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Nonhydrostatic internal waves in the presence of mean currents and rotation

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ABSTRACT

In this paper we present a new exact solution that represents a Pollard-like, three-dimensional, nonlinear internal wave propagating on a non-uniform zonal current in a nonhydrostatic ocean model. The solution is presented in Lagrangian coordinates, and in the process we derive a dispersion relation for the internal wave which is subjected to a perturbative analysis which reveals the existence of two distinct modes of wave motion.

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I. INTRODUCTION

In this paper we present an exact, nonlinear, three-dimensional Pollard-like internal wave solution to the governing equations for geophysical flows. The constructed solution models internal ocean waves in the presence of a non-uniform zonal current. The ocean model being considered is a more sophisticated model than the one recently presented in Ref. 41 which employed the simplified assumption of particles having no motion immediately below the thermocline, and we now incorporate a series of transitional layers beneath the thermocline.

In 1970 Pollard[42] extended the two dimensional explicit solution presented in 1802 by Gerstner,[17] to include the effects of the Earth's rotation, thereby constructing a nonlinear, three-dimensional solution to the governing equations of geophysical ocean flows. The planetary rotation present in Pollard's solution introduces a cross-wave tilt to the particle trajectories. The magnitude of this tilt is proportional to displacement from the Equator, thus Pollard-like solutions are more suitable for describing non-Equatorial flows. The solution is given in the f-plane approximation and therefore is valid for arbitrary but approximately fixed latitudes, excluding the poles. Instability analyses have been carried out for both Gerstner[6,27] and Pollard-like[26] waves, while mass-transport properties of Pollard waves are investigated in Ref. 32.

A number of recent papers have used such explicit solutions to model a multitude of ocean flows near to the Equator,[3–5,19,21,28,33,39] as well as for modeling geophysical internal wave solutions[4,29,31,34,40,45] and azimuthally propagating flows.[11,23,35,36] The solution presented here is that of an internal wave propagating zonally in the presence of a non-uniform zonal current. Therefore, the azimuthally propagating[12,18,37,38,43] Antarctic Circumpolar Current (ACC) is a relevant physical example possessing many of the important features for which this paper is relevant. Moreover, a variety of wave-current interactions have been modeled in Refs. 7–10, 14, 19, 20, 22, and 44.

The aim of this paper is to extend the model used in Ref. 41 to include transitional, nonhydrostatic layers beneath the thermocline as in Ref. 30, thus constructing a new internal wave solution. A perturbation analysis of the resulting wave solution shows that the inertial wave modes found in Ref. 41 are retained, however the dynamics of the particle trajectories are altered by the incorporation of the transitional layers beneath the thermocline. We note that a recent paper[46] has appeared subsequent to Ref. 41 which presents a similar nonhydrostatic model, however this work exhibits a number of fundamental errors in its derivation of the related dispersion relation, and does not provide a perturbative analysis revealing the resulting wave solutions.

II. DESCRIPTION OF THE MODEL

The model that is to be investigated is that of a stably stratified ocean on a rotating Earth. In a stable stratification, there are two layers of constant but differing densities $\rho_+ > \rho_0$, separated by a thermocline. The thermocline oscillates in the shape of a trochoid, and the internal

