# A Distributed Asynchronous Solver for Nash Equilibria in Hypergraphical Games [1]

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**Abstract.** Hypergraphical games provides a compact model of a network of self-interested agents, each involved in simultaneous subgames with its neighbors. The overall aim is for the agents in the network to reach a Nash Equilibrium, in which no agent has an incentive to change their response, but without revealing all their private information. Asymmetric Distributed constraint satisfaction (ADisCSP) has been proposed as a solution to this search problem. In this paper, we propose a new model of hypergraphical games as an ADisCSP based on a new global constraint, and a new asynchronous algorithm for solving ADisCSP that is able to find a Nash Equilibrium. We show empirically that we significantly reduce both message passing and computation time, achieving an order of magnitude improvement in messaging and in non-concurrent computation time on dense problems compared to state-of-the art algorithms.

## 1 Introduction

In many multi-agent problems, agents must interact with each other to achieve a global goal while maximising their own individual preferences. The *hypergraphical games* model [13] provides a compact representation of the problem, in which agent interactions are represented as normal-form strategic subgames, and the relationship topology between the agents is represented as a hypergraph. Each agent has a set of strategies, and a utility function specifying the agent's payoff under each possible combination of its own and neighboring agents' strategies in each subgame. A solution to a hypergraphical game is the selection of a strategy for each agent, such that the network is in equilibrium. Typically, the aim is to find a Nash Equilibrium (NE), in which no agent can improve its payoff by changing its strategy. To model more realistic problems, $\epsilon$-approximate Nash Equilibria ($\epsilon$-NE) are considered, in which no agent can improve its payoff by more than some minimum threshold $\epsilon$.

Given the multi-agent setting, algorithms to compute solutions should be distributed, to avoid the need for agents to reveal potentially private information. Early work focused on identifying graph topologies which allowed polynomial-time solutions, based on algorithms for Bayesian Network inference. More recent approaches focus on arbitrary graphs with cyclic dependencies, and represent the problem as *asymmetric distributed constraint satisfaction* (ADisCSP), maintaining the individual utility functions as extensional table constraints [5]. ADisCSP allows agents to keep their strategic information private while optimizing their local utility, and allows the system to stabilize at an equilibrium by coordinating agents' decisions. However, for dense graphical games with large strategy sets, the encoding of the table constraints ([5]) becomes expensive in the number or size of messages that need to be exchanged.

Our contributions in this paper are as follows. We develop new approaches to finding approximate Nash Equilibria for hypergraphical games using the distributed constraint satisfaction framework. We develop a new model of a hypergraphical game as ADisCSP using a new global constraint, $\epsilon\text{-BRConstraint}$, to represent an agent's requirement to find an approximate best strategy given the other decisions in its neighborhood. We introduce asymmetric asynchronous backtracking, AABT, a new algorithm for solving ADisCSP with global constraints using intelligent backtracking to avoid thrashing. AABT is then used to solve the problem of finding an $\epsilon$-NE in hypergraphical games formulated as an ADisCSP. We compare the new model and algorithm empirically to previous state-of-the-art algorithms, and we show that we achieve significant reductions in non-concurrent computation time and an order of magnitude improvement in message passing.

The paper is organized as follows. Section 2 gives a brief overview of related works on game theory and distributed CSP. Section 3 introduces the necessary background, basic notation and terminology. We present our model of the problem of finding $\epsilon$-NE in hypergraphical games as ADisCSP in Section 4. Section 5 introduces our new algorithm, AABT, for solving ADisCSP with global constraints, and we show our empirical results in Section 6.

## 2 Related Work

[8] introduced *graphical games*, a compact representation of $n$-player normal-form games and proposed NashTree, a dynamic programming algorithm for computing Nash equilibria in graphical games for which the underlying graph is a tree. NashTree consists of two phases: a table passing phase from the leaves to the root and an assignment passing phase from the root to the leaves. [15] proposed a constraint satisfaction generalization of NashTree for general graphical games using variable elimination. They transform the graphical game into a tree via triangulation, and then subsequently run NashTree algorithm on the resulting junction tree.

[12] introduced the NashProp algorithm, another generalization of NashTree for general graphical games based on belief propagation requiring no triangulation. In the table passing phase, NashProp proceeds in a series of rounds to maintain (generalized) arc consistency in the constraints network. In each round, every node will send a different binary-valued table to each of its neighbors in the graph. Each table represents the value combination of the sender and the receiver

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that the sender believes could be in an $\epsilon$-NE. In the assignment passing phase, NashProp performs a synchronous search to find $\epsilon$-NE [5, Section 5.1].

The pioneering algorithm for symmetric DisCSP was *asynchronous backtracking* (ABT) [18, 2]. ABT is an asynchronous algorithm executed autonomously by each agent, and is guaranteed to converge to a global consistent solution (or detect inconsistency) in finite time. [3] proposed two varieties of ABT for solving ADisCSP, namely ABT-2ph and ABT-1ph. ABT-2ph alternates the execution of ABT considering constraints in one direction following a total ordering on agents until the problem is solved or inconsistency is proved. ABT-1ph checks constraints asynchronously in both directions, by agents sending their proposed assignments to all their neighbors. In ABT-1ph, an agent only changes its assignment when it is inconsistent with a higher neighbor assignment. When it is inconsistent with a lower neighbor assignment, the conflict is reported to the lower neighbor to change its assignment. However, both algorithms were restricted to solving problems with binary constraints.

Grubshtein and Meisels proposed in [5] a model of graphical games as an ADisCSP with a unique private (global) table constraint for each agent. The table constraint contains all tuples (joint strategies) of the neighbors that satisfy the $\epsilon$-NE condition. They also proposed *asynchronous Nash backtracking* (ANT), the first asynchronous algorithm for solving ADisCSP with global constraints capable of finding $\epsilon$-NE. ANT is an extension of ABT-1ph that handles asymmetric global (non-binary) constraints. ANT achieves orders of magnitude improvements over NashProp. However, ANT only uses the global constraints as checkers. In addition its handling of global constraints produces chronological backtracks (thrashing), as all value assignments for agents in the constraint are considered, even if they are not the cause of the conflict. Recent developments in DisCSP have shown how to make more effective use of global constraints, exploiting their pruning power, and backjumping closer to the point of conflict [1, 16].

## 3 Preliminaries

In this section, we introduce some basic notation and terminology for game theory, and describe the framework of hypergraphical games before presenting the distributed constraint satisfaction formalism.

### 3.1 Game Theory

An $n$-player game in *normal-form* is a tuple $(\mathcal{P}, \{S_i, U_i\}_{p_i \in \mathcal{P}})$ where $\mathcal{P} = \{p_1, \ldots, p_n\}$ is a set of $n$ players (agents). For each agent $p_i \in \mathcal{P}$, $S_i$ is a finite set of actions or pure strategies, and $U_i : S \to \mathbb{R}$ where $S = \overset{n}{\underset{i=1}{\times}} S_i$, is a local utility hypermatrix/function that specifies the payoff for player $p_i$ under each strategy profile $s \in S$. The joint strategic choice of all agents other than agent $p_i$ is denoted by $s_{-i}$, and $s = (s_i, s_{-i})$. $U_i(s)$ is the utility that player $p_i$ receives when, for all players $p_j \in \mathcal{P}$, $p_j$ plays $s_j \in s$. The neighbors of agent $p_i$ are all other agents in the normal-form game, i.e., $\mathcal{N}_i = \mathcal{P} \setminus p_i$. In the rest of the paper, we assume all agents have the same number of strategies, i.e., $\forall p_i \in \mathcal{P}, |S_i| = d$. The representation size of each local utility hypermatrix is exponential in the number of players, i.e., $O(d^n)$. Motivated by scenarios where an agent's utility is directly dependent on only a subset of the total number of agents, researchers devoted a considerable effort to develop compact representations following the graphical game model [8].

