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HOMOMORPHIC FELLER COCYCLES ON A C^*-ALGEBRA

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Abstract. When a Fock-adapted Feller cocycle on a C^*-algebra is regular, completely positive and contractive it possesses a stochastic generator that is necessarily completely bounded. Here necessary and sufficient conditions are given, in the form of a sequence of identities, for a completely bounded map to generate a weakly multiplicative cocycle. These are derived from a product formula for iterated quantum stochastic integrals. Under two alternative assumptions, one of which covers all previously considered cases, the first identity in the sequence is shown to imply the rest.

0. Introduction

In a previous paper we showed that completely bounded mapping matrices on a C^*-algebra (or indeed any operator space) stochastically generate Markovian cocycles on the algebra. Since stochastic generators of (Fock-adapted Markov-regular) completely positive contraction cocycles on a C^*-algebra are necessarily completely bounded, this entails an existence theorem for such cocycles. In this paper we revisit the question of which completely bounded operators stochastically generate Markov-regular *-homomorphic cocycles. In the case of finite dimensional noise this was settled by M P Evans ([Eva]), and was extended to a class of unbounded operators by Fagnola and Sinha ([FaS]). Here we are interested in the case of infinite dimensional noise.

The history of the problem is as follows. Mohari and Sinha found sufficient conditions for the generation of unital *-homomorphic stochastic flows when the component maps of the putative stochastic generator are bounded. Their conditions are analytic assumptions, together with formally derived necessary algebraic conditions on the generator which make sense under their analytic assumptions ([MoS]). Subsequently Meyer observed that the Mohari-Sinha existence proof for quantum stochastic differential equations is valid under weaker (more natural) regularity than they had assumed ([Mey], p185). However a continuity property of Picard approximants, needed in the proof of sufficiency for *-homomorphic solutions, is not implied by Meyer's regularity condition ([LW1], p533). Next, following Lindsay and Parthasarathy ([LiP]), the present authors found necessary and sufficient conditions on the generator of a stochastic flow for the flow to be completely positive and contractive, under the assumption that the components of the generator are bounded ([LW1], Proposition 5.1). See also [Bel], and references therein. These conditions entail (global) complete boundedness of the generator ([LW1], Theorem 5.2) and in turn complete boundedness implies Meyer's regularity condition ([LW3], Theorem 2.2) so that existence of the (completely positive, contractive) flow is ensured. The question therefore remains (in the infinite dimensional case): which completely bounded maps generate *-homomorphic stochastic flows?

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For completely bounded generators on a von Neumann algebra the analytic conditions of Mohari and Sinha are equivalent to ultraweak continuity ([LW1], Proposition 1.3), which in turn is equivalent to ultraweak continuity of the flow, when this is bounded ([LW1], Proposition 3.2). In order to obtain a stochastic dilation, for any normal norm continuous completely positive contraction semigroup on a von Neumann algebra, Goswami and Sinha extended the existence theorem to non-separable noise dimension spaces. This was achieved by means of a coordinate-free stochastic calculus using Hilbert W^*-modules ([GoS]).

One reason why one should be concerned with the infinite dimensional situation is that the representation of stochastic flows even in classical probability demands infinite dimensional noise ([Kun]). As mentioned above the dilation of dynamical semigroups requires infinite dimensional noise too. Applications of the noncommutative theory to classical infinite state Markov chains may be found in [Par].

In this paper we give necessary and sufficient conditions for multiplicativity in the form of a sequence of identities. We then isolate natural conditions under which the first of these identities implies the whole sequence. Our conditions are in turn implied by those considered respectively by Evans and Mohari-Sinha; the removal of ad hoc assumptions begun in [LW3] is thereby completed. Our criteria also enable a direct approach to the dilation of dynamical semigroups on a C^*-algebra ([GPS]).