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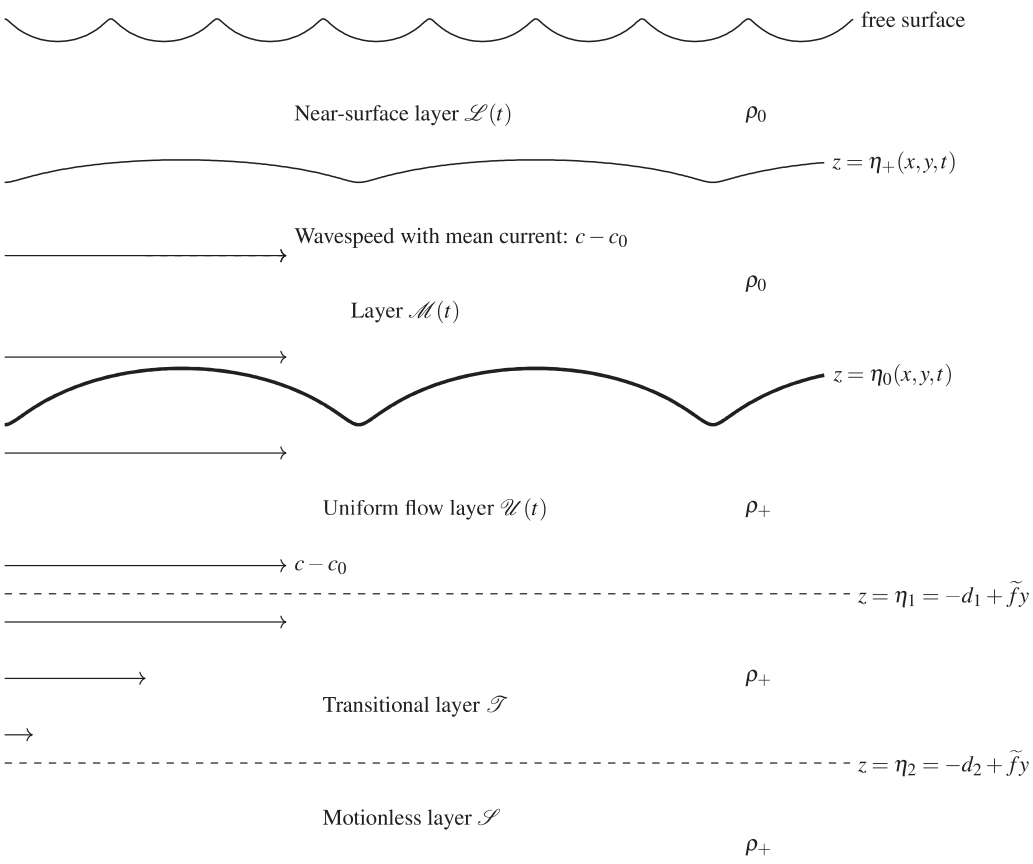


FIG. 1. Schematic of the nonhydrostatic model for a given latitude. The interface $z = \eta_0(x, y, t)$ represents the thermocline. The oscillation of the thermocline induces wave motion in the layer $\mathscr{M}(t)$, and propagates with wavespeed c. Furthermore, there exists a uniform zonal current c_0 in the layer $\mathscr{M}(t)$. Beneath the thermocline there is no wave motion, but a continuous transition to a uniformly flowing horizontal current in the layer $\mathscr{U}(t)$. The layer $\mathscr{T}$ allows for a linear decay to particle motion with depth to a still deep-water region.

wave solution presented here represents this oscillation and propagates above the thermocline. This paper presents a nonhydrostatic model, which incorporates a transition in particle velocities from the uniform horizontal speed experienced on the thermocline to being motionless as we extend downwards toward the deep-water region. The motion in the layer $\mathscr{M}(t)$ is dictated by zonally propagating waves induced by the oscillation of the thermocline. In the following we construct a solution which prescribes fluid motion in the layer $\mathscr{M}(t)$, while essentially disregarding the fluid motion in the layer $\mathscr{L}(t)$, based on the physically realistic assumption that the motion induced by the thermocline is likely to be (relatively) insignificant in the near-surface region. Of course, wave motion and wave-current interactions in the near-surface region are subjects of vast importance in their own right, and we note that the derivation of exact solutions with the potential to model such fluid motion has been the subject of much recent mathematical developments.[11,13,23,35,36] The novelty in this nonhydrostatic model compared with the model presented by the author in Ref. 41 is the addition of transitional layers below the layer $\mathscr{M}(t)$. Immediately below the thermocline we introduce a uniform current layer, $\mathscr{U}(t)$, where particles move with a uniform horizontal velocity in the direction of the internal wave propagation. Beneath $\mathscr{U}(t)$ exists the transitional layer $\mathscr{T}$, where the horizontal velocity of particles decreases linearly with depth until the motionless layer $\mathscr{S}$ is reached, where particles are still (Fig. 1).