A *hypergraph* is a pair $(V, E)$ where $V$ is a set of vertices, and $E$ is a set of non-empty subsets of $V$ called hyperedges. [13] introduced
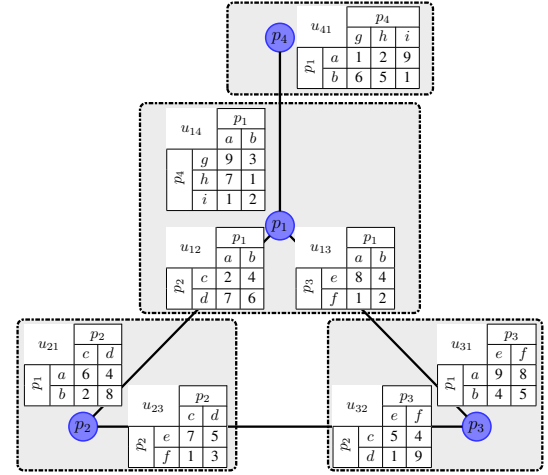


(a) Graphical polymatrix games
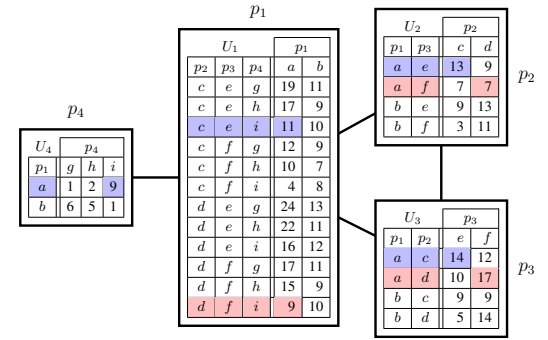


(b) Graphical game representation

**Figure 1.** An example of a hypergraphical game.

*hypergraphical games*, in which each agent $p_i$ is involved in simultaneous (local) normal-form subgames, $\mathcal{G}_i$. A hypergraphical game is described by a hypergraph $(\mathcal{P}, E)$ where each hyperedge $h \in E$ represents an explicit subgame involving the players in $h \subseteq \mathcal{P}$. The strategy set of each agent $p_i$ is the same in all subgames in $\mathcal{G}_i$. The payoff function of $p_i$ is the sum of all $p_i$'s payoffs in all $\mathcal{G}_i$. The neighbors of agent $p_i$, $\mathcal{N}_i$, is the union of its neighbors in all subgames in $\mathcal{G}_i$. The degree of agent $p_i$ is denoted by $\kappa_i = |\mathcal{N}_i|$. The utility of an agent is directly dependent on its neighbors, i.e., $U_i : S_i \times \underset{p_j \in \mathcal{N}_i}{\times} S_j \to \mathbb{R}$. From now on, $s_{-i}$ is the joint strategy of $\mathcal{N}_i$.

Hypergraphical games is a generalization of *graphical games* [8] where each agent is involved in exactly one subgame. In a graphical game, the representation size of $p_i$' utility is exponential in the degree of the agent $O(d^{\kappa_i})$. Hypergraphical games is also a generalization of *graphical polymatrix games* where each agent is involved in simultaneous 2-player games [6]. The representation size of $p_i$'s utilities is $O(\kappa_i \cdot d^2)$.

In graphical polymatrix games, we can represent the utility function of agent $p_i$, $U_i$, by a set of (binary) utility functions $u_{ij}$ for each $p_j \in \mathcal{N}_i$. The utility function $u_{ij}(s_i, s_j)$ represents the gain in utility of agent $p_i$ when playing strategy $s_i \in S_i$ and agent $p_j$ plays strategy $s_j \in S_j$, and $p_i$'s overall utility function becomes:

$$U_i(s) = U_i(s_{-i}, s_i) = \sum_{p_j \in \mathcal{N}_i, s_j \in s} u_{ij}(s_i, s_j) \qquad (1)$$

For a strategy profile $s$, the *regret* $\delta_i(s)$ of an agent $p_i$ is the highest additional reward $p_i$ could have gained by changing its strategy,

assuming its neighbors' strategy choices remain the same:

$$\delta_i(s) = \max_{s'_i \in S_i} \{U_i(s_{-i}, s'_i) - U_i(s)\} \qquad (2)$$

In strategic games, each rational agent would ideally play its best strategy given a fixed joint strategy of other agents. A configuration in which all agents selected a best strategy is called a *Nash Equilibrium* (NE) . Specifically, a NE is a strategy profile $s$ where each player's regret is 0 (i.e., $\forall p_i \in \mathcal{P}, \delta_i(s) = 0$). Given a joint strategy of other players $s_{-i}$, the *best response* of agent $p_i$, $BR_i(s_{-i})$, is the strategies which produces the maximal gain for agent $p_i$ (i.e., $BR_i(s_{-i}) = \{s_i | \delta_i(s_{-i}, s_i) = 0\}$).

An *approximate Nash Equilibrium* ($\epsilon$-NE) represents scenarios where agents are satisfied with choices whose payoff is sufficiently close to the maximum. Formally, a strategy profile $s$ is an $\epsilon$-NE if each player's regret is at most $\epsilon$, i.e., $\forall p_i \in \mathcal{P}, \delta_i(s) \leq \epsilon$. Given a joint strategy of other players $s_{-i}$, the approximate best response set of agent $p_i$ is defined as: $\epsilon\text{-}BR_i(s_{-i}) = \{s_i | \delta_i(s_{-i}, s_i) \leq \epsilon\}$.

Figure 1 shows a simple hypergraphical game with 4 agents: $p_1$, $p_2$, $p_3$ and $p_4$ having the following strategy sets $S_1 = \{a, b\}$, $S_2 = \{c, d\}$, $S_3 = \{e, f\}$ and $S_4 = \{g, h, i\}$. Figure 1(a) shows the original representation in graphical polymatrix games and Figure 1(b) shows a representation of the same instance in graphical games. Agent $p_1$ is involved in three 2-player subgames with $p_2$, $p_3$ and $p_4$ represented respectively by the utilities $u_{12}$, $u_{13}$, and $u_{14}$. Agent $p_2$ is involved in two subgames with $p_1$ ($u_{21}$) and $p_3$ ($u_{23}$). Agent $p_3$ is involved in two subgames with $p_1$ ($u_{31}$) and $p_2$ ($u_{32}$). Agent $p_4$ is involved in one 2-player subgame with $p_1$ represented by utility $u_{41}$. This problem has one NE $[p_1 = a, p_2 = c, p_3 = e, p_4 = i]$ and one $\epsilon$-NE in addition to the NE where $\epsilon = 3$, that is, $[p_1 = a, p_2 = d, p_3 = f, p_4 = i]$.