Our aim is to highlight the algebraic imperative of the result, somewhat in the spirit of [L2], in contrast to previous approaches which involve showing "that some complicated sums of scalar products tend to 0" ([Mey], p187). We do this by working coordinate-free, exploiting complete-boundedness of stochastic generators, and using matrix spaces over C^*-algebras. We deduce the necessary and sufficient conditions from an elementary product formula for iterated quantum stochastic integrals which has independent interest and applicability.

In the context of finite dimensional quantum stochastic calculus the product formula is proved in [HPu] where noncommutative functional Itô formulae and chaotic expansions in the universal enveloping algebra of a finite dimensional Lie algebra are investigated. It is closely related to the product formula for four-argument integral-sum kernel operators ([L1]). Indeed operators of this type with two-argument kernels were introduced by Maassen precisely for displaying the simple explicit action of operator solutions of quantum stochastic differential equations ([Maa]). The potential for using iterated quantum stochastic integrals for establishing multiplicativity of Markovian cocycles was recognised by Hudson and coworkers in [CEH] and was implemented for finite dimensional quantum noise in [Hud] where its applicability to Fermionic theories is also highlighted.

Our proofs are elementary, using only the fundamental formulae and estimate of quantum stochastic calculus (2.4–6), and some simple combinatorics of finite sets. We use the language of Markovian and Feller cocycles, following [LW2] and [LW3], to further emphasise algebraic structure. For a discussion of the term 'Feller' see Section 3. In the context of noncommutative probability the terminology of Markovian cocycles was introduced by Accardi ([Acc], [AFL]).

General notation. We use $\odot$ to denote algebraic tensor products of spaces and operators, $\otimes$ for the tensor product of Hilbert spaces and their operators, and $\otimes_{\rm sp}$ for spatial tensor products of C^*-algebras. Tensor symbols between Hilbert space vectors *will usually be dropped.* Hilbert space inner products here are linear in their second argument. Operator spaces arising in this paper are all concrete — that is each is a closed subspace of $B(\mathsf{H})$ for some Hilbert space H. For a linear map $\phi : \mathsf{V} \to \mathsf{W}$ between operator spaces and a Hilbert space h, if *either* h is finite dimensional and ϕ is bounded, *or* h is infinite dimensional and ϕ is completely bounded, then ϕ is said to be h-*bounded*. The collection of such operators is denoted h-$B(\mathsf{V};\mathsf{W})$.

1. Preliminaries

We fix once and for all a Hilbert space $\mathfrak{h}$, a C^*-algebra $\mathcal{A}$ acting nondegenerately on $\mathfrak{h}$, and a separable Hilbert space k, and let $\mathcal{F}$ denote the symmetric Fock space over $L^2(\mathbb{R}_+;\mathsf{k}) = L^2(\mathbb{R}_+)\otimes\mathsf{k}$. The notation

$$\widehat{\mathsf{k}} := \mathbb{C}\oplus\mathsf{k}, \quad \text{and, for } c\in\mathsf{k}, \quad \widehat{c} := \begin{pmatrix} 1 \\ c \end{pmatrix} \tag{1.1}$$

will be used, and the orthogonal projection in $B(\widehat{\mathsf{k}})$ with range k will be denoted P. Whenever $\eta = (e_i)_{i\geq 0}$ is an orthonormal basis for k the basis $(e_\alpha)_{\alpha\geq 0}$ for $\widehat{\mathsf{k}}$ obtained by setting $e_0 = 1\in\mathbb{C}$ is denoted $\widehat{\eta}$. For any Hilbert space h, $I_{\mathsf{h}}\otimes P$ will be written Δ_{h} with $\Delta_{\mathfrak{h}}$ simply denoted by Δ; also

$$\mathsf{h}_n := \mathsf{h}\otimes\widehat{\mathsf{k}}^{\otimes n}, \text{ and } \mathsf{h}_n^0 := \mathsf{h}\odot\widehat{\mathsf{k}}^{\odot n}. \tag{1.2}$$

The identity operator on $\widehat{\mathsf{k}}^{\otimes n}$ is denoted I_n. For any bounded Hilbert space operator T, $\iota(T)$ denotes its ampliation $T\otimes I_1$.