III. GOVERNING EQUATIONS

In this paper we consider a geophysical fluid dynamics (GFD) model which incorporates a density stratification throughout the fluid on a rotating Earth. The Earth is approximated in this model as a sphere with radius $R = 6371$ km that rotates with a constant angular velocity of $\Omega = 7.29 \times 10^{-5}$ rad s^{-1} from west to east. The flow is then described with respect to a rotating Cartesian coordinate system that is fixed locally to the Earth's surface. In this framework we take the zonal coordinate x to point eastwards, the meridional coordinate y to point northwards, and the vertical coordinate z to point upwards normal to the Earths surface. This reference frame is not inertial so the Coriolis force acts on all particles within the rotating frame of reference. The ocean is assumed inviscid, and so the governing equations of motion for geophysical

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ocean waves are the Euler equations[15]

$$\begin{aligned} u_t + uu_x + vu_y + wu_z + 2\Omega w\cos\phi - 2\Omega v\sin\phi &= -\frac{1}{\rho}P_x, \\ v_t + uv_x + vv_y + wv_z + 2\Omega u\sin\phi &= -\frac{1}{\rho}P_y, \\ w_t + uw_x + vw_y + ww_z - 2\Omega u\cos\phi &= -\frac{1}{\rho}P_z - g, \end{aligned} \tag{1}$$

where ϕ denotes latitude, P denotes pressure and $g = 9.81\ \mathrm{ms}^{-2}$ is the constant gravitational acceleration on the Earth's surface. Since the density is constant in each layer of the stratified ocean, the continuity equation,

$$\nabla\cdot\boldsymbol{u} = u_x + v_y + w_z = 0, \tag{2}$$

must be satisfied in each layer. The f-plane approximation, which is a geometrically motivated simplification of the Coriolis terms in (1), is invoked to make the problem more tractable. This approximation is applicable away from the poles and in a relatively narrow ocean strip less than a few degrees of latitude wide. Furthermore, additional restrictions on the latitudinal region of applicability are necessary due to the structure of the prescribed internal wave solution presented here, and will be outlined in Sec. IV B. Introducing the Coriolis parameters f and $\hat{f}$ where

$$f = 2\Omega\sin\phi, \quad \hat{f} = 2\Omega\cos\phi, \quad \widetilde{f} = \frac{f}{\hat{f}} = \tan\phi,$$

allows the Euler equations to be written in the f-plane approximation as

$$\begin{aligned} P_x &= -\rho\left[\frac{Du}{Dt} + \hat{f}w - fv\right], \\ P_y &= -\rho\left[\frac{Dv}{Dt} + fu\right], \\ P_z &= -\rho\left[\frac{Dw}{Dt} - \hat{f}u + g\right]. \end{aligned} \tag{3}$$

The pressure is assumed to be continuous across the thermocline and thus the pressure in the uniform flow layer must equal the pressure in the layer $\mathscr{M}(t)$ when evaluated at $z = \eta_0$. Finally, to prevent the mixing of particles between layers, we impose the kinematic condition

$$w = \frac{\partial\eta_i}{\partial t} + u\frac{\partial\eta_i}{\partial x} + v\frac{\partial\eta_i}{\partial y} \qquad \text{on } z = \eta_i(x,y,t) \qquad \text{for } i = 0,1,2,$$

and so, particles on the boundary remain on the boundary for all time.

IV. MAIN RESULTS

A. The motionless layer $\mathscr{S}$

The upper boundary of the motionless layer, $\mathscr{S}$, is defined as

$$z = \eta_2 = -d_2 + \widetilde{f}y, \tag{4}$$

where $d_2 > 0$. The fluid is motionless, thus by the Euler equations yields

$$P(x,y,z,t) = P_0 - \rho_+ gz, \qquad z \le -d_2 + \widetilde{f}y,$$

where P_0 is a constant.

