Analytical Limitation for Time-Delayed Feedback Control in Autonomous Systems

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(Received 25 August 2011; revised manuscript received 30 July 2012; published 9 October 2012)

We prove an analytical limitation on the use of time-delayed feedback control for the stabilization of periodic orbits in autonomous systems. This limitation depends on the number of real Floquet multipliers larger than unity, and is therefore similar to the well-known odd number limitation of time-delayed feedback control. Recently, a two-dimensional example has been found, which explicitly demonstrates that the unmodified odd number limitation does not apply in the case of autonomous systems. We show that our limitation correctly predicts the stability boundaries in this case.

DOI: 10.1103/PhysRevLett.109.154101 PACS numbers: 05.45.Gg, 02.30.Ks

When chaotic systems started to get wider scientific attention during the 1960s, chaos was considered to be a mathematically interesting concept with little practical applications. This changed dramatically in the 1990s when Ott, Grebogi, and Yorke [1] introduced a method to stabilize unstable periodic orbits (UPOs) within the chaotic attractor using small perturbations. Since then, the subject of chaos control has been vigorously developed [2,3].

One simple method to stabilize a particular UPO within a chaotic attractor is via the time-delayed feedback control due to Pyragas [4]. Because no detailed knowledge of the chaotic system or its attractor is required, this method proved to be easy to implement and widely applicable [5–12]. However, it was claimed by Nakajima [13] that the time-delayed feedback control is not able to stabilize a UPO with an odd number of real Floquet multipliers larger than unity. While this odd number limitation was proved in Ref. [13] for the case of hyperbolic UPOs in nonautonomous systems, it was also stated that the same restriction should apply for the autonomous case "with a slight revision" (footnote 2 of Ref. [13]). Over the following years the odd number limitation was used by many researchers, and it seemed to be supported by experimental and numerical evidence even for autonomous systems, although in this case a strict proof was missing. Recently, Fiedler *et al.* [14] discovered a UPO in an autonomous two-dimensional system, which has precisely one Floquet multiplier larger than one, and can be stabilized by the time-delay feedback control scheme. This directly refuted the common belief that the odd number limitation is also valid for systems without explicit time dependence. Autonomous systems are by far the most dominating type of systems considered in nonlinear science, and time-delayed feedback control is one of the most practical methods for stabilizing (or destabilizing) periodic orbits. Therefore any limitation on the use of time-delayed feedback control is not only important from an academic point of view, but also has practical implications for the many applications of time-delayed feedback in real-world systems.

In this Letter we give an analytical condition under which the time-delayed feedback control is not successful in autonomous systems. Similarly to the odd number limitation, this condition involves the number of real Floquet multipliers larger than unity, but it is now modified by a term which takes the action of the control force in the direction of the periodic orbit into account. We will also connect this modification to the response of the system to changes in the delay time. Our proof follows to a large extent the proof of the original odd number limitation given in Ref. [13] but now implements the necessary modification for the autonomous case. As a first application we show that our limitation correctly reproduces the boundaries of stability for the two dimensional system studied in Refs. [14–16], which originally served as a counterexample of the unmodified odd number limitation.

Let us start with an uncontrolled dynamical system $\dot{x}(t) = f(x(t))$ with $x(t) \in \mathbb{R}^n$ and $f: \mathbb{R}^n \to \mathbb{R}^n$ and implement the time-delayed feedback control in the form

$$\dot{x}(t) = f(x(t)) + K[x(t-\tau) - x(t)], \tag{1}$$

where K is an $n \times n$ control matrix, and τ is a positive parameter. If the uncontrolled system has a τ-periodic solution $x^*(t) = x^*(t+\tau)$, then the form of (1) implies that $x^*(t)$ is also a solution of (1) for any choice of the control matrix K.