Bra- and -ket notation. Let H and H' be Hilbert spaces. For any vector x in a Hilbert space h, the bounded operator $\mathsf{H}\otimes\mathsf{H}'\to\mathsf{H}\otimes\mathsf{h}\otimes\mathsf{H}'$ determined by

$$\xi\xi' \mapsto \xi x\xi'$$

is denoted $|\mathsf{H},x,\mathsf{H}'\rangle$, and its adjoint is denoted $\langle\mathsf{H},x,\mathsf{H}'|$. The combination $|x\rangle\langle y|$ (case $\mathsf{H}=\mathsf{H}'=\mathbb{C}$) yields the rank one operator (Dirac dyad) $z\mapsto\langle y,z\rangle x$, and $|\mathsf{h}\rangle$ denotes the operator space $B(\mathbb{C};\mathsf{h}) = \{|x\rangle : x\in\mathsf{h}\}$.

When $\mathsf{H}'=\mathbb{C}$ we abbreviate these operators to E_x and E^x respectively.

Matrix spaces and induced maps. For an operator space V in $B(\mathsf{H})$, and a Hilbert space h,

$$\mathrm{M}(\mathsf{h};\mathsf{V})_{\mathrm{b}} := \{T\in B(\mathsf{H}\otimes\mathsf{h}) : E^xTE_y\in\mathsf{V} \text{ for all } x,y\in\mathsf{h}\}$$

defines an operator space in $B(\mathsf{H}\otimes\mathsf{h})$ called the h-*matrix space over* V. Even though we are here interested in processes on the C^*-algebra $\mathcal{A}$, operator spaces are relevant since in general $\mathrm{M}(\mathsf{h};\mathcal{A})_{\mathrm{b}}$ is *not* an algebra. There are two ways in which matrix spaces arise naturally here: firstly regular *-homomorphic Markovian cocycles map $\mathcal{A}$ into $\mathrm{M}(\mathcal{F};\mathcal{A})_{\mathrm{b}}$, secondly their stochastic generators map $\mathcal{A}$ into $\mathrm{M}(\widehat{\mathsf{k}};\mathcal{A})_{\mathrm{b}}$. For an operator space V in $B(\mathsf{H})$, the natural (*-algebra) isomorphism $B((\mathsf{H}\otimes\mathsf{h}_1)\otimes\mathsf{h}_2)\to B(\mathsf{H}\otimes(\mathsf{h}_1\otimes\mathsf{h}_2))$ restricts to a (completely isometric) isomorphism $\mathrm{M}(\mathsf{h}_2;\mathrm{M}(\mathsf{h}_1;\mathsf{V})_{\mathrm{b}})_{\mathrm{b}}\to\mathrm{M}(\mathsf{h}_1\otimes\mathsf{h}_2;\mathsf{V})_{\mathrm{b}}$. If W is another operator space in $B(\mathsf{K})$ then, for any h-bounded linear map $\phi:\mathsf{V}\to\mathsf{W}$,

$$E^x\phi^{(\mathsf{h})}(A)E_y = \phi(E^xAE_y), \quad A\in\mathrm{M}(\mathsf{h};\mathsf{V})_{\mathrm{b}},\ x,y\in\mathsf{h}, \tag{1.3}$$

determines an h-bounded operator $\phi^{(\mathsf{h})}:\mathrm{M}(\mathsf{h};\mathsf{V})_{\mathrm{b}}\to\mathrm{M}(\mathsf{h};\mathsf{W})_{\mathrm{b}}$, which satisfies

$$\|\phi^{(\mathsf{h})}\| \leq (\dim\mathsf{h})\|\phi\|, \text{ respectively } \|\phi^{(\mathsf{h})}\|_{\mathrm{cb}} = \|\phi\|_{\mathrm{cb}},$$

when h is finite (resp. infinite) dimensional ([LW3]).

