpha=\left(\frac{d\tau}{dC}\right)^{-1}N\frac{dh}{dC}\bigg|_r=\frac{2m-R_*}{2R_*^3-8mR_*^2+9m^2R_*}\left(\frac{2R_*^3-8mR_*^2+9m^2R_*}{2m-R_*}\right)^{3/2}\frac{\delta}{R_*^2}=\left(\frac{2R_*^2-8mR_*+9m^2}{2mR_*^3-R_*^4}\right)^{1/2}\delta. \tag{80}$$

On substituting in Eq. (78), we get

$$\alpha=B\exp\left[-\left(\frac{2R_*^2-8mR_*+9m^2}{R_*^4}\right)^{1/2}\tau\right], \tag{81}$$

where B is a constant which equals

$$B=\left(\frac{2R_*^2-8mR_*+9m^2}{2mR_*^3-R_*^4}\right)^{1/2}\times\exp\left[\left(\frac{2R_*^2-8mR_*+9m^2}{R_*^4}\right)^{1/2}A\right]. \tag{82}$$

We have not evaluated A, so we cannot compute B.

It is clear from Eq. (78) that A sets the zero of τ. When we compute the collapse of the lapse for the maximal case, the moment of time symmetry slice sets a natural zero for the time function. In the CMC case we discuss here, we cannot use the Killing time at infinity because it is infinite. As discussed in the beginning of Sec. V, we have to normalize it by setting the zero of time to be that of the umbilical slice, the slice which has $C=0$. In some sense, this is the analogue of the moment of time symmetry slice, but at the same time it is somewhat arbitrary. This arbitrariness will be reflected in the constant A. It is also quite difficult to compute. We would need to evaluate the integral (43) as we approach the critical slice. The leading term should agree with Eq. (77), but we also need to compute the next term, which will give us A.

This indicates a different way of computing the exponent. Let us look at h' as given by Eq. (35), or rather, let us look at $1/h'^2$ near the critical point,

$$\frac{1}{h'^2}\approx\frac{2R_*^2-8mR_*+9m^2}{R_*^4}(y^2-\delta^2). \tag{83}$$

When this is substituted into Eq. (43), we get

$$\tau(\delta)\approx\int_\delta\frac{R_*^2}{\sqrt{2R_*^2-8mR_*+9m^2}}\frac{dy}{\sqrt{y^2-\delta^2}}. \tag{84}$$

We have

$$\int_\delta\frac{dy}{\sqrt{y^2-\delta^2}}=\ln|y+\sqrt{y^2-\delta^2}|_\delta=-\ln\delta. \tag{85}$$

Therefore, we reproduce Eq. (77).

There are many ways of deriving the critical exponent. In what follows, we offer a very different derivation, based on an explicit formula for the lapse function for spherical CMC slices. The computation given here and the calculation there entirely agree.

The calculation here is closely modeled on the computation of the critical exponent for the maximal foliation given in [6]. This allows us to perform a number of internal consistency checks at various points in the calculation by reducing to the $K=0$ situation. We get agreement at each stage. This agreement is not trivial because the key integration by parts to obtain an explicitly finite derivative of J differs in the two cases.

VII. CMC FOLIATIONS AS DYNAMICAL SOLUTIONS

In this sequel, we get CMC foliations by solving Einstein equations in a particular gauge. A crucial role is played by a condition [Eq. (90) below] imposed on the lapse. While this method is completely equivalent to the preceding more geometric approach, it seems to be more straightforward and technically simpler. We focus on the concise derivation of the explicit CMC foliation near the critical point of the CMC foliation. The final result is identical to the result derived earlier.

The constant mean curvature foliations have been recently investigated numerically in the simulation of a single spherically symmetric black hole [11]. We hope that our analytic results appear helpful in the verification of the numerical schemes.