In order to assess the stability of the periodic orbit $x^*(t)$ in the controlled case, it is convenient to first introduce the the fundamental matrix $\Phi(t)$ for the *uncontrolled* system as the solution of the initial value problem

$$\dot{\Phi}(t) = Df(x^*(t))\Phi(t); \qquad \Phi(0) = \mathbb{I}, \tag{2}$$

where $Df(x^*(t))$ denotes the Jacobian of f evaluated at $x^*(t)$, and $\mathbb{I}$ is the $n \times n$ identity matrix. The generalized eigenvalues $\{\mu_1, \ldots, \mu_n\}$ of $\Phi(\tau)$ are the Floquet multipliers associated with the periodic orbit $x^*(t)$. We also define the matrix $W(t) = (v_1(t), \ldots, v_n(t))$ such that its kth column $v_k(t) \in \mathbb{C}^n$ is given by $v_k(t) = \Phi(t)v_k(0)$ and the set $\{v_1(0), \ldots, v_n(0)\}$ is a Jordan basis of generalized eigenvectors of $\Phi(\tau)$. For each t, the set $\{v_1(t), \ldots, v_n(t)\}$ provides a local (but in general not τ-periodic) basis at the position $x^*(t)$ along the orbit. Since we consider an autonomous system we also observe that $\dot{x}^*(0) = \Phi(\tau)\dot{x}^*(0)$; i.e., one of the Floquet multipliers is equal to unity. It is therefore convenient to choose $v_1(t) = \dot{x}^*(t)$ and $\mu_1 = 1$. By defining

$$\hat{K}(t) = [W(t)]^{-1} K W(t), \tag{3}$$

we transform the control matrix to this local basis. As we will see in the following the (1, 1) component of the matrix $\hat{K}(t)$, which we denote by $\hat{K}_{11}(t)$, plays a decisive role in assessing the stability of the controlled orbit $x^*(t)$. Some intuition for the quantity $\hat{K}_{11}(t)$ can be obtained if we expand the result of applying the control matrix to $\dot{x}^*(t)$ in the local basis via

$$K\dot{x}^*(t) = \hat{K}_{11}(t)\dot{x}^*(t) + \sum_{k=2}^{n} \hat{K}_{k1}(t) v_k(t). \tag{4}$$

In loose terms we can therefore interpret the quantity $\hat{K}_{11}(t)$ as the action of the control matrix K projected in the tangential direction of the orbit at time t. Note that $\hat{K}_{11}(t)$ is well defined and in particular not affected by any reordering or rescaling of the $v_k(t)$ for $k \geq 2$. Using this definition of $\hat{K}_{11}(t)$ we are now in a position to formulate the main result.

Theorem.—Let $x^*(t)$ be a τ-periodic orbit of (1) which for $K = 0$ possesses m real Floquet multipliers larger than unity and precisely one Floquet multiplier equal to unity. Then $x^*(t)$ is an unstable solution of the time-delayed system (1) if the condition

$$(-1)^m \left(1 + \int_0^\tau \hat{K}_{11}(t) dt\right) < 0, \tag{5}$$

is fulfilled. Here $\hat{K}_{11}(t)$ is defined as in (4).

Before we proceed with the proof of the theorem, we briefly discuss its significance and reformulate it in a way which is more useful for practical applications. The theorem provides an analytical limitation on the use of time-delayed feedback control, and states that time-delayed feedback can *only* successfully stabilize a periodic orbit, if the condition (5) is violated. We stress, however, that the converse is not implied by the theorem; i.e., a violation of (5) alone does *not* guarantee that time-delayed feedback will successfully stabilize a given periodic orbit. The theorem is only applicable to periodic orbits with exactly one Floquet multiplier equal to one.

In practice the integral over the matrix element $\hat{K}_{11}(t)$ in (5) is difficult to perform, even if the system is analytically known. A practically useful reformulation of condition (5) can be obtained from studying the response of the system to changes in the delay time. We consider a variant of the system (1)

$$\dot{x}(t) = f(x(t)) + K[x(t - \hat{\tau}) - x(t)], \tag{6}$$

where the delay time $\hat{\tau}$ is slightly different from the period τ of the uncontrolled orbit. For $\hat{\tau}$ sufficiently close to τ the system (6) will possess a (possibly unstable) induced periodic orbit $\tilde{x}^*(t)$ with period $\tilde{\tau}(\hat{\tau})$. In general $\tilde{\tau}$ is different from both τ and $\hat{\tau}$; however, one can show that the period of the induced orbit is connected with the matrix element $\hat{K}_{11}(t)$ via

$$\lim_{\hat{\tau}\to\tau} \frac{\tilde{\tau}(\hat{\tau}) - \tau}{\hat{\tau} - \tilde{\tau}(\hat{\tau})} = \int_0^\tau \hat{K}_{11}(t) dt. \tag{7}$$