In fact it is a modification of this amplification procedure that we need here. For Hilbert spaces h and K, and an h-bounded operator $\phi:\mathcal{A}\to\mathrm{M}(\mathsf{K};\mathcal{A})_{\mathrm{b}}$, define an h-bounded operator ϕ^{h} by

$$\phi^{\mathsf{h}}:\mathrm{M}(\mathsf{h};\mathcal{A})_{\mathrm{b}}\to\mathrm{M}(\mathsf{K};\mathrm{M}(\mathsf{h};\mathcal{A})_{\mathrm{b}})_{\mathrm{b}}, \quad \phi^{\mathsf{h}} = \tau\circ\phi^{(\mathsf{h})} \tag{1.4}$$

where τ is the tensor flip $B(\mathfrak{h} \otimes \mathsf{K} \otimes \mathsf{h}) \to B(\mathfrak{h} \otimes \mathsf{h} \otimes \mathsf{K})$. The following identity enjoyed by these operators will be useful later:

$$\phi^{\mathsf{h}}\big((I_{\mathfrak{h}} \otimes S)A(I_{\mathfrak{h}} \otimes T)\big) = (I_{\mathfrak{h}} \otimes S \otimes I_{\mathsf{K}})\phi^{\mathsf{h}}(A)(I_{\mathfrak{h}} \otimes T \otimes I_{\mathsf{K}}) \tag{1.5}$$

for $A \in \mathrm{M}(\mathsf{h}; \mathcal{A})_{\mathrm{b}}$ and $S, T \in B(\mathsf{h})$.

2. A product formula

In this section we establish a product formula for iterated quantum stochastic integrals. First we recall some elements of quantum stochastic calculus ([Par],[Mey]) as developed in [LW1] and [LW3] for infinite dimensional noise. After dispensing with bases and defining iterated quantum stochastic integrals we introduce some notations for permuting tensor components in order to state the formula. After the proof we establish simple convergence and injectivity criteria for series of iterated integrals.

Quantum stochastic processes. A convenient exponential domain for present purposes is $\mathcal{E} := \{\varepsilon(f) : f \in \mathbb{S}\}$ where $\mathbb{S}$ is the (*admissible*) set consisting of all functions in the linear span of $\{c\mathbf{1}_{[0,t[} : c \in \mathsf{k}, 0 < t < \infty\}$. The collection of (adapted, weakly measurable) $\mathcal{E}$-processes on a Hilbert space h is denoted $\mathbb{P}(\mathsf{h}, \mathcal{E})$. A process X in $\mathbb{P}(\mathsf{h}, \mathcal{E})$ is *continuous* or *locally square-integrable* if, for each $\zeta \in \mathsf{h} \odot \mathcal{E}$, the map $X(\cdot)\zeta : \mathbb{R}_+ \to \mathsf{h} \otimes \mathcal{F}$ enjoys that property.

If $L \in \mathbb{P}(\mathsf{h} \otimes \widehat{\mathsf{k}}, \mathcal{E})$ is locally square-integrable then, for any orthonormal basis for $\widehat{\mathsf{k}}$ of the form $\widehat{\eta}$, where $\eta = (e_i)_{i \geq 1}$, the matrix of processes $[L^\alpha_\beta]$ on h defined by

$$L^\alpha_\beta := \langle \mathsf{h}, e_\alpha, \mathcal{F} | L(\cdot) | \mathsf{h}, e_\beta, \mathcal{F} \rangle, \quad \alpha, \beta \geq 0 \tag{2.1}$$

is quantum stochastically integrable (in the sense of [LW1], (2.7)) against the matrix of fundamental processes of (Boson Fock) quantum stochastic calculus, defined with respect to the same basis:

$$\begin{bmatrix} \Lambda^0_0(t) & \Lambda^j_0(t) \\ \Lambda^0_i(t) & \Lambda^j_i(t) \end{bmatrix} := I_{\mathsf{h}} \odot \begin{bmatrix} tI_{\mathcal{F}} & a(e_j \mathbf{1}_{[0,t]}) \\ a^*(e_i \mathbf{1}_{[0,t]}) & d\Gamma(|e_i\rangle\langle e_j| \otimes M_{[0,t]}) \end{bmatrix}, \tag{2.2}$$