A. CMC slicing of the Schwarzschild spacetime

The notation is the same as in the preceding part. We define

$$(pR)^2=4\left[1-\frac{2m}{R}+\left(\frac{KR}{3}-\frac{C}{R^2}\right)^2\right], \tag{86}$$

$$\gamma(R,t)=1+8\partial_tC\int_R^\infty dr\frac{1}{r^5p^3} \tag{87}$$

and

$$N=\gamma\frac{pR}{2}. \tag{88}$$

Here m is the mass, K (the trace of the extrinsic curvature) is a constant, and C is a time-dependent parameter which measures the transverse part of the extrinsic curvature.

The Schwarzschild line element, expressed in terms of coordinates adapted to the constant mean curvature foliation, is given by [10]

$$ds^2 = -dt^2\left[N^2 - \gamma^2\left(\frac{KR}{3} - \frac{C}{R^2}\right)^2\right] + 4N\frac{\frac{C}{R^3} - \frac{K}{3}}{p^2 R}dtdR + \frac{4}{(pR)^2}dR^2 + R^2 d\Omega^2. \tag{89}$$

The hypersurfaces of constant time are CMC slices, asymptotic to the CMC slices of Minkowskian geometry.

B. Elliptic slicing condition

A minimal surface is a locus of points defined by the condition $p=0$. Choose a CMC Cauchy hypersurface Σ_C of the extended Schwarzschild manifold corresponding to a parameter C and let R_0 be an areal radius corresponding to a simple zero of p^2; that is, $p^2(R_0)=0$ but $\partial_R p^2|_{R_0}\neq 0$. Furthermore, assume that

$$\left.\frac{\partial_r N}{\sqrt{a}}\right|_{R_0} = 0 \tag{90}$$

at R_0. The condition (90) yields

$$\partial_t C = \frac{1}{8I(R_0)}. \tag{91}$$

Here

$$I(R_0) \equiv \int_{R_0} \frac{dr}{pr} \frac{6\frac{C^2}{r^4} + \frac{K^2 r^2}{3}}{\left(2m + \frac{2KC}{3} + \frac{2K^2 r^2}{9} - \frac{4C^2}{r^3}\right)^2}. \tag{92}$$

The value of the lapse function N at the minimal surface, that is, at the areal radius R_0, can be shown to be equal [using Eqs. (86)–(88)] to

$$N = \frac{dC}{dt} \frac{1}{m + \frac{KC}{3} + \frac{K^2 R_0^3}{9} - 2\frac{C^2}{R_0^3}}. \tag{93}$$

The lapse N is strictly positive at the minimal surface corresponding to a simple zero R_0. Equations (86)–(88) imply that $N(R) > N(R_0)$ if $R > R_0$ and therefore the lapse exists on all of Σ_C. Equation (91) dictates the rate of change of the parameter C. It is clear that one can uniquely construct a foliation of a part of the extended Schwarzschild geometry by imposing the condition (90) at minimal surfaces on all slices to the future of a given one. The leaves of the resulting foliation connect two null infinities of the extended Schwarzschild spacetime. This gives us a curve $R_0(t)$ of zeros of the mean curvature p. It is evident, just by inspecting the explicit solution presented above, that the line running along the locations of minimal surfaces $R_0(t)$ can be arranged to be smooth. It can be chosen to coincide with the "vertical" $t=0$ axis in standard Schwarzschild coordinates.

This construction breaks down when R_0 ceases to be a simple zero of p^2, since expressions appearing in Eqs. (91) and (93) become unboundedly large. The goal of this paper is to show the asymptotic behavior of the lapse at the critical minimal surface.

C. The evolution of *C* near critical point

Let C_* and R_* be degenerate, that is, such that the zero of p^2 ceases to be simple. In this case, both p and its derivative $\partial_R p$ vanish; that means that

$$1 - \frac{2m}{R_*} - \frac{2KC_*}{3R_*} + \frac{K^2 R_*^2}{9} + \frac{C^2}{R_*^4} = 0,$$

$$2m + \frac{2KC_*}{3} + \frac{2K^2 R_*^3}{9} - \frac{4C_*^2}{R_*^3} = 0. \tag{94}$$

One can easily show, if C_* and R_* are critical, that the sign of

$$\beta \equiv -2C_* + \frac{2}{3}KR_*^3 \tag{95}$$

is the same as the sign of $-C_*$.