## 3.2 CSP & Asymmetric Distributed CSP

The *constraint satisfaction problem* (CSP) is a triple $(\mathcal{X}, \mathcal{D}, \mathcal{C})$, where $\mathcal{X} = \{x_1, \ldots, x_n\}$ is a set of $n$ variables, $\mathcal{D} = \{D(x_1), \ldots, D(x_n)\}$ is a set of domains, where $D(x_i)$ is a finite set of values from which one value must be assigned to variable $x_i$, and $\mathcal{C}$ is a set of constraints. A constraint $c_k(X) \in \mathcal{C}$, on the ordered subset of variables $X = (x_{j_1}, \ldots, x_{j_k}) \subseteq \mathcal{X}$, is $c_k(X) \subseteq D(x_{j_1}) \times \cdots \times D(x_{j_k})$, and specifies the tuples of values which may be assigned simultaneously to the variables in $X$. $c_k(X)$ can be represented extensionally or intensionally. $|X|$ is the *arity* of $c_k(X)$, and $X$ is its *scope*. A global constraint is defined on a set of variables and thus an instance of the constraint may have arbitrary arity. A *solution* is an assignment to each variable of a value from its domain, satisfying all the constraints.

*Asymmetric distributed CSP* (ADisCSP) [3] models problems where variables and constraints are held by distinct agents. ADisCSP is a 4-tuple $(\mathcal{A}, \mathcal{X}, \mathcal{D}, \mathcal{C})$, where $\mathcal{X}, \mathcal{D}$ and $\mathcal{C}$ are as above, and $\mathcal{A} = \{a_1, \ldots, a_m\}$ is a set of $m$ agents. Each variable $x_i \in \mathcal{X}$ is controlled by a single agent in $\mathcal{A}$. During a solution process, only the agent which controls a variable can assign a value to this variable. In an ADisCSP, constraints are private and only the agent, $a_i$, that holds a constraint knows it while other agents involved in that constraint are only aware that $a_i$ constrains their variable without knowing the nature of that constraint or its scope. We denote by $\mathcal{C}_i \subseteq \mathcal{C}$ all constraints held by $a_i$. As in CSP, a *solution* to an ADisCSP is an assignment to each variable of a value from its domain, satisfying all the constraints. For simplicity and without loss of generality, we assume each agent controls exactly one variable and use the two terms interchangeably (i.e., $m = n$).

## 4 Hypergraphical Games as ADisCSP

We now present a model of the problem of finding $\epsilon$-NE in hypergraphical games as ADisCSP, where agents can control a local CSP that allows them to maintain the approximate best responses.

The straightforward modeling of a hypergraphical game $(\mathcal{P}, \{S_i, U_i\}_{p_i \in \mathcal{P}})$ into an ADisCSP is to represent each player $p_i \in \mathcal{P}$ by an agent $a_i \in \mathcal{A}$ having a single local variable $x_i$ that can take its value from the strategy set of player $p_i$, i.e., $D(x_i) = S_i$. In the following we use the terms agent, player and variable interchangeably, (i.e., $a_k = p_k = x_k$). In addition, we consider a generic agent $a_i$. $\mathcal{X}_i = \{x_i, x_j | x_j \in \mathcal{N}_i\}$ denotes the variables (or copies) maintained by $a_i$. Agent $a_i$ encodes the problem of finding an $\epsilon$-NE (finding an $\epsilon$-approximate best strategy) by a single constraint $c_i(\mathcal{X}_i)$ requiring that the regret, Eq. (2), is less than or equals $\epsilon$, i.e., $c_i(\mathcal{X}_i)$: $\delta_i(s) \leq \epsilon$ where $s = (s_i, s_{-i})$ is a joint strategy (the assignments) of agents in $\mathcal{X}_i$. Thus, $\mathcal{C} = \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ where $\mathcal{C}_i = \{c_i(\mathcal{X}_i)\}$. The agent $a_i$ that holds the constraint $c_i(\mathcal{X}_i)$ is the only agent that knows it, and must ensure it is satisfied, given the joint assignment of all its neighbors. To evaluate this constraint agent $a_i$ needs to maintain local copies of its neighbors' variables in $\mathcal{X}_i$. Finding $\epsilon$-NE distributively in a hypergraphical game is then equivalent to solving the ADisCSP model above.

**Proposition 1.** *Let $\mathcal{M}$ be an ADisCSP model of a hypergraphical game $\mathcal{H}$. A solution of $\mathcal{M}$ is an $\epsilon$-NE of $\mathcal{H}$.*

*Proof.* (Sketch) A solution to $\mathcal{M}$ is an assignment to each variable of a value from its domain, satisfying all the constraints. Thus, each player is assigned a strategy and every agent $a_i$'s private constraint, $c_i(\mathcal{X}_i)$, is satisfied. Satisfying each constraint $c_i(\mathcal{X}_i)$ means that each agent regret is less than $\epsilon$ ($\delta_i(s) \leq \epsilon$). Thus, each agent assignment is an approximate best response to its neighbors assignments. Hence, a solution of $\mathcal{M}$ is an $\epsilon$-NE of $\mathcal{H}$. $\square$

If agents are allowed mixed strategies (i.e. a probability distribution over deterministic strategies), then the discretization scheme proposed in [8] guarantees that an $\epsilon$-NE always exists. Thus, the ADisCSP model of the hypergraphical game is always satisfiable. For more details about mixed strategies and the existence of an $\epsilon$-NE, we refer the reader to [8, 12, 11, 5]. If the agents are restricted to pure strategies, it is possible that no $\epsilon$-NE exists. In such circumstances, the ADisCSP model would have no solution, and any algorithm should report failure.

In the following we propose $\epsilon$-BRConstraint, a new incremental global constraint to ensure an agent's value is an approximate best response, that is efficient in memory and allows efficient propagation. The $\epsilon$-BRConstraint enforces $\delta_i(s) \leq \epsilon$ for each agent $a_i$ (i.e., it is an implementation of $c_i(\mathcal{X}_i)$). For simplicity, we restrict our attention to graphical polymatrix games. However, our constraint and algorithm will represent any hypergraphical game including graphical games and graphical polymatrix games. Our experiments are against the state-of-the-art distributed algorithm for graphical games, and we experiment with the same class of problems that that algorithm was tested on. These problems happen to be graphical polymatrix games.

## The Constraint $\epsilon$-BRConstraint

When the system is at an $\epsilon$-NE, each agent has assigned to its variable an approximate best strategy with respect to its neighbors assignments. Thus, at each step in the distributed algorithm, an agent $a_i$ should prune all strategies that are dominated by others. In agent

$a_i$, $\epsilon-\texttt{BRConstraint}$ filters the domain of $x_i$ to prune dominated strategies with respect to the (subset of) decisions of other agents in $\mathcal{N}_i$. $\epsilon-\texttt{BRConstraint}(\epsilon, x_i, \{x_j\}_{x_j \in \mathcal{N}_i}, \{u_{ij}\}_{x_j \in \mathcal{N}_i})$ takes as parameters all variables in $\mathcal{X}_i$ and utility functions $u_{ij}$ in addition to $\epsilon$.[3] Before presenting the constraint behavior we first consider the auxiliary variables used inside $\epsilon-\texttt{BRConstraint}$ to filter dominated strategies. In addition to $x_i$ and copies of neighbors' variables $x_j \in \mathcal{N}_i$, $a_i$ has in $\mathcal{X}_i$ the following integer variables:

- $v_j[s_i]$ for each strategy $s_i \in D(x_i)$, and each utility $u_{ij}$. The variable $v_j[s_i]$ represents the gain in utility of agent $a_i$ when playing strategy $s_i$ with respect to $x_j$ possible strategies. This variable ranges over the possible utilities in $u_{ij}$ when $a_i$ plays $s_i$, i.e., $D(v_j[s_i]) = \{u_{ij}(s_i, s_j) | s_j \in D(x_j)\}$.