$a^*, d\Gamma$ and a being respectively Fock space creation, differential second quantisation and annihilation, and $M_{[0,t]}$ being the operator of multiplication by the indicator function $\mathbf{1}_{[0,t]}$ on $L^2(\mathbb{R}_+)$. In the notation (1.1) the resulting integral satisfies

$$\langle x\varepsilon(f), \int_r^t L^\alpha_\beta(s)\, d\Lambda^\beta_\alpha(s)\, y\varepsilon(g) \rangle = \int_r^t ds\, \langle x\widehat{f}(s)\varepsilon(f), L(s) y\widehat{g}(s)\varepsilon(g) \rangle, \tag{2.3}$$

for all $0 \leq r \leq t$, $x, y \in \mathsf{h}$ and $f, g \in \mathbb{S}[\eta] := \{h \in \mathbb{S} : \mathrm{Ran}\, h \subset \mathrm{Lin}\, \eta\}$, in which $\widehat{f}(s) := \widehat{f(s)}$. To obtain a basis-independent integral note that if η_1 and η_2 are orthonormal bases of k, then there is a third basis η_3 for which $\mathbb{S}[\eta_1] \cup \mathbb{S}[\eta_2] \subset \mathbb{S}[\eta_3]$. Thus if X^i is the stochastic integral process constructed from L and the basis η_i for $i = 1, 2, 3$, then (2.3) implies that X^3_t extends both X^1_t and X^2_t. It follows that there is a process $X \in \mathbb{P}(\mathsf{h}, \mathcal{E})$ satisfying $X(t)x\varepsilon(f) = \int_0^t L^\alpha_\beta(s)\, d\Lambda^\beta_\alpha(s)\, x\varepsilon(f)$ whenever $f \in \mathbb{S}[\eta]$ and $[L^\alpha_\beta]$ is obtained from the process L and basis η via (2.1); $X(t)$ is written $\int_0^t L(s)\, d\Lambda(s)$.

If $X = \int_0^\cdot L(s)\, d\Lambda(s)$ and $Y = \int_0^\cdot M(s)\, d\Lambda(s)$, for locally square-integrable processes L and M in $\mathbb{P}(\mathsf{h} \otimes \widehat{\mathsf{k}}, \mathcal{E})$, then the two fundamental formulae and the fundamental estimate take the following form. Let $\zeta = x\varepsilon(f), \eta = y\varepsilon(g) \in \mathsf{h} \odot \mathcal{E}$ and

$0 \le r \le t \le T$ then, in the notation (2.12) introduced below,

$$\langle \zeta, X(t)\eta \rangle = \int_0^t ds \, \langle \zeta(s), L(s)\eta(s) \rangle, \tag{2.4}$$

$$\langle X(t)\zeta, Y(t)\eta \rangle = \int_0^t ds \, \big\{ \langle L(s)\zeta(s), \widehat{Y}(s)\eta(s) \rangle + \langle \widehat{X}(s)\zeta(s), M(s)\eta(s) \rangle + \langle L(s)\zeta(s), (\Delta_{\mathsf{h}} \otimes I_{\mathcal{F}})M(s)\eta(s) \rangle \big\}, \tag{2.5}$$

$$\|[X(t) - X(r)]\zeta\|^2 \le C(f,T) \int_r^t ds \, \|L(s)\zeta(s)\|^2 \tag{2.6}$$

where $\zeta(s) = x\widehat{f}(s)\varepsilon(f)$, $\eta(s) = y\widehat{g}(s)\varepsilon(g)$, and $C(f,T)$ is a constant depending only on f and T. In particular $\int_0^{\cdot} L(s)\, d\Lambda(s)$ is a continuous (and therefore locally square-integrable) process on h.