There exist critical values of C_* that are positive (C_*^+) or negative (C_{*-}). For definiteness, we shall consider only the case when $C(t=0) > C_{*-}$, therefore the only limiting case we consider is that with $C \to C_*^+$. (That choice corresponds to a foliation formed by leaves connecting two null infinities which moves forward in time—see the discussion in Sec. IV.) For simplicity, we will drop the "+" suffix and C_* will mean a positive critical parameter. From the dynamical equation (91) it follows that C can only increase.

Next, let us introduce the notation

$$\epsilon \equiv C_* - C,$$

$$R_0 \equiv R_* + \delta, \tag{96}$$

where both δ and ϵ are positive and small.

The equation $p(R_0)=0$ yields a nonlinear algebraic equation whose truncation gives

$$\delta^2 A + \epsilon\beta = 0. \tag{97}$$

Here $A \equiv 2R_*^2 + K^2 R_*^4$. Equation (97) is in fact the Lyapunov-Schmidt reduced equation constructed according to the standard rules [12]. Therefore, in the vicinity of the critical point we have

$$\delta = \sqrt{\frac{-\beta\epsilon}{A}}. \tag{98}$$

The function p can be expressed in a form

$$\frac{pr}{2}=\sqrt{1-\frac{R_0}{r}}\left[\frac{\kappa\delta}{R_0}+\frac{K^2}{9}(rR_0+r^2-2R_0^2)\right.$$
$$\left.-\frac{C^2}{R_0^4}\left(\frac{R_0}{r}+\frac{R_0^2}{r^2}+\frac{R_0^3}{r^3}-3\right)\right]^{1/2}. \tag{99}$$

The insertion of Eqs. (96), (98), and (99) into Eq. (92) and the change of the integration variable to $y=\sqrt{1-(R_0/r)}$ yield after a simple but tedious algebra

$$I(R_0)\approx\sqrt{R_*}\int_0^1 dy\,\frac{F_1}{\sqrt{\kappa\delta+y^2F_2}(\kappa\delta+y^2F_3)^2}. \tag{100}$$

Here the functions F_1, F_2, and F_3 are given by

$$F_1(y)=\frac{K^2R_*^3}{3(1-y^2)^4}+\frac{6C_*^2}{R_*^3}(1-y^2)^2,$$

$$F_2(y)=\frac{K^2R_*^3(3-2y^2)}{9(1-y^2)^2}+\frac{C_*^2}{R_*^3}(6-4y^2+y^4),$$

$$F_3(y)=(3-3y^2+y^4)\left(\frac{2K^2R_*^3}{9(1-y^2)^3}+\frac{4C_*^2}{R_*^3}\right), \tag{101}$$

while κ reads

$$\kappa=\frac{2K^2R_*^2}{3}+\frac{12C_*^2}{R_*^4}. \tag{102}$$

D. Limiting behavior of the foliation

The asymptotic behavior of C will be dominated by the $1/\delta^2$ part of $I(R_0)$. As will be shown later, C tends exponentially to C_*; the attenuation factor in the exponent depends only on the leading term of $I(R_0)$. It is useful to define $z=y/\sqrt{\kappa\delta}$. Then one obtains $I(R_0)=(\sqrt{R_*}/\kappa^2\delta^2)\times I_d$, where

$$I_d\equiv\int_0^{1/\sqrt{\kappa\delta}}dz\,\frac{F_1(\sqrt{\kappa\delta}z)}{\sqrt{1+z^2F_2(\sqrt{\kappa\delta}z)}(1+z^2F_3(\sqrt{\kappa\delta}z))^2}. \tag{103}$$