B. The transitional layer $\mathscr{T}$

The transitional layer $\mathscr{T}$ is bounded below by $z = \eta_2 = -d_2 + \widetilde{f}y$ and above by the boundary

$$z = \eta_1 = -d_1 + \widetilde{f}y, \tag{5}$$

where $d_2 > d_1 > 0$. A suitable value at the Equator for the thermocline is 120 m, while suitable values at the Equator for d_1 and d_2 are 160 and 200 m respectively.[5] The dependence of the fluid boundaries η_1 and η_2 on $\widetilde{f}y$ is necessary for the continuity of pressure across the thermocline,

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and presents itself physically as the boundaries of the transitional layer $\mathscr{T}$ being inclined at an angle of ϕ with respect to the local meridional axis. Furthermore, the form of the interfaces η_1 and η_2 prescribed in (5) and (4) respectively, presents additional restrictions due to the steep tilt of the boundaries experienced at high latitudes. Moreover, high values of $\widetilde{f}$ result in an additional consideration which will further restrict the latitudinal variability of solutions at higher latitudes due to the raising (or lowering) of the fluid boundaries. In this layer, the horizontal component of the particle's velocities increase linearly with height, with the following conditions on the boundaries to ensure continuity across each interface,

$$\begin{aligned} u &= 0 \qquad \text{on } z = -d_2 + \widetilde{f}y, \\ u &= c - c_0 \qquad \text{on } z = -d_1 + \widetilde{f}y, \end{aligned}$$

leading to the following velocity field

$$(u, v, w) = \left(u = \frac{c - c_0}{d_2 - d_1}\left(z + d_2 - \widetilde{f}y\right), 0, 0\right).$$

To find the pressure in the transitional layer, the above velocities are substituted into the governing Eq. (3),

$$P = \frac{1}{2}\rho_+ \hat{f}\frac{c - c_0}{d_2 - d_1}\left(z + d_2 - \widetilde{f}y\right)^2 - \rho_+ gz + P_0.$$

C. The uniform flow layer $\mathscr{U}(t)$

Existing between the transitional layer and the thermocline is a layer in which all particles experience a uniform horizontal motion. This purely zonal current has the following velocity field

$$(u, v, w) = (c - c_0, 0, 0),$$

which by way of the Euler equations for geophysical ocean flows gives rise to the pressure distribution

$$P = \rho_+ \hat{f}(c - c_0)\left[z - \widetilde{f}y\right] - \rho_+ gz + \rho_+ \hat{f}\frac{1}{2}(c - c_0)(d_1 + d_2) + P_0,$$

in the layer $\mathscr{U}(t)$.

D. The layer $\mathscr{M}(t)$, above the thermocline

The layer $\mathscr{M}(t)$ is bounded below by the thermocline, and above by $z = \eta_+(x, y, z)$. The internal wave motion in this layer is due to the oscillation of the thermocline, and propagates above the thermocline. The solution that is constructed includes a uniform horizontal current and is a Pollard-like internal wave in that it represents a periodic zonally traveling wave with a vertical amplitude that decays with height above the thermocline. Furthermore, the positions of fluid particles (x, y, z) are described in the Lagrangian labeling variables (q, r, s), where

$$\begin{aligned} x &= q - be^{-ms}\sin\left[k(q - ct)\right] - c_0 t, \\ y &= r - de^{-ms}\cos\left[k(q - ct)\right], \\ z &= -d_0 + s - ae^{-ms}\cos\left[k(q - ct)\right], \end{aligned} \tag{6}$$

with $d_0 > 0$ such that $d_0 < d_1 < d_2$, determining the depth of the thermocline, and where the Lagrangian variables are given by

$$(q, r, s) \in \mathbb{R} \times [-r_0, r_0] \times [s_0(r), s_+(r)],$$

with r_0 chosen to satisfy the restraints imposed by the f−plane approximation and where $s = s_0(r) \geq s^* > 0$ represents the thermocline and $s = s_+(r) > s_0(r)$ represents the interface between the layers $\mathscr{M}(t)$ and $\mathscr{L}(t)$. For convenience, let

$$\theta = k(q - ct).$$

The mapping from the particles position in our rotating Cartesian reference frame to the Lagrangian labeling variables has Jacobian

$$\begin{pmatrix} \frac{\partial x}{\partial q} & \frac{\partial y}{\partial q} & \frac{\partial z}{\partial q} \\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \end{pmatrix} = \begin{pmatrix} 1 - kbe^{-ms}\cos\theta & kde^{-ms}\sin\theta & kae^{-ms}\sin\theta \\ 0 & 1 & 0 \\ mbe^{-ms}\sin\theta & mde^{-ms}\cos\theta & 1 + mae^{-ms}\cos\theta \end{pmatrix}, \tag{7}$$