Iterated quantum stochastic integrals. For $F \in B(\mathsf{h}_n)$ define $\Lambda^n(F) \in \mathbb{P}(\mathsf{h}, \mathcal{E})$ recursively as follows:

$$\Lambda_t^0(F) = F \odot I|_{\mathcal{E}}, \quad \text{and} \quad \Lambda_t^n(F) = \int_0^t \Lambda_s^{n-1}(\widetilde{F})\, d\Lambda(s) \text{ for } n \ge 1,$$

where $\widetilde{F} := U^* F U$ with U being the natural unitary operator $(\mathsf{h} \otimes \widehat{\mathsf{k}})_{n-1} \to \mathsf{h}_n$ determined by

$$x\chi \otimes \chi_1 \cdots \chi_{n-1} \mapsto x \otimes \chi\chi_1 \cdots \chi_{n-1}; \tag{2.7}$$

thus $\widetilde{F} \in B((\mathsf{h}_1)_{n-1})$. The apparently pedantic distinction between $\widetilde{F}$ and F is designed to facilitate acceptance of the elementary combinatorial arguments to follow.

Successive application of (2.4) and (2.6) yields:

$$\langle x\varepsilon(f), \Lambda_t^n(F) y\varepsilon(g) \rangle = e^{\langle f,g \rangle} \int_0^t ds_1 \cdots \int_0^{s_{n-1}} ds_n \, \langle x\widehat{f}^{\otimes n}(\mathbf{s}), Fy\widehat{g}^{\otimes n}(\mathbf{s}) \rangle, \tag{2.8}$$

and

$$\|\Lambda_t^n(F) x\varepsilon(f)\|^2 \le C(f,T)^n \|\varepsilon(f)\|^2 \int_0^t ds_1 \cdots \int_0^{s_{n-1}} ds_n \, \|Fx\widehat{f}^{\otimes n}(\mathbf{s})\|^2, \tag{2.9}$$

for $x, y \in \mathsf{h}$, $f, g \in \mathbb{S}$ and $t \le T$, where $\widehat{f}^{\otimes n}(\mathbf{s}) := \widehat{f}(s_1) \cdots \widehat{f}(s_n)$.

Shuffle notations. For a Hilbert space h, an operator $F \in B(\mathsf{h}_n)$ and an integer $p \ge n$ we need to refer to the operator on h_p which acts as F does on n particular tensor components and as the identity on the others. More precisely, for $\nu \subset \{1, \dots, p\}$ with $\#\nu = n$, define an operator in $B(\mathsf{h}_p)$ by

$$F(\nu; p) := \Pi[\mathsf{h}, \nu; p]^* (F \otimes I_{\mathsf{K}}) \Pi[\mathsf{h}, \nu; p] \tag{2.10}$$

where $\mathsf{K} = \widehat{\mathsf{k}}^{\otimes(p-n)}$ and $\Pi[\mathsf{h}, \nu; p]$ is the unitary operator on h_p which effects the following permutation of tensor components:

$$\xi\chi(1) \cdots \chi(p) \mapsto \xi\chi(\nu_1) \cdots \chi(\nu_n)\chi(\overline{\nu}_1) \cdots \chi(\overline{\nu}_{p-n}).$$

Here $\overline{\nu} := \{1, \dots, p\} \setminus \nu$ and the elements of ν are labelled in increasing order, thus $\nu = \{\nu_1 < \cdots < \nu_n\}$, and likewise for $\overline{\nu}$. Also let

$$\Delta[\mathsf{h}, \nu; p] := (I_{\mathsf{h}} \otimes P^{\otimes n})(\nu; p) \quad \text{and} \quad \Delta[\nu; p] := \Delta[\mathfrak{h}, \nu; p].$$

where P is the orthogonal projection introduced with (1.1).