- $y[s_i]$ for each strategy $s_i \in D(x_i)$. The variable $y[s_i]$ is used to maintain the minimum reward gained when $a_i$ chooses to play strategy $s_i$.

$$y[s_i] = \sum_{x_j \in \mathcal{N}_i} \min\{D(v_j[s_i])\} \qquad (3)$$

- $z[s_i]$ for each strategy $s_i \in D(x_i)$. The variable $z[s_i]$ is used to maintain the maximum reward gained when $a_i$ chooses to play strategy $s_i$.

$$z[s_i] = \sum_{x_j \in \mathcal{N}_i} \max\{D(v_j[s_i])\} \qquad (4)$$

For each $s_i \in D(x_i)$, we need to represent the relation between the values of the variables $x_i$, $x_j$ and $v_j[s_i]$, and maintain $v_j[s_i] = u_{ij}(s_i, s_j)$. As soon as utility $v_j[s_i]$ is updated, some now inconsistent values of $x_j$ can be removed from consideration. Similarly, when $D(x_j)$ is updated (e.g., a value is removed from $D(x_j)$), the utility variable $v_j[s_i]$ can be updated correspondingly. Each time a domain of $x_j$ or $v_j[s_i]$ is changed, the domain of the other variable is updated to keep only values having a support on other variables, i.e., we need to ensure that $\exists s_j \in D(x_j) \land r \in D(v_j[s_i])$, $r = u_{ij}(s_i, s_j)$.[4]

We need also to make the correspondence between the values of $v_j[s_i]$ and those of $y[s_i]$ and $z[s_i]$ for each $s_i \in D(x_i)$ by ensuring Eq. (3) and Eq. (4). We maintain a support for $y[s_i]$ (resp. $z[s_i]$) on variable $v_j[s_i]$ and we only update that support and its corresponding reward on $y[s_i]$ (resp. $z[s_i]$) when the lower bound (resp. the upper bound) of the domain of $v_j[s_i]$ has changed.

By maintaining the above variables and properties on their domain changes, $\epsilon-\texttt{BRConstraint}$ is able to detect and remove from $D(x_i)$ the dominated strategies using the $y[s_i]$ and $z[s_i]$ variables. We say that a strategy $s_i \in D(x_i)$ is *dominated* if the largest utility that $a_i$ can gain when choosing strategy $s_i$ (i.e., $z[s_i]$) is lower than the minimal reward that $a_i$ can gain by choosing another strategy $s_i'$ where $s_i' = \arg \max_{s_i \in D(x_i)} \{y[s_i]\}$ taking $\epsilon$ in consideration. Specifically, a strategy $s_i \in D(x_i)$ is dominated iff:

$$z[s_i] < \max_{s_i' \in D(x_i)} \{y[s_i']\} - \epsilon \qquad (5)$$

Removing dominated strategies $s_i$ from $D(x_i)$ is safe because $a_i$ will never play $s_i$, Eq. (5). Filtering in $\epsilon-\texttt{BRConstraint}$ is based on pruning all dominated strategies each time they are detected with changes on domains of variables in $\mathcal{X}_i$ without needing their full joint strategy (assignments).

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[3] In the general hypergraphical games, $u_{ij}$ will be replaced by payoffs matrix of each subgame.

[4] Maintaining $v_j[s_i]$ values follows the same scheme as the `element` constraint [14]. However, in the general case the index variable is a combination of indexes of all neighborhood values in the subgame.



| $x_2$ | $x_3$ | $x_4$ | $x_1$ |
|-------|-------|-------|-------|
| $c$ | $e$ | $g$ | $b$ |
| $c$ | $e$ | $h$ | $b$ |
| $c$ | $e$ | $i$ | $b$ |
| $c$ | $f$ | $g$ | $b$ |
| $c$ | $f$ | $h$ | $b$ |
| $c$ | $f$ | $i$ | $a$ |
| $d$ | $e$ | $g$ | $b$ |
| $d$ | $e$ | $h$ | $b$ |
| $d$ | $e$ | $i$ | $b$ |
| $d$ | $f$ | $g$ | $b$ |
| $d$ | $f$ | $h$ | $b$ |
| $d$ | $f$ | $i$ | $a$ |

$D(v_2[a]) = \{2, 7\}$
$D(v_2[b]) = \{4, 6\}$   $D(y[a]) = \{4..24\}$
$D(v_3[a]) = \{1, 8\}$   $D(y[b]) = \{7..13\}$
$D(v_3[b]) = \{2, 4\}$   $D(z[a]) = \{4..24\}$
$D(v_4[a]) = \{1, 7, 9\}$   $D(z[b]) = \{7..13\}$
$D(v_4[b]) = \{1, 2, 3\}$

$v_2[a] \quad = u_{12}(a, s_2) \quad = [2, 7]$
$v_2[b] \quad = u_{12}(b, s_2) \quad = [4, 6]$
$v_3[a] \quad = u_{13}(a, s_3) \quad = [8, 1]$
$v_3[b] \quad = u_{13}(b, s_3) \quad = [4, 2]$
$v_4[a] \quad = u_{14}(a, s_4) \quad = [9, 7, 1]$
$v_4[b] \quad = u_{14}(b, s_4) \quad = [3, 1, 2]$

(a) Table constraint encoding $c_1(\mathcal{X}_1)$ used in [5].

(b) $\epsilon-\texttt{BRConstraint}$ for $c_1(\mathcal{X}_1)$.

**Figure 2.** The encoding of ANT and AABT in $a_1$ of the constraint $c_1(\mathcal{X}_1)$ in the problem shown in Figure 1.

## 4.1  Memory requirements

In a hypergraphical game, in [5], $c_i(\mathcal{X}_i)$ is represented in extensional form using a table constraint containing all tuples (joint strategy) $s \in D(x_i) \times_{x_j \in \mathcal{N}_i} D(x_j)$ that satisfies $c_i(\mathcal{X}_i)$. A tuple $s$ satisfies $c_i(\mathcal{X}_i)$ if it yields a maximal gain to agent $a_i$, i.e., $\delta_i(s) \leq \epsilon$. Thus, the utility functions of all subgames of $a_i$ are encoded and represented by a large table constraint requiring a memory size of $O(d^{\kappa_i + 1})$. The table constraint encoding the constraint $c_1(\mathcal{X}_1)$ of agent $a_i$ of the example presented in Figure 1 is shown in Figure 2(a). All tuples satisfying $c_1(\mathcal{X}_1)$ are represented in that table.

In our new model, $\epsilon-\texttt{BRConstraint}$ maintains $a_i$'s utility in each subgame in $\mathcal{G}_i$. Thus, the representation size in $\epsilon-\texttt{BRConstraint}$ is similar to that required by the hypergraphical game in each agent, i.e., $O(|\mathcal{G}_i| \cdot d^{\gamma + 1})$ where $\gamma$ is largest neighborhood in subgames $\mathcal{G}_i$ and $|\mathcal{G}_i|$ is the number of subgames in $\mathcal{G}_i$. In polymatrix games, this representation is polyspace $O(\kappa_i \cdot d^2)$. The encoding of the constraint $c_1(\mathcal{X}_1)$ of agent $a_i$ of the example presented in Figure 1 is shown in Figure 2(b).

In ANT the handling of the global constraint does not exploit its filtering power. In addition, when a dead-end occurs, a no-good is produced from all neighbors' assignments and then sent to the lowest neighbor in the ordering. This produces chronological backtracks (thrashing). In the following, we propose a new algorithm for solving ADisCSP with global constraints, that exploits the pruning power of global constraints and produces no-goods closer to the (real) point of conflict.