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with determinant

$$J = 1 + (am - bk)e^{-ms}\cos\theta - kmabe^{-2ms}.$$

For the flow to be incompressible[2] $J_t/J = 0$ and so

$$am = bk, \tag{8}$$

and

$$ambke^{-2ms} \neq 1.$$

The above constraint that $J \neq 0$ ensures that (6) is a valid local diffeomorphic change of coordinates by means of the inverse function theorem. Since $s \geq s^* > 0$, and we have that $am = bk$ from (8), then

$$a^2m^2e^{-2ms^*} < 1, \tag{9}$$

and thus the maximum amplitude of a particle is given by

$$a_{\max} = \frac{1}{m}. \tag{10}$$

Due to the explicit formulation of (6), it is possible to plot trajectories for the particles induced by this nonlinear wave-current interaction, something which is much more involved in the Eulerian formulation.[24,25] The solution is a parameterization of an internal wave propagating zonally, with wave crests parallel to the meridional direction and with the particles moving in trochoidal orbits. The solution is Pollard-like and thus describes particle paths as closed circles, that are inclined at an angle $\arctan(-d/a)$ to the local vertical due to the Earth's rotation. We will later show that d is proportional to f, implying that the magnitude of the tilt increases with distance from the Equator. In what follows we uncover relations that determine the values of b, d, c in terms of $a > 0, m > 0, k > 0$. The velocities and accelerations of a particle are given respectively through direct differentiation[1] by

$$\begin{aligned} u &= \frac{Dx}{Dt} = kcbe^{-ms}\cos\theta - c_0, \\ v &= \frac{Dy}{Dt} = -kcde^{-ms}\sin\theta, \\ w &= \frac{Dz}{Dt} = -kcae^{-ms}\sin\theta, \end{aligned}$$

and

$$\begin{aligned} \frac{Du}{Dt} &= k^2c^2be^{-ms}\sin\theta, \\ \frac{Dv}{Dt} &= k^2c^2de^{-ms}\cos\theta, \\ \frac{Dw}{Dt} &= k^2c^2ae^{-ms}\cos\theta. \end{aligned}$$

Which, upon substitution into the governing Eq. (3), yields

$$\begin{aligned} P_x &= -\rho_0\left[kc\left(bkc - a\hat{f} + df\right)e^{-ms}\sin\theta\right], \\ P_y &= -\rho_0\left[kc(dkc + bf)e^{-ms}\cos\theta - fc_0\right], \\ P_z &= -\rho_0\left[kc\left(akc - \hat{f}b\right)e^{-ms}\cos\theta + \hat{f}c_0 + g\right]. \end{aligned}$$

The pressure gradient in the Lagrangian labeling variables is given by the change of variables

$$\begin{pmatrix} P_q \\ P_r \\ P_s \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial q} & \frac{\partial y}{\partial q} & \frac{\partial z}{\partial q} \\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \end{pmatrix} \cdot \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix},$$

$$\begin{aligned} P_q &= -\rho_0\left[k^3c^2\left(a^2 + d^2 - b^2\right)e^{-2ms}\cos\theta\sin\theta + (bk^2c^2 - a\hat{f}kc + a\hat{f}kc_0 - dfkc_0 + dkfc + gak)e^{-ms}\sin\theta\right], \\ P_r &= -\rho_0\left[kc(dkc + bf)e^{-ms}\cos\theta - fc_0\right], \\ P_s &= -\rho_0\left[\left(ak^2c^2 + \hat{f}amc_0 - fdmc_0 - \hat{f}bkc + gam\right)e^{-ms}\cos\theta + mkc(kcb^2 - \hat{f}ab + fbd)e^{-2ms}\right. \\ &\quad \left. + mk^2c^2(a^2 + d^2 - b^2)e^{-2ms}\cos^2\theta + g + \hat{f}c_0\right]. \end{aligned} \tag{11}$$