From the condition (5) of our theorem it then follows that $x^*(t)$ is an unstable solution of the system (1) if the condition

$$(-1)^m \lim_{\hat{\tau}\to\tau} \frac{\hat{\tau} - \tau}{\hat{\tau} - \tilde{\tau}(\hat{\tau})} < 0 \tag{8}$$

holds. Since condition (8) only requires the knowledge of the period of the induced orbit $\tilde{\tau}$ as a function of the delay time $\hat{\tau}$, it is often more convenient in practice than the equivalent but more technical condition (5).

The proof of the theorem uses many ideas from Ref. [13] but now takes particular consideration of the autonomous case. The essential tools are the two functions $F(\nu)$ and $G(\nu)$ defined by [13]

$$G(\nu) = \det[\nu \mathbb{I} - \Phi(\tau)], \tag{9}$$

$$F(\nu) = \det[\nu \mathbb{I} - \Psi_\nu(\tau)], \tag{10}$$

where $\Psi_\nu(t)$ solves the initial value value problem

$$\dot{\Psi}_\nu(t) = [Df(x^*(t)) + (\nu^{-1} - 1)K]\Psi_\nu(t), \qquad \Psi_\nu(0) = \mathbb{I}. \tag{11}$$

By direct differentiation it can be verified that the solution of (11) can also be expressed as [13]

$$\Psi_\nu(t) = \Phi(t)\left[\mathbb{I} + (\nu^{-1} - 1)\int_0^t \Phi^{-1}(u) K \Psi_\nu(u) du\right]. \tag{12}$$

We first show the following lemma:

Lemma.—If for a given τ-periodic orbit $x^*(t)$ with precisely one Floquet multiplier equal to unity the condition $F'(1) < 0$ holds, then $x^*(t)$ is an unstable solution of the time-delayed system (1).

Proof of the Lemma.—From (11) it follows that $\Psi_\nu(t)$ is bounded in the limit of $\nu \to +\infty$, and therefore (10) implies that $\lim_{\nu\to+\infty} F(\nu) = +\infty$. In an autonomous system we know that $\mu_1 = 1$ is an eigenvalue of $\Phi(\tau)$ and it therefore follows from (9) that $G(1) = 0$. But since from (11) it follows that $\Psi_1(t) = \Phi(t)$ we also have $F(1) = G(1) = 0$. Thus $F(\nu)$ is a continuous function, which vanishes at $\nu = 1$, has a negative slope at $\nu = 1$, and diverges to $+\infty$ for large ν. By the intermediate value theorem there exists at least one $\nu_c > 1$ with $F(\nu_c) = 0$. From (10) it then follows that there exists a vector $w_c(0) \in \mathbb{R}^n$ with $\Psi_{\nu_c}(\tau) w_c(0) = \nu_\nu w_c(0)$ and we can define $w_c(t) = \Psi_{\nu_c}(t) w_c(0)$. Then $w_c(t)$ is a growing solution of the linearized equation $\dot{w}_c(t) = Df(x^*(t))w_c(t) + K[w_c(t-\tau) - w_c(t)]$ and therefore the original orbit $x^*(t)$ is an unstable solution of the time-delayed system (1). This completes the proof of the lemma.