## 5  Asymmetric Asynchronous BackTracking

*Asymmetric asynchronous backtracking* (AABT) is an asynchronous algorithm for solving ADisCSP that allows agents to keep their constraints private. In AABT agents operate asynchronously, but are subject to a known total priority order. AABT combines a distributed search procedure with a failure learning mechanism to perform intelligent backtracking. Intelligent backtracking techniques usually store an explanation for each value removal. Such explanations are computed on-the-fly on each domain reduction.

In AABT, each agent $a_i$ tries to solve its local $CSP_i$ defined by $\mathcal{X}_i$, their domains, and $\mathcal{C}_i$. Solving a $CSP_i$ is achieved by interleaving search with propagation. The search can be regarded as the dy-

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**Algorithm 1:** AABT algorithm running by agent $a_i$.

**procedure** AABT()
1. $E_i^+ \leftarrow \{a_j \mid a_i \in \mathcal{N}_j\}$; $\ end \leftarrow$ **false**; $\ x_i \leftarrow nil$ ;
2. assignVariable() ;
3. **while** ( $\neg end$ ) **do**
4. $\quad msg \leftarrow$ getMsg() ;
5. $\quad$ **switch** ( $msg.type$ ) **do**
6. $\qquad$ ***ok?*** : processOk ($msg.var$, $msg.tag$) ;
7. $\qquad$ ***ngd*** : processNogood ($msg.sender$, $msg.ngd$) ;
8. $\qquad$ ***adl*** : processAddLink ($msg.sender$) ;
9. $\qquad$ ***stp*** : $end \leftarrow$ **true**;

**procedure** assignVariable()
10. **while** ( $x_i = nil \wedge \neg end$ ) **do**
11. $\quad$ propagate ($x_i = D(x_i).peek()$) ;
12. $\quad$ **if** ( $\exists\, x_k \in \mathcal{X}_i \mid D(x_k) = \emptyset$ ) **then**
13. $\qquad$ repair() ; // An empty domain has been found
14. $\quad$ **else**
15. $\qquad$ $t_i \leftarrow t_i{+}1$ ;
16. $\qquad$ sendMsg: ***ok?***$\langle x_i = s_i, t_i \rangle$ **to** $E_i^+$ ;

**procedure** propagate ($x_k = s_k'$)
17. remove ($expl(x_l {\neq} s_l)$) **s.t.** $x_k = s_k \in expl(x_l {\neq} s_l)$ ;
18. $D(x_k) \leftarrow \{s_k'\}$ **s.t.** $expl(x_k {\neq} s_k) \leftarrow x_k = s_k'$;
19. $\mathcal{C}_i$.propagate() ;

**procedure** processOk ($x_j'$, $t_j'$)
20. **if** ( $t_j' \geq t_j$ ) **then**
21. $\quad$ propagate ($x_j = x_j'$) ;
22. $\quad$ **if** ( $\exists\, x_k \in \mathcal{X}_i \mid D(x_k) = \emptyset$ ) **then** repair() ;
23. $\quad$ assignVariable() ;

**procedure** processNogood ($a_j$, $ng$)
24. **if** ( $\forall x_l \in \{ng \cap \mathcal{X}_i\}, ng[x_l] = \mathcal{X}_i[x_l]$ ) **then**
25. $\quad$ $Links \leftarrow \{ng \cup \mathcal{X}_i\} \setminus \mathcal{X}_i$ ;
26. $\quad$ $\mathcal{X}_i \leftarrow \mathcal{X}_i \cup Links$ ;
27. $\quad$ sendMsg: ***adl***$\langle\rangle$ **to** $Links$ ;
28. $\quad$ learn ($ng$) ;
29. **else if** ( $ng[x_i] = x_i$ ) **then**
30. $\quad$ sendMsg: ***ok?***$\langle x_i = s_i, t_i \rangle$ **to** $a_j$;

**procedure** repair()
31. $ng \leftarrow \bigwedge\limits_{s_k \in D(x_k)} expl(x_k {\neq} s_k)$ ; $\qquad$ /* $D(x_k) = \emptyset$ */
32. **if** ( $ng = \emptyset$ ) **then**
33. $\quad$ sendMsg: ***stp***$\langle\rangle$ **to** $\mathcal{A} \setminus a_i$ ;
34. $\quad$ $end \leftarrow$ **true**;
35. **else** learn ($ng$) ;

**procedure** learn ($ng$)
36. Let $x_t$ be the lowest variable in $ng$ ;
37. **if** ( $x_t \neq x_i$ ) **then** sendMsg: ***ngd***$\langle ng \rangle$ **to** $a_t$ ;
38. $expl(x_t {\neq} s_t) \leftarrow \{ng \setminus x_t = s_t\}$ ;
39. remove ($expl(x_l {\neq} s_l)$) **s.t.** $x_t = s_t \in expl(x_l {\neq} s_l)$ ;
40. $\mathcal{C}_i$.propagate() ;
41. **if** ( $\exists\, x_k \in \mathcal{X}_i \mid D(x_k) = \emptyset$ ) **then** repair() ;
42. assignVariable() ;

**procedure** processAddLink ($a_j$)
43. $E_i^+ \leftarrow E_i^+ \cup a_j$ ;
44. sendMsg: ***ok?***$\langle x_i = s_i, t_i \rangle$ **to** $a_j$ ;

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namic addition of constraints (decision constraints) to $\mathcal{C}_i$ and retractions (backtracks) [7]. For simplicity, we restrict ourselves to decision constraints that are of the form $x_k = s_k$, i.e., assignments of values to variables.[5] Each assignment is followed by the propagation of $\mathcal{C}_i$ with respect to $\mathcal{X}_i$'s domains. This propagation may result in a value removal ($x_k {\neq} s_k$) for a variable $x_k \in \mathcal{X}_i$. We define an *explanation*, $expl(x_k {\neq} s_k)$, of that value removal (i.e., $x_k {\neq} s_k$) by the set of decision constraints $x_j = s_j, \ldots, x_l = s_l$ (assignments), such that $(x_j = s_j \wedge \ldots \wedge x_l = s_l \wedge x_k = s_k)$ is globally inconsistent. During search, a failure (a domain wipeout) may occur leading to *no-goods* computations. A no-good can be regarded as a subset of the assignments made so far that caused a failure (i.e., $\bigwedge\limits_{s_k \in D(x_k)} expl(x_k {\neq} s_k)$ where $D(x_k) = \emptyset$).

AABT agents exchange the following message types:

**ok?**: used to notify its recipients of a new assignment associated with the current counter $t_i$ of agent $a_i$, used to discard obsolete assignments.

**ngd**: used to report a no-good to another agent, requesting the removal of its value.

**adl**: used to request the additional of a link to the receiver.

The pseudo-code of AABT executed by each agent $a_i$ is presented in Algorithm 1. In the following, $s_i$ will represent the current value assigned to $x_i$ and $t_i$ the counter tagging $s_i$ used for the timestamp mechanism. "$x_i = nil$" means that $x_i$ is unassigned. After initialization, each agent assigns a value and informs all agents in $E_i^+$ of its decision (assignVariable call, line 2) by sending them **ok?** messages. $E_i^+$ are agents having a constraint involving $a_i$'s variable (line 1). Then, a loop considers the reception of the possible message types. If no message is traveling through the network, the state of quiescence is reached meaning that a global solution is found. The solution is given by the current variables' assignments. The quiescence state can be detected by a specialized algorithm [4].