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Assuming symmetry of second derivatives leads to the relations

$$dkc + bf = 0. \tag{12}$$

and

$$mkc^2 b + mcdf = k^2 c^2 a. \tag{13}$$

From the pressure gradient (11), the pressure is found as

$$\begin{aligned} P(q,r,s) = \widetilde{P}_0 - \rho_0\Big[-fc_0 r + \Big(g + \hat{f}c_0\Big)s - \frac{1}{2}\Big(b^2k^2c^2 - \hat{f}abkc + fbdkc\Big)e^{-2ms} \\ - (bkc^2 - a\hat{f}c + a\hat{f}c_0 - dfc_0 + dfc + ga)e^{-ms}\cos\theta - \frac{1}{2}k^2c^2\big(a^2 + d^2 - b^2\big)e^{-2ms}\cos^2\theta\Big]. \end{aligned} \tag{14}$$

It is required that the pressure be continuous across the thermocline, that is the pressure in $\mathscr{M}(t)$ must equal the pressure in $\mathscr{U}(t)$ when evaluated on the thermocline. Thus for a given latitude r, and the pressure evaluated on the thermocline $s = s_0(r)$, we define that

$$\begin{aligned} P_0 - \widetilde{P}_0 = \frac{1}{2}\rho_0 bkc\Big(bkc - a\hat{f} + df\Big)e^{-ms_0} - \Big(\rho_+\hat{f}c - (\rho_+ - \rho_0)g\Big)s_0 + (\rho_+ - \rho_0)\hat{f}c_0 s_0 \\ + \rho_+ fcr - (\rho_+ - \rho_0)fc_0 r - \frac{1}{2}(c - c_0)\rho_+\hat{f}(d_2 + d_1) + d_0\rho_+\Big(\hat{f}(c - c_0) - g\Big), \end{aligned}$$

where the thermocline at a given latitude is found by solving the above implicit equation for $s_0(r)$. Furthermore, continuity of the pressure distribution across the thermocline also leads to the relations

$$a^2 + d^2 = b^2, \tag{15}$$

$$bkc^2 = \frac{\rho_+ - \rho_0}{\rho_0}\Big[-a\hat{f}c + a\hat{f}c_0 - dfc_0 + dfc + ga\Big]. \tag{16}$$

Relations (8), (12) and (13), can be rearranged to yield the following

$$b = \frac{ma}{k}, \tag{17}$$

$$d = -\frac{fma}{k^2 c}, \tag{18}$$

$$m^2 = \frac{k^4 c^2}{k^2 c^2 - f^2}. \tag{19}$$

It is noteworthy that the third equation above implies that on the equator (where $f = 0$) $m = k$ and hence m can be thought of as a "odified wavenumber." The dispersion relation for nonhydrostatic internal water waves in the presence of a uniform zonal current induced by the oscillations of the thermocline is found through substitution of the above parameters into the continuity Eq. (16), and takes the form:

$$\Big[k^2c^3 + \frac{\rho_+ - \rho_0}{\rho_0}f^2(c - c_0)\Big]^2 = \Big(\frac{\rho_+ - \rho_0}{\rho_0}\Big)^2\big(k^2c^2 - f^2\big)\Big(g - \hat{f}(c - c_0)\Big)^2. \tag{20}$$

Note that in the absence of a mean horizontal current this result agrees with the dispersion relation found in Ref. 30.

V. PERTURBATIVE ANALYSIS OF THE DISPERSION RELATION

To analyze this dispersion relation further we first nondimensionalise it by introducing the nondimensionalised variables

$$X = c\sqrt{\frac{k}{\widetilde{g}}}, \quad X_0 = c_0\sqrt{\frac{k}{\widetilde{g}}}, \quad \varepsilon = \frac{f}{\sqrt{\widetilde{g}k}}, \quad F = \frac{\hat{f}}{f}, \tag{21}$$

where

$$\widetilde{g} = g\frac{\rho_+ - \rho_0}{\rho_0}. \tag{22}$$

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A typical value[16] for $\widetilde{g}$ at the Equator is 4×10^{-3} ms^{-2}. This allows the dispersion relation (20) to be rewritten as

$$\left[X^3 + \varepsilon^2\left(\frac{\rho_+ - \rho_0}{\rho_0}\right)(X - X_0)\right]^2 = \left(X^2 - \varepsilon^2\right)\left[1 - \varepsilon F\left(\frac{\rho_+ - \rho_0}{\rho_0}\right)(X - X_0)\right]^2,$$