One can split the integral $\int_0^{1/\sqrt{\kappa\delta}}$ into two parts: $\int_0^{1/\sqrt{\kappa\delta}}=\int_0^{1/\sqrt{10^4\kappa\delta}}+\int_{1/\sqrt{10^4\kappa\delta}}^{1/\sqrt{\kappa\delta}}$. It is easy to check that the contribution coming from the second integral goes to zero as δ approaches zero. Therefore, $F_1\approx R_*\kappa/2$, $F_2\approx R_*\kappa/2$, and $F_3\approx R_*\kappa$. Thus the first integral (and also the integral I_d) is well approximated by

$$I=\frac{\kappa R_*}{2}\int_0^\infty dz\,\frac{1}{\sqrt{1+\frac{R_*\kappa}{2}z^2}(1+R_*\kappa z^2)^2}$$
$$=\frac{\sqrt{\kappa R_*}}{2}\int_0^\infty dz\,\frac{1}{\sqrt{1+\frac{z^2}{2}}(1+z^2)^2}. \tag{104}$$

The integral

$$I_z=\int_0^\infty dz\,\frac{1}{\sqrt{1+\frac{z^2}{2}}(1+z^2)^2}$$

can be explicitly evaluated and gives $\sqrt{2}/2$.

In summary, near the critical point we have

$$I(R_0)=\frac{\sqrt{R_*}}{\kappa^2\delta^2}I\approx\frac{\sqrt{2}}{4\epsilon}\frac{R_*A}{\kappa^{3/2}|\beta|}. \tag{105}$$

The insertion of Eqs. (96) and (105) into Eq. (91) yields

$$\partial_t\epsilon=-\Gamma\epsilon, \tag{106}$$

where

$$\Gamma=\frac{|\beta|\kappa^{3/2}}{2\sqrt{2}AR_*}. \tag{107}$$

Equation (106) immediately implies that ϵ approaches 0 exponentially as

$$\epsilon(t)=\epsilon_0e^{-t\Gamma}, \tag{108}$$

where ϵ_0 is an initial value of the parameter. Taking into account relations (96) and (98), one can conclude that the parameter C and the minimal radius R_0 tend exponentially to their critical values C_* and R_*, respectively.

Finally, collecting the above information and putting it into Eq. (93), we obtain the asymptotic behavior of the lapse function near the critical point,

$$N=N_0e^{-t\Gamma/2}. \tag{109}$$

This is exactly the same result as that obtained in Sec. VI; in order to show equivalence, use expressions for the extrinsic curvatures (valid in the upper quadrant of the extended Schwarzschild geometry), which imply $K=(2R_*-3m)/\sqrt{2mR_*^3-R_*^4}$ and $C_*=(3mR_*^3-R_*^4)/\sqrt{2R_*^3-R_*^4}$. In the case of maximal slicing ($K=0$), the decay constant $\Gamma/2$ is equal to $4/(3\sqrt{6})$, in agreement with the analytic derivation of [6] and close to the numerical result of [13]. The asymptotic behavior of γ and p in a region close to the line R_0 is given by

$$\gamma = \gamma_0 \frac{e^{-t\Gamma/4}}{\sqrt{1-\frac{R_0}{r}}},$$

$$\frac{pr}{2} = p_0 \sqrt{1-\frac{R_0}{r}} e^{-t\Gamma/4}. \tag{110}$$

Here γ_0 and p_0 are initial values of $\gamma\sqrt{1-(R_*/r)}$ and $pr/[2\sqrt{1-(R_*/r)}]$, respectively. The four constants (ϵ_0, N_0, γ_0, and p_0) can be expressed in terms of one free parameter (say, ϵ_0) and A,β,κ. Equations (88), (100), (108)–(110), and $C=C_*-\epsilon_0 e^{-t\Gamma}$ suffice to construct the metric (89) near the line $R_0(t)$ of minimal surfaces.