When an agent $a_i$ receives an **ok?** message from $a_j$ it calls procedure processOk (line 6). If the received assignment has a larger counter than that received beforehand, it is accepted, otherwise it is discarded (line 20). If the assignment is accepted, $a_i$ calls procedure propagate to propagate $\mathcal{C}_i$ after adding the newly received assignment as a decision constraint $x_j = s_j'$, line 21. If the propagation results in a domain wipeout procedure repair is called to resolve the conflict, line 22. Finally, assignVariable is called (line 23) to assign a new consistent value to $x_i$ if its value has been pruned by the propagation of $\mathcal{C}_i$.

When calling procedure propagate ($x_k = s_k'$), the value of $x_k$ is set to $s_k'$. Next, all explanations not relevant to this new assignment, i.e., $expl(x_k {\neq} s_k)$ containing $x_k = s_k$ where $s_k' {\neq} s_k$, are removed (line 17).[6] Then, the domain of $x_k$ is reduced to a singleton $D(x_k) = \{s_k'\}$ with $x_k = s_k'$ as explanation, line 18. Finally, the constraints in $\mathcal{C}_i$ that might be affected by the domains' changes above are propagated, line 19.

When every value ($s_k$) of a variable $x_k \in \mathcal{X}_i$ is ruled out by an explanation $expl(x_k {\neq} s_k)$, the procedure repair is called to resolve the conflict (lines 13, 22 and 41). The conflict is resolved by computing a new no-good $ng$ from the conjunction of these explanations, i.e., $expl(x_k {\neq} s_k)$, line 31. If the new no-good $ng$ is empty, $a_i$ terminates execution after sending a **stp** message to all agents in the system meaning that the problem is unsolvable (line 32). Otherwise,

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[5] Agent $a_i$ can not take decision for other agents variables $x_j$, i.e., $i {\neq} j$.
[6] The removal of some explanations $expl(x_k {\neq} s_k)$ does not imply the restoration of $s_k$ to $D(x_k)$ unless the propagation of constraints involving $x_k$ allows that.

it calls procedure `learn(ng)`, line 35. Let $x_t$ be the variable having the lowest priority in $ng$. If $x_t$ is different than $x_i$, $ng$ is reported to $a_t$ through a **ngd** message, lines 36 and 37. In AABT, the backtracking target $x_t$ is always that having the lowest priority in $ng$. Agent $x_t$ can be a higher (as in ABT) but also a lower priority agent. Thus, we guarantee that the conflict is always reported to the agent with lowest priority in the conflict. Next, a new explanation from $ng$ is used to justify the removal of the value of $x_t$, i.e., $expl(x_t \neq s_t) \leftarrow \{ng \setminus x_t = s_t\}$, line 38. All explanations containing the assignment of $x_t$ ($x_t = s_t$) are removed because they are not valid anymore, line 39. The constraints in $\mathcal{C}_i$ are then propagated to check the new changes and if a failure occurs again `repair` is called, line 41. Finally, `assignVariable` is called to check if $x_i$ needs to be assigned (line 42).

When a **ngd** message is received (line 7), $a_i$ checks the validity of the received no-good (procedure `processNogood` call). If the assignments of the received no-good are consistent with those stored locally, this no-good is valid (line 24). Then, agent $a_i$ sends a link request to non linked agents having variables in $ng$ (lines 25 to 27). Next, $a_i$ calls procedure `learn(ng)`, line 28. If the $ng$ is not valid but is consistent with the current assignment of $x_i$, $a_i$ sends an **ok?** message to the generator of $ng$, lines 29 and 30.

When receiving an **adl** message (lines 43 and 44) $a_i$ adds a new link to the request sender ($a_j$) and sends an **ok?** message to inform $a_j$ of its assignment.

## 5.1 Privacy

In AABT, to use the power of the filtering algorithms of global constraints, each agent maintains a copy of its neighborhood domains. However, agents in AABT only require to know the initial domain of their neighbors and do not need their actual domain. For the case of the hypergraphical game as ADisCSP, an agent can construct the domain of a neighbor based on the payoff matrices of local sub-games. At each stage of AABT, the privacy of agents' current domains is preserved. In AABT, agents exchange their values with their neighbors. Thus, in AABT there is no privacy of assignments. In AABT, agents do not have to share their constraints/scopes to solve the problem. Thus, AABT preserves privacy of constraints and privacy of agents topologies. However, during the solving process some information is exchanged (value assignments and no-goods) between agents, thus leaking information about agents' constraints.

A method to evaluate the constraint privacy using entropy as a quantitative measure for privacy loss has been proposed in [10, 3]. This method measures the percentage of the conflicts in a table constraint held by an agent that are revealed to another agent involved in a conflict induced from **ok?** and **ngd** messages. However, this method only applies to ADisCSP with binary table constraints. Thus, this method can not be used to evaluate the privacy loss in AABT. We believe that studying the privacy loss in distributed algorithms for solving DisCSP with global constraints (e.g., ANT and AABT) is an open research area that needs to be investigated.

## 5.2 Theoretical Analysis

Here we prove that AABT is sound, complete and terminates.

**Theorem 1.** *AABT terminates.*

*Proof.* AABT stops its execution in two cases: when an empty no-good has been generated meaning that the problem is unsolvable or when the network reaches a quiescent state reporting a solution. To prove that AABT terminates, we need to prove that AABT reaches one of these two cases in finite time, i.e., agents can never fall into an infinite loop cycling among their possible values. In the following, we prove by induction on the agent ordering that agents can never fall into an infinite loop. Let assume a lexicographic ordering on agents. The base case for induction ($i = 1$) is obvious. Unlike ABT, $a_1$ can receive **ok?** messages from its neighbors (having a lower priority). When receiving these **ok?** messages $a_1$ may generate new no-goods as results of propagating its constraints. Agent $a_1$ may generate three categories of no-goods. The first are empty no-goods. This category stops the algorithm execution. The second are no-goods containing a lower neighbor. Those no-goods are transmitted to the lowest agent involved in. The third category are singleton no-goods containing a value of $a_1$. Agent $a_1$ may also receive **ngd** messages from lower priority agents. Now, all no-goods contained in **ngd** messages $a_1$ receives are singleton because in AABT the generated no-good is always sent to the agent having the lowest priority in it (lines 36 and 37). Hence, when agent $a_1$ proposes a possible value, it will not change it unless it receives or itself produces singleton no-goods ruling out this value. Values in singleton no-goods are removed once and for all from the domain of $a_1$. Because its domain is finite, $a_1$ cannot fall into an infinite loop.