which can be expanded to yield the following polynomial of degree six

$$\begin{aligned}
P(X) = {} & X^6 + \varepsilon^2\left(\frac{\rho_+ - \rho_0}{\rho_0}\right)\left[2 - F^2\left(\frac{\rho_+ - \rho_0}{\rho_0}\right)\right]X^4 \\
& + 2\varepsilon\left(\frac{\rho_+ - \rho_0}{\rho_0}\right)\left[F - \varepsilon X_0 + \varepsilon F^2\left(\frac{\rho_+ - \rho_0}{\rho_0}\right)X_0\right]X^3 \\
& + \left[\left(\frac{\rho_+ - \rho_0}{\rho_0}\right)^2\varepsilon^4 - 1 - 2\varepsilon F\left(\frac{\rho_+ - \rho_0}{\rho_0}\right)X_0 - \varepsilon^2 F^2\left(\frac{\rho_+ - \rho_0}{\rho_0}\right)^2 X_0^2 + \varepsilon^4 F^2\left(\frac{\rho_+ - \rho_0}{\rho_0}\right)^2\right]X^2 \\
& - 2\varepsilon^3\left(\frac{\rho_+ - \rho_0}{\rho_0}\right)\left[\varepsilon\left(\frac{\rho_+ - \rho_0}{\rho_0}\right)X_0 + F + \varepsilon F^2\left(\frac{\rho_+ - \rho_0}{\rho_0}\right)X_0\right]X \\
& + \varepsilon^2\left[\left(\frac{\rho_+ - \rho_0}{\rho_0}\right)^2\varepsilon^2 X_0^2 + 1 + 2\varepsilon F\left(\frac{\rho_+ - \rho_0}{\rho_0}\right)X_0 + \varepsilon^2 F^2\left(\frac{\rho_+ - \rho_0}{\rho_0}\right)^2 X_0^2\right] = 0.
\end{aligned}$$

The roots to the dispersion relation at mid-latitudes cannot be found analytically, however Fig. 2 suggests that roots may be found near to $X = \pm 1$ and $X = \pm\varepsilon$. With this in mind we search for accurate approximations to the roots of $P(X)$ through a perturbation expansion in the small parameter ε. Let

$$X = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \cdots$$

Subbing into $P(X)$ and collecting different powers of ε, it is found that to $O(\varepsilon^0)$

$$x_0^6 - x_0^2 = 0,$$

and thus

$$x_0 = \{0, \pm 1, \pm i\}.$$

The roots to $P(X)$ near to $X = \pm 1$ correspond to internal Gerstner-like waves slightly modified by the Earth's rotation. Since we are only interested in physical roots, for the following analysis we assume that $x_0 = 0$, which results in the existence of two possible small roots mediated by the presence of a nonhydrostatic mean zonal current, and are approximately given by

$$X = \pm\varepsilon\left(1 + \varepsilon^2\left(\frac{\rho_+ - \rho_0}{\rho_0}\right)^2\frac{X_0^2}{2}\right). \tag{23}$$

This gives an approximation of the wavespeed to $O(\varepsilon^2)$ of

$$c \approx \pm\frac{f}{k}, \tag{24}$$

and therefore the period of this wave motion is

$$T = \frac{\lambda}{c} \approx \frac{2\pi}{f} = T_i,$$

where T_i is the inertial period of the Earth, and so the wave motion described by (23) is that of an inertial wave. In light of the relation (19) it is important to calculate further an approximation of the wavespeed to $O(\varepsilon^3)$ to find that