Proof of the Theorem.—To complete the proof of the theorem, it now remains to show that the condition (5) implies that $F'(1) < 0$. Under this condition the above lemma then implies that the orbit $x^*(t)$ is unstable and thereby proves the theorem. Since $F(1) = 0$ we can write $F'(1) = \lim_{\epsilon\to 0} F(1+\epsilon)/\epsilon$. To assess the sign of $F'(1)$ it is therefore necessary to evaluate $F(1+\epsilon)$ up to first order in ϵ. According to (10) we can achieve this by first evaluating $\Psi_\nu(t)$ at $\nu = 1+\epsilon$. Using the representation (12) we write

$$\begin{aligned}\Psi_{1+\epsilon}(t) &= \Phi(t)\left[\mathbb{I} - \frac{\epsilon}{1+\epsilon}\int_0^t \Phi^{-1}(u)K\Psi_{1+\epsilon}(u)du\right] \\ &= \Phi(t)\left[\mathbb{I} - \epsilon\int_0^t \Phi^{-1}(u)K\Phi(u)du\right] + O(\epsilon^2).\end{aligned}$$

Then to first order of ϵ we obtain from (10)

$$F(1+\epsilon) = \det[M^0 + \epsilon M^1], \tag{13}$$

where we have defined the two matrices

$$\begin{aligned}M^0 &= \mathbb{I} - \Phi(\tau); \\ M^1 &= \mathbb{I} + \Phi(\tau)\int_0^t \Phi^{-1}(u)K\Phi(u)du.\end{aligned}$$

Using the previously defined matrix $W(0)$ we transform the argument of the determinant in (13) as

$$F(1+\epsilon) = \det[W(0)^{-1}(M^0 + \epsilon M^1)W(0)], \tag{14}$$

$$= \det\left[\mathbb{I} - \hat{\Phi}(\tau) + \epsilon\left(\mathbb{I} + \hat{\Phi}(\tau)\int_0^\tau \hat{K}(u)du\right)\right], \tag{15}$$

where we have used $W(0)^{-1}\Phi^{-1}(u)K\Phi(u)W(0) = W(u)^{-1}KW(u) = \hat{K}(u)$. Using the Jordan normal form of $\hat{\Phi}(\tau)$ and the fact that $\mu_1 = 1$ we find

$$\mathbb{I} - \hat{\Phi}(\tau) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1-\mu_2 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & * \\ 0 & \cdots & \cdots & 0 & 1-\mu_n \end{pmatrix}, \tag{16}$$

where the entries indicated by * can be either 0 or -1. Since all entries in the first column and first row of the matrix $\mathbb{I} - \hat{\Phi}(\tau)$ vanish, the only contribution to the determinant (15) up to first order in ϵ is given by

$$\begin{aligned}F(1+\epsilon) &= \epsilon\left(\mathbb{I} + \hat{\Phi}(\tau)\int_0^\tau \hat{K}(u)du\right)_{(1,1)}\prod_{k=2}^n(1-\mu_k) \\ &= \epsilon\left(1 + \int_0^\tau \hat{K}_{11}(u)du\right)\prod_{k=2}^n(1-\mu_k).\end{aligned}$$

The product $\prod_{k=2}^n(1-\mu_k)$ does not vanish, since it was assumed that $\mu_1 = 1$ is the only Floquet multiplier equal to unity. Each real Floquet multiplier larger than unity contributes a negative factor to this product, while pairs of complex conjugated Floquet multipliers or real Floquet multipliers smaller than unity do not change its sign. We can therefore write $\mathrm{sgn}[\prod_{k=2}^n(1-\mu_k)] = (-1)^m$ and conclude that

$$F'(1) < 0 \Leftrightarrow (-1)^m\left(1 + \int_0^\tau \hat{K}_{11}(u)du\right) < 0.$$

This means that condition (5) implies a negative $F'(1)$ and it then follows from the lemma that $x^*(t)$ is unstable. This concludes the proof of the theorem.

Let us now compare our theorem and its proof with the proof of the original odd number limitation (theorem 2 in Ref. [13]), which states that a *hyperbolic* UPO of a nonautonomous system with an odd number m of real Floquet multipliers larger than unity can not be stabilized using time-delayed feedback control. We stress that the term *hyperbolic orbit* in the context of a nonautonomous system means that the orbit has *no* Floquet multipliers equal to unity. In contrast, for an autonomous system the term *hyperbolic orbit* denotes an orbit with *precisely one* Floquet multiplier equal to unity [17]. Therefore any hyperbolic orbit in the autonomous system becomes a nonhyperbolic orbit in the associated nonautonomous system. The proof in Ref. [13], however, makes explicit use of the fact that *all* Floquet multipliers differ from one, and is therefore only correct if the term *hyperbolic* is understood in the context of non-autonomous systems.