Now, assume that agents higher that agent $a_i$ ($i > 2$), i.e., $a_1$ to $a_{i-1}$ ($i > 2$), are in a stable state, i.e., they are all assigned values to their variables and do not change their values. In the following, we show that agent $a_i$ never falls into an infinite loop. After processing **ok?** and **ngd** messages it receives, agent $a_i$ may generate contradictions (no-goods) as results of propagating its local constraints. The categories of no-goods agent $a_i$ may generate are: (i) empty no-goods, (ii) no-goods containing lower neighbors, (iii) no-goods containing the agents $a_1$ to $a_{i-1}$, and (iv) no-goods containing the agents $a_1$ to $a_i$. No-goods of (i) causes the algorithm to stop, those of (ii) are forwarded to a lower agent and can not cause $a_i$ to fall in infinite loop. No-goods of (iii) breaks our assumption because they are forwarded to a higher agent that we assumed to be in a stable state, causing it to change its value. Thus, only (iv) may affect the termination of $a_i$. Agent $a_i$ may also receive **ngd** messages from other agents (higher or lower). However, all no-goods contained in **ngd** messages $a_i$ receives contain only the agents $a_1$ to $a_i$, i.e., category (iv), because in AABT no-goods are always sent to the lowest agent involved in. Since agents $a_1$ to $a_{i-1}$ are in a stable state, these no-goods are valid for $a_i$ (the assignments on these no-goods are consistent with those stored locally).[7] Thus, $a_i$ will remove its value and will not assign it again while at least one of agents $a_1$ to $a_{i-1}$ does not change its value. Because its domain is finite, $a_i$ will either eventually change its assignment with a different value or exhaust the possible values and send a no-good to a higher agent, i.e., one of $a_1 \ldots a_{i-1}$. This no-good will cause an agent that we assumed to be in a stable state, not to be in a stable state. Hence, agent $a_i$ can never fall into an infinite loop for a given stable state of $a_1$ to $a_{i-1}$. By induction we have that agents can never fall into an infinite loop and AABT is thus guaranteed to terminate. $\square$

A global solution is reported when the network has reached quiescence, meaning that all agents are idle and no message is transmitting through the network. To prove that AABT is sound, we should first establish the two following lemmas about when the network reaches the quiescent state $\sigma$.

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[7] Note here that we discuss only no-goods consistent with $a_1$ to $a_i$, because inconsistent ones are discarded.
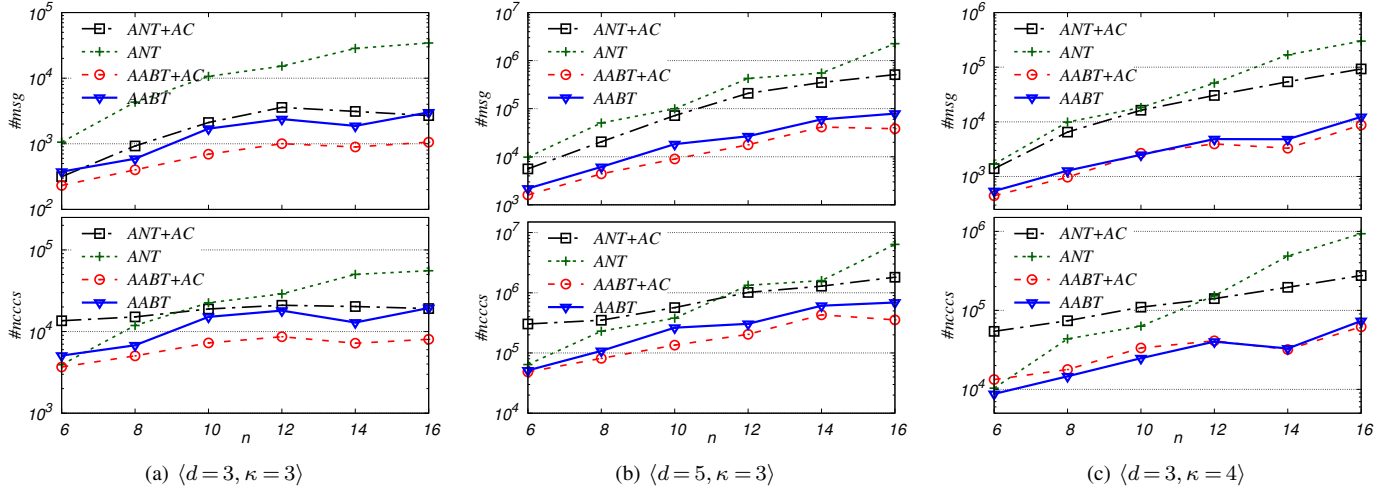
**Figure 3.** The $\#msg$ and $\#ncccs$ for solving random regular graphical polymatrix games when varying the number of agents in logarithmic scale.

**Lemma 1.** *When reaching a quiescent state $\sigma$, every agent is assigned, and its assignment is known by all its neighbors.*

*Proof.* (Sketch) Assume the system reaches a quiescent state $\sigma$. According to Algorithm 1, each time an agent $a_i$ is unassigned, it immediately calls procedure `assignVariable` (lines 23 and 42). Before exiting its loop and being idle allowing the system to reach $\sigma$, `assignVariable` guarantees that either (i) an empty domain has been found when calling `repair` (line 13) meaning that the problem is unsolvable or (ii) a value is assigned to $x_i$. (i) contradicts our assumption. Thus, when reaching $\sigma$ every agent $a_i$ is assigned. Now, let $v_i$ be the value assigned to $x_i$ when reaching $\sigma$. Each neighbor of $a_i$, say $a_j$, should receive an **ok?** message from $a_i$ containing its decision, i. e., $x_i = v_i$ (line 16). The only case where agent $a_j$ removes $x_i = v_i$ is when it sends a no-good message to $a_i$. We can easily see that this message is either not obsolete, in which case $a_i$ will change its value $v_i$ and breaks our assumption, or obsolete, which means that an **ok?** message has not yet reached $a_j$ which breaks our quiescence assumption. Hence, when reaching $\sigma$ all agents know the assignments of all their neighbors and no two agents store different assignments for the same variable. □

**Lemma 2.** *When reaching $\sigma$, all constraints are satisfied.*

*Proof.* In AABT, each time a new value is assigned or a new message is received by an agent $a_i$, it immediately propagates its constraints $\mathcal{C}_i$. When generating an empty domain, $a_i$ calls procedure `repair` to repair inconsistent assignments. Hence, after processing each message all assignments stored locally satisfy all constraints of the agent. □

**Theorem 2.** *AABT is sound.*

*Proof.* Direct from Lemmas 1 and 2. □

**Theorem 3.** *AABT is complete.*

*Proof.* In AABT, agents only store valid explanations and no-goods. In addition, all explanations and no-goods are generated by logical inferences from existing constraints. Thus, the empty no-good cannot be inferred if the network is satisfiable. □

**Corollary 1.** *AABT is a correct and complete solver for hypergraphical games.*

*Proof.* From Proposition 1 and Theorems 1 to 3. □

## 6 Empirical Study

In this section we experimentally compare AABT to ANT [5], AABT+AC and ANT+AC [5]. In AABT+AC and ANT+AC arc consistency (AC) is performed in a preprocessing phase before running AABT and ANT. This processing phase is the table passing phase of NashProp [12]. We do not compare the algorithms to NashProp because in [5] it has been shown that ANT achieves orders of magnitude improvements over NashProp. All the problems used for evaluating both algorithms were generated randomly and then modified to ensure that at least one pure strategy Nash equilibrium exists. All experiments were performed on the DisChoco 2.0 platform [17], in which agents are simulated by Java threads that communicate only through message passing. We evaluate the performance of the algorithms by communication load and computation effort. Communication load is measured by the total number of exchanged messages among agents during algorithm execution ($\#msg$) [9]. Computation effort is measured by the number of non-concurrent constraint checks ($\#ncccs$) [19]. $\#ncccs$ is the metric used in distributed constraint solving to simulate the computation time. Algorithms are evaluated on two random benchmarks: random regular graphical polymatrix games, and random graphical polymatrix games.