The inductive steps in Theorems 2.2 and 4.1 below require the construction of various subsets of $\{1,\dots,p+1\}$ from a given subset ν of $\{1,\dots,p\}$. Writing $\nu = \{\nu_1 < \cdots < \nu_n\}$ as above, these subsets are

$$\overset{\circ\to}{\nu}, \text{ where } \overset{\circ\to}{\nu_i} = 1+\nu_i, \qquad \overset{\bullet\to}{\nu} := \{1\} \cup \overset{\circ\to}{\nu}, \qquad \overset{\leftarrow\circ}{\nu} := \nu, \qquad \overset{\leftarrow\bullet}{\nu} := \overset{\leftarrow\circ}{\nu} \cup \{p+1\}. \tag{2.11}$$

A second type of shuffling applies to an operator T on a Hilbert space of the form $\mathsf{h}_n \otimes \mathsf{H}$, where H is either $\mathcal{F}$ or $\mathbb{C}$, and delivers an operator on $(\mathsf{h} \otimes \widehat{\mathsf{k}})_n \otimes \mathsf{H}$ according to the prescription

$$\operatorname{Dom} \widehat{T} = V^*(\widehat{\mathsf{k}} \odot \operatorname{Dom} T), \quad \widehat{T} = V^*(I_1 \odot T)V, \tag{2.12}$$

where V is the tensor flip $(\mathsf{h} \otimes \widehat{\mathsf{k}})_n \otimes \mathsf{H} \to \widehat{\mathsf{k}} \otimes (\mathsf{h}_n \otimes \mathsf{H})$. Thus if T is bounded then $\widehat{T} \in B((\mathsf{h}_1)_n \otimes \mathsf{H})$.

Product formula. The above operations on sets correspond to the hat and tilde operations on quantum stochastic integrands defined through (2.12) and (2.7) respectively.

Lemma 2.1. *Let $F \in B(\mathsf{h}_n)$ and let $p \geq n$. Then*

(a) $\widehat{\Lambda^n_t(F)} \subset \Lambda^n_t(\widehat{F})$, *and for $\nu \subset \{1,\dots,p\}$ with $\#\nu = n$,*

$$\widehat{F}(\nu; p) = F(\overset{\circ\to}{\nu}; p+1).$$

(b) *If $n \geq 1$ then $\Lambda^n_t(F) = \int_0^t \Lambda^{n-1}_s(\widetilde{F})\, d\Lambda(s)$, and for $\mu \subset \{1,\dots,p\}$ with $\#\mu = n-1$,*

$$\widetilde{F}(\mu; p) = F(\overset{\bullet\to}{\mu}; p+1).$$

Proof. Let $x, y \in \mathsf{h}$, $\chi, \delta \in \widehat{\mathsf{k}}$, $f, g \in \mathbb{S}$ and $\varphi, \psi \in \widehat{\mathsf{k}}^{\otimes p}$.
(a) By (2.8), the first part holds by identity

$$\begin{aligned}
&\langle \chi x \varepsilon(f), (I_1 \odot \Lambda^n_t(F))\delta y \varepsilon(g)\rangle \\
&\quad = \langle \chi, \delta\rangle \int_0^t ds_1 \dots \int_0^{s_{n-1}} ds_n \, \langle x \widehat{f}^{\otimes n}(\mathbf{s}), F y \widehat{g}^{\otimes n}(\mathbf{s})\rangle e^{\langle f, g\rangle} \\
&\quad = \int_0^t ds_1 \dots \int_0^{s_{n-1}} ds_n \, \langle x \chi \widehat{f}^{\otimes n}(\mathbf{s}), \widehat{F} y \delta \widehat{g}^{\otimes n}(\mathbf{s})\rangle e^{\langle f, g\rangle}.
\end{aligned}$$

For the second part note that

$$\begin{aligned}
&\langle \Pi[\mathsf{h}_1, \nu; p] x\chi \otimes \varphi, (\widehat{F} \otimes I_{p-n}) \Pi[\mathsf{h}_1, \nu; p] y\delta \otimes \psi\rangle \\
&\quad = \langle \chi, \delta\rangle \langle \Pi[\mathsf{h}, \nu; p] x \otimes \varphi, (F \otimes I_{p-n}) \Pi[\mathsf{h}, \nu; p] y \otimes \psi\rangle \\
&\quad = \langle \Pi[\mathsf{h}, \overset{\circ\to}{\nu}; p+1] x \otimes \chi\varphi, (F \otimes I_{p+1-n}) \Pi[\mathsf{h}, \overset{\circ\to}{\nu}; p+1] y \otimes \delta\psi\rangle.
\end{aligned}$$