Let us now study condition (8) for odd m. If we increase the delay time to $\hat{\tau} > \tau$ the system will respond with a period $\tilde{\tau}$ of the induced orbit. One might now be tempted to assume that the period of the induced orbit should always be less than the delay time, i.e., $\tilde{\tau} < \hat{\tau}$. This is however not the case, and it is possible to find dynamical systems, which respond to an increased delay time with an induced period which is even bigger than the delay time itself. In this case (8) is violated and it might be possible to stabilize the corresponding periodic orbit using time-delayed feedback control. This consideration leads to an important practical consequence for the design of a successful control scheme for a UPO with an odd number of Floquet multipliers larger than 1. The control term needs to be constructed in such a way that for increasing delay time the period of the induced orbit grows faster than the delay time itself.

The first autonomous example, where a UPO with odd m was stabilized using time-delayed feedback control was

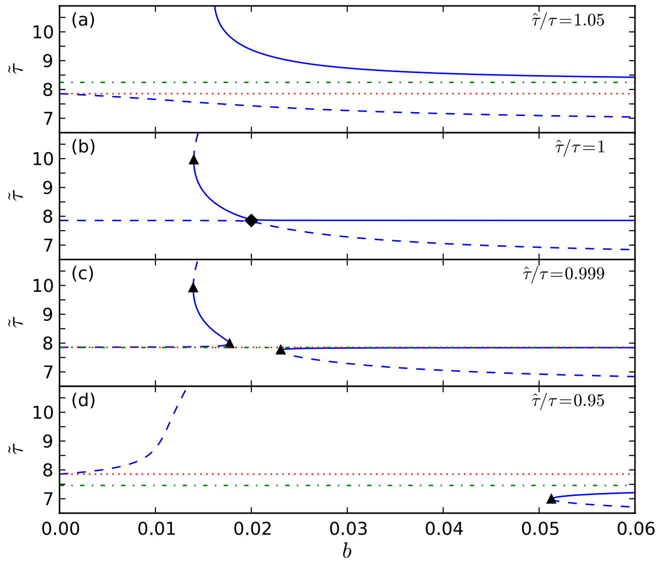

FIG. 1 (color online). The periods $\tilde{\tau}$ of stable (solid lines) and unstable (dashed lines) periodic orbits of the system (17) as a function of b for different values of $\hat{\tau}$. Calculations were performed using Knut [18]. Triangles and a diamond indicate saddle-node and transcritical bifurcations, respectively. The values of τ and $\hat{\tau}$ are indicated by the horizontal dotted (red) and dashed-dotted (green) lines, respectively.

given in Ref. [14] and provided a counterexample which showed that the original odd number limitation can not be applied to the autonomous case without modification. It is therefore important to check that our conditions (5) and (8) correctly handle this case. Let us consider the dynamical system for $z(t) \in \mathbb{R}^2$ given by [14,15]

$$\dot{z}(t) = \begin{pmatrix} (|z(t)|^2 - R^2) & -(1+\gamma|z(t)|^2) \\ (1+\gamma|z(t)|^2) & (|z(t)|^2 - R^2) \end{pmatrix} z(t) + b\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}[z(t-\hat{\tau}) - z(t)], \quad (17)$$

with the main bifurcation parameters $\hat{\tau} > 0$ and $b \geq 0$. For the remaining parameters we choose $R^2 = 0.02$, $\gamma = -10$ and $\beta = \pi/4$. For $b = 0$ we find a periodic orbit $z^*(t) = R[\cos(2\pi t/\tau), \sin(2\pi t/\tau)]^T$ with period $\tau = 2\pi/(\gamma R^2 + 1) > 0$. A short calculation shows that the two Floquet multipliers are given by $\mu_1 = 1$ and $\mu_2 = \exp(2R^2\tau) > 1$ which implies $m = 1$.