**Random regular graphical polymatrix games:** are characterized by $\langle n, d, \kappa \rangle$, where $n$ is the number of players/agents, $d$ is the number of strategies per agent, and $\kappa$ represents the degree of each agent. For each value combination a utility was uniformly selected in $\{0, \ldots, 9\}$. We solved and report the average over 25 instances of four settings. The first and the fourth settings cover the experiments presented in [5]. In the three first settings, we fixed the number of strategies and the degree of each agent and varied the number of agents in the range 6..16 by a step of 2 to guarantee the graphs are $\kappa$-regular. In the first setting we generated 3 random connections for each agent ($\kappa = 3$) and fixed the number of strategies of each agent at $d = 3$. We then increase the number of strategies to $d = 5$ in the second setting and increase the degree of each agent to $\kappa = 4$ in the third setting.

Results are presented in Figure 3. Regarding the number of required messages to solve the problem ($\#msg$), AABT improves ANT by an order of magnitude in almost all instances of Figure 3.
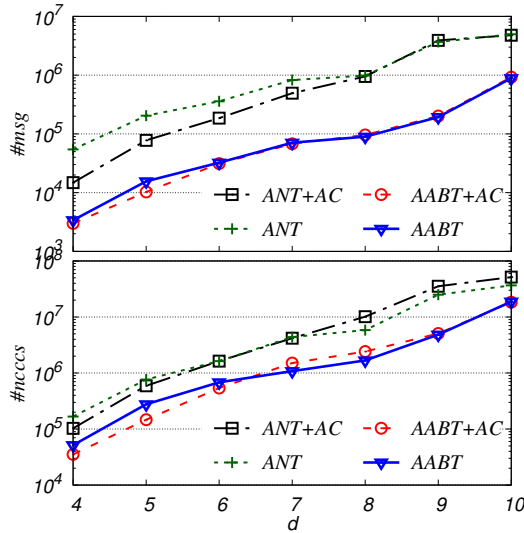
**Figure 4.** The $\#msg$ and $\#ncccs$ for solving random regular graphical polymatrix games when varying the number of strategies in logarithmic scale.



**Figure 5.** The $\#ncccs$ performed and $\#msg$ exchanged for solving problems where $\langle n = 10, d = 5, \rho \rangle$ in logarithmic scale.

In the first setting (Figure 3(a)), the improvement factor ranges between 3 for small problems and 15 on larger ones. In the second setting (Figure 3(b)), the factor of improvement ranges between 5 for small problems and 29 on larger ones. In the third setting (Figure 3(c)), the factor of improvement ranges between 3 for small problems and 35 on larger ones. For $\#ncccs$, AABT outperforms ANT, although the improvement is less significant than in $\#msg$. In small problems the two algorithms perform similarly. However, in larger problems, the factor of improvement ranges from 4 in the first setting ($d = 3, \kappa = 3$) to 9 in the second ($d = 5, \kappa = 3$) and 15 in the third ($d = 5, \kappa = 4$). Regarding the AC, ANT+AC (resp. AABT+AC) always improves ANT (resp. AABT) in $\#msg$. For $\#ncccs$, AABT+AC always improves AABT, however, ANT+AC only improves ANT on problems with larger number of agents ($n > 10$). AABT always improves ANT+AC and the improvement is more significant (an order of magnitude improvement) in larger problems of the second and the third setting when increasing the number of strategies or the agents degrees. AABT+AC is clearly the best algorithm for solving all instances in the three settings.

In the fourth setting (Figure 4) we generated random 3-regular graphical polymatrix games ($\kappa = 3$) where we fixed the number of agents at $n = 10$ and varied the number of strategies of all agents in the range 4..10. Again, the results show that AABT outperforms ANT and ANT+AC in all instances. Regarding $\#ncccs$, the improvement factor ranges between 3 to 7. For $\#msg$, in instances with larger number of strategies the AABT improvement over ANT and ANT+AC is again an order of magnitude. AABT and AABT+AC algorithms perform almost similarly in all instances. ANT+AC shows small improvement over ANT on $\#msg$ for problems with smaller number of strategies.

**Random graphical polymatrix games:** are characterized by $\langle n, d, \rho \rangle$, where $n$ is the number of players/agents, $d$ is the number of strategies per player, $\rho$ is the network connectivity defined as the ratio of existing binary utility functions. For each value combination a utility was uniformly selected from the set $\{0, \ldots, 9\}$. For each $\rho \in \{0.4, \ldots, 0.9\}$, we generated 25 instances in the class $\langle n = 10, d = 5, \rho \rangle$. We report the average over these instances in Figure 5. The results demonstrate that AABT improves ANT algorithm
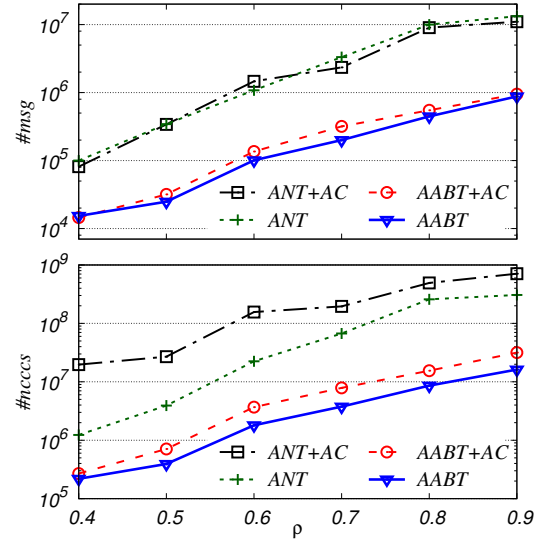
in both $\#msg$ and $\#ncccs$. This improvement is over an order of magnitude on dense problems, $\rho \geq 6$. In contrast to $\kappa$-regular games, the improvement on $\#ncccs$ is more significant than on $\#msg$. Specifically, the improvement factor on $\#ncccs$ when $\rho = .8$ is 30 while it is 22 regarding the $\#msg$. Regarding AC, our results confirm those of [5] that if pruning is limited, AC can create a significant overhead.

**Summary:** In all our experimentation, AABT always improves ANT and ANT+AC in both metrics $\#msg$ and $\#ncccs$. This improvement is more significant on harder problems (when increasing the number of players/agents and/or the number of strategies of each player and/or the degree of each agent). In sparse problems of $\kappa$-regular graph $\kappa = 3$, AABT improves the $\#ncccs$ slightly compared to ANT.[8] However, this improvement is more significant on dense graphical polymatrix games. AABT+AC only improves AABT on sparse problems. AC is harmful for hard problems because it does not lead to domain filtering when problems are dense and/or large with a large number of strategies.

## 7 Conclusion

We studied the problem of finding an approximate Nash Equilibrium in hypergraphical games, an elegant framework for modeling collaboration in multi-agent systems within strategic environments. Our study is based on asymmetric distributed constraint satisfaction, an efficient tool for distributed problem solving allowing agents to keep their utilities private. We proposed a new model of hypergraphical games as an asymmetric DisCSP based on $\epsilon-\texttt{BRConstraint}$, a new global constraint modeling hypergraphical games using the original compact representation of the subgames with a filtering algorithm of dominated strategies. Finally we introduced a new asynchronous algorithm for solving (asymmetric) DisCSP, in order to find Nash Equilibria for hypergraphical games. This achieves an order of magnitude improvement in messaging and in non-concurrent computation time on dense problems compared to state-of-the art algorithms.

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[8]    In our implementation, table constraints are presented in lexicographic ordering of tuples and we use a dichotomic search ($\log(\kappa_i + 1)$) to check the consistency of each tuple.

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