VIII. FOLIATIONS WITH NONCONSTANT *K*?

As is clear from the discussion, one has three parameters to play with in constructing spherical CMC foliations or slicings in the Schwarzschild solution. One can change *K*, one can change *C*, and one can drag a slice along the ''timelike'' Killing vector. Of course, one can change more than one of these at once. The foliation we have focused on is one where we kept *K* fixed, changed *C*, and eliminated the Killing freedom by considering the slices where the minimal surface coincided with the $t=0$ axis in standard coordinates.

We could consider any one of these slices and drag it along the Killing vector. For the slices which run from null infinity to null infinity, one with $|C|$ small, one would get a slicing which looks somewhat like the standard $t=$const slicing of the Schwarzschild solution. It would rise up along one null infinity and sink down on the other while the throat ran along one of the $R=$const lines in either the upper or lower quadrant. These would not form a foliation. Each slice crosses each other in the interior, and the lapse function will vanish on the throat and be positive on one side and negative on the other. If we drag one of the slices which runs from null infinity to the singularity (one with $|C|$ large) along the Killing vector, we get a foliation. The slices do not cross, but each one ends on the $R=0$ singularity. Such a foliation would cover the upper half of the right-hand quadrant and all of the upper quadrant if one picked $K>0$ and $C>0$. Other patches could be covered by choosing other options for *K* and *C*.

In closed cosmologies, on the other hand, we are used to CMC foliations where the value of *K* changes. If the cosmology goes from a big bang to a big crunch, we might expect to have a foliation which goes from $K=+\infty$ at the big bang through $K=0$, the moment of maximum expansion, and monotonically to $K=-\infty$, the big crunch. It can be shown that no such foliations with varying *K* exist in the Schwarzschild solution.

A first try would be to consider the slicing where one changes *K* while keeping *C* fixed. Such slices always cross each other. Consider two slices, one with $K=0, C=0$ (this is the standard moment-of-time-symmetry, $t=0$, slice through the Schwarzschild solution) and the other with $K=1, C=0$. As we discussed previously, this slice starts at future null infinity, crosses in the middle below the bifurcation point, and rises up again to the other future null infinity. This slice crosses the first slice twice. Choosing a different, fixed value of *C* will not change this behavior. Therefore, if we want a foliation with varying *K*, we need a nonconstant *C*.

One almost has such a foliation. Consider the slicing where one changes *K* while simultaneously changing *C* such that $C=8m^3K/3$. Each of these slices has its throat at $R=2m$. Each of these slices runs through the bifurcation point and so they must all touch there. They do not cross, however, and this is their only point of contact. This slicing covers all of the left and right quadrants. One might hope that by slightly changing *C*, one could spread the slices apart along the vertical $t=0$ axis and thus convert this slicing into a foliation. This cannot be done, as we show below, if we want to allow *K* to be unboundedly large.

The standard work on the way CMC slices act as barriers was written by Brill and Flaherty [14]. Among other results, they show that two slices with the same value of *K* cannot touch at a single point. Further, if two CMC slices do touch at one point, the slice to the future must have a larger value of *K* than the other one. This restriction strongly restricts the behavior of CMC slices in the Schwarzschild solution.

Let us assume that we have a CMC foliation of a Schwarzschild solution which starts off at the moment of time symmetry slice and moves up. Consider one slice of this foliation. This slice will have positive $K=K_S$ and $C>8m^3K/3$. This slice will have a throat with some radius R_S. The cylindrical slice with this given radius is also a CMC slice (with $K=K_1$, say) and touches the other CMC slice at one point, the throat. Therefore, we must have that $K_S<K_1$. Since we assume we have a foliation, we must have that R_S monotonically decreases and hence also K_1. It passes through zero at $R_S=3m/2$. However, we expect K_S to be increasing as the foliation moves forward in time so we will eventually run into a situation where $K_S=K_1$, which Brill and Flaherty forbid. Therefore, any spherical CMC foliation of the Schwarzschild solution cannot have unboundedly large values of *K*.

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