For $\hat{\tau} = \tau$ and increasing b the unstable orbit $z^*(t)$ is stabilized via a transcritical bifurcation at a critical value b_c [14]. This scenario is shown in Fig. 1(b), where the transcritical bifurcation is indicated by a diamond. We now change the delay time $\hat{\tau}$ and study the period $\tilde{\tau}$ of the induced orbit which continuously connects to the orbit $z^*(t)$. For $\hat{\tau} > \tau$, we observe that the transcritical bifurcation evolves into an avoided crossing of two branches, as shown in Fig. 1(a). For $b < b_c$ the periodic orbit $z^*(t)$ at $\hat{\tau} = \tau$ evolves into an orbit with period $\tilde{\tau} < \hat{\tau}$. Therefore

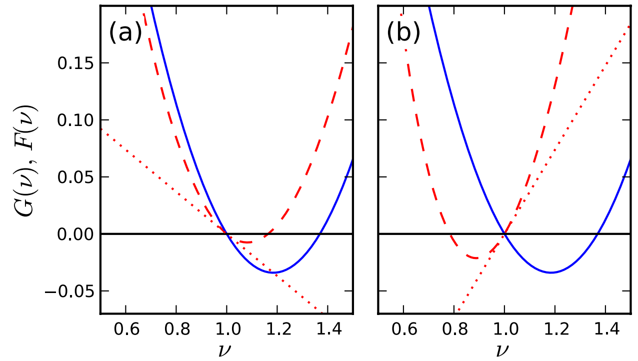

FIG. 2 (color online). The functions $G(\nu)$ (solid blue lines) and $F(\nu)$ (dashed red lines) and the slopes $F'(1)$ (dotted red lines) for system (17) for $b = 0.01$ (a), and $b = 0.04$ (b) and with $\hat{\tau} = \tau$.

condition (8) is fulfilled, and our theorem guarantees that the orbit $z^*(t)$ is unstable for $\hat{\tau} = \tau$ and $b < b_c$. For $\hat{\tau} > \tau$ and $b > b_c$ we observe that $\tilde{\tau} > \hat{\tau}$. This means that as we increase the delay time, the period of the induced orbit becomes even larger than the new delay time. In this intuitively unusual case the condition (8) is violated. Thus our theorem does not apply for $b > b_c$ and stabilization is possible. Similar considerations apply for the case $\hat{\tau} < \tau$ [Fig. 1(c) and 1(d)] where the transcritical bifurcation evolves into a pair of saddle-node bifurcations. Again condition (8) is only fulfilled for $b < b_c$.

For this simple example we can also explicitly calculate $\hat{K}_{11}(t) = b(\cos\beta + \gamma\sin\beta)$. For the possibility of successful stabilization we need to violate the condition (5), which leads to the necessary condition

$$b(\cos\beta + \gamma\sin\beta)\tau < -1. \quad (18)$$

This agrees with the previous stability condition given in Ref. [15] and the location of the transcritical bifurcation in Fig. 1(b).

It is also illustrative to study the functions $F(\nu)$ (10) and $G(\nu)$ (10) for the current example. In Fig. 2(a) the corresponding plots are shown for $b < b_c$. In this case we observe that $F(1) = 0$ and the slope $F'(1)$ is negative. Therefore the function $F(\nu)$ needs to cross the zero axis at a point larger than unity and the periodic orbit is unstable. In the case of $b > b_c$, as shown in Fig. 2(b), the slope $F'(1)$ is positive and the function $F(\nu)$ does not cross the zero axis at values larger then unity. This is a necessary condition for the stability of the periodic orbit under time-delayed feedback control.

In conclusion, we have proved an analytical limitation on the use of time-delayed feedback control in autonomous systems. This limitation depends on the number of real Floquet multipliers larger than unity, and on the properties of the induced orbits as the delay time is varied. While the limitation is valid for arbitrary dimensions, we have demonstrated its usefulness in a well studied two-dimensional system, for which the original odd number limitation does not apply. The knowledge of this limitation will provide important guidance for the design of time-delayed feedback implementations in practical applications.

This work was supported by Science Foundation Ireland under Grant No. 09/SIRG/I1615.

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