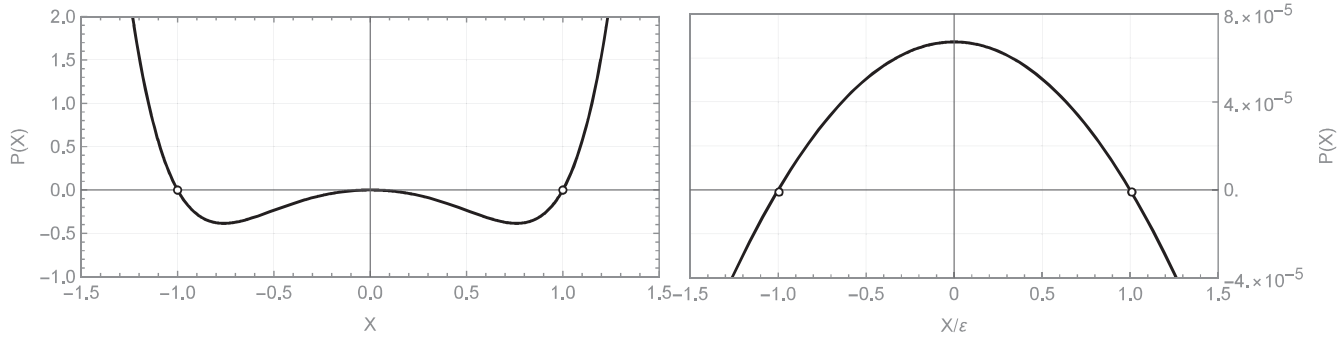
$$c^2 = \frac{f^2}{k^2}\left(1 + \varepsilon^2\left(\frac{\rho_+ - \rho_0}{\rho_0}\right)^2 X_0^2 + O(\varepsilon^4)\right).$$

Substitution of the above wavespeed into (19) yields

$$m \approx \frac{k^2 c}{f\varepsilon\left(\frac{\rho_+ - \rho_0}{\rho_0}\right)X_0}.$$

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(a) (b)

FIG. 2. (a) Plot of $P(X)$ shows the existence of two $O(1)$ roots. (b) Plot of $P(X)$ shows the existence of two $O(\varepsilon)$ roots. $P(X)$ was evaluated with parameters $f \approx -1.3 \times 10^{-4}$ s^{-1}, $c_0 \approx -0.3$ ms^{-1} and $k = 6.28 \times 10^{-2}$ m^{-1}, chosen to match the antarctic circumpolar current.

Reversing the nondimensionalisation (21), and substituting (22) and (24) into the above yields a value for the modified wavenumber

$$m \approx \pm \frac{gk}{f c_0}, \tag{25}$$

quantifying the rate of decay of the amplitude of the internal wave with height above the thermocline. The appearance of g in this relation represents a significant change from the hydrostatic model[41] where m is instead proportional to the reduced gravity coefficient $\widetilde{g}$. Since $g \gg \widetilde{g}$, the amplitude of wave motion in the layer $\mathscr{M}(t)$ decays vertically above the thermocline much faster when transitional layers are introduced beneath the thermocline. The velocity of a particle is calculated as

$$u^2 + v^2 + w^2 = c_0^2 + a^2 m^2 c^2 e^{-2ms} - 2c_0 amce^{-ms} \cos\theta,$$

with maximum particle velocity of

$$\boldsymbol{v}_{\max} = c_0 + \frac{ag}{c_0},$$

through utilisation of the relations (8), (15) and (25). We can estimate the maximum lateral and longitudinal amplitudes. The lateral amplitude can be expressed through the manipulation of (18), (24) and (25) to yield

$$d \approx \pm a \frac{g}{f c_0}.$$

Similarly, the longitudinal displacement is found to be

$$b \approx \pm a \frac{g}{f c_0}.$$

Therefore, the lateral and longitudinal displacement of a particle undergoing this wave motion greatly exceeds its vertical displacement. An upper bound for the vertical displacement of particle motion was established in (10) and can be used along with (25) to estimate the maximum lateral (and longitudinal) displacement as follows,

$$d_{\max} \approx \pm \frac{g}{m f c_0} \approx \pm \frac{1}{k} (\approx b_{\max}).$$

In comparision with the hydrostatic model presented in Ref. 41, an artifact of this more physically realistic model is that the presence of transitional layers beneath the thermocline compels the amplitude of particle motion to decrease more rapidly with height above the thermocline. Furthermore, recalling relation (10), the maximum vertical amplitude attainable by a particle on the thermocline is diminished in the presence of nonhydrostatic layers beneath the thermocline.

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AUTHOR DECLARATIONS

Conflict of Interest

The author has no conflicts to disclose.

Author Contributions

Jordan McCarney: Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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