(b) The first part holds by definition, and the second follows from the identity

$$\begin{aligned}
&\langle \Pi[\mathsf{h}_1, \mu; p] x\chi \otimes \varphi, (\widetilde{F} \otimes I_{p-(n-1)}) \Pi[\mathsf{h}_1, \mu; p] y\delta \otimes \psi\rangle \\
&\quad = \langle \Pi[\mathsf{h}, \overset{\bullet\to}{\mu}; p+1](x \otimes \chi\varphi), (F \otimes I_{p+1-n}) \Pi[\mathsf{h}, \overset{\bullet\to}{\mu}; p+1](y \otimes \delta\psi)\rangle. \qquad \square
\end{aligned}$$

Theorem 2.2. *Let $F_n, G_n \in B(\mathsf{h}_n)$ for $n = 0, 1, \dots$. Then, for each N,*

$$\sum_{l,m=0}^{N} \Lambda^l_t(F^*_l)^* \Lambda^m_t(G_m) = \sum_{n=0}^{2N} \Lambda^n_t(H_n) \tag{2.13}$$

where

$$H_n = \sum_{\substack{\lambda \cup \mu = \{1,\ldots,n\} \\ \#\lambda, \#\mu \le N}} F_{\#\lambda}(\lambda; n) \Delta[\mathsf{h}, \lambda \cap \mu; n] G_{\#\mu}(\mu; n).$$

Proof. Let $\mathcal{P}_{l,m}$ be the following proposition: for $F \in B(\mathsf{h}_l)$ and $G \in B(\mathsf{h}_m)$,

$$\Lambda_t^l(F^*)^* \Lambda_t^m(G) = \sum_{n \ge 0} \sum_{\substack{\lambda \cup \mu = \{1,\ldots,n\} \\ \#\lambda = l, \#\mu = m}} \Lambda_t^n \big(F(\lambda; n) \Delta[\mathsf{h}, \lambda \cap \mu; n] G(\mu; n) \big). \tag{2.14}$$

Since the right-hand side of (2.13) may be expressed in the form $\sum_{l,m=0}^N \alpha_{l,m}$ where $\alpha_{l,m}$ is the right-hand side of (2.14), the linearity of quantum stochastic integration implies that it suffices to show that $\mathcal{P}_{l,m}$ is true for all $l, m \ge 0$.

If $F \in B(\mathsf{h})$ and $G \in B(\mathsf{h}_m)$ then

$$\begin{aligned} \Lambda_t^0(F^*)^* \Lambda_t^m(G) &= (F \otimes I_{\mathcal{F}}) \Lambda_t^m(G) \\ &= \Lambda_t^m \big((F \otimes I_m) G \big) = \Lambda_t^m \big(F(\emptyset; m) G(\{1, \ldots, m\}; m) \big). \end{aligned}$$

It follows that $\mathcal{P}_{0,m}$ holds; similarly $\mathcal{P}_{l,0}$ holds. Thus $\mathcal{P}_{l,m}$ is true if l or m equals 0; in particular $\mathcal{P}_{0,0}$, $\mathcal{P}_{0,1}$ and $\mathcal{P}_{1,0}$ hold. Suppose that $\mathcal{P}_{l,m}$ holds for $l + m < p$, where $p \ge 2$, and let $F \in B(\mathsf{h}_l)$ and $G \in B(\mathsf{h}_m)$ where $l + m = p$ and $l, m \ge 1$. Let $\zeta = x\varepsilon(f)$, $\eta = y\varepsilon(g) \in \mathsf{h} \odot \mathcal{E}$ then, by (2.5) and part (a) of Lemma 2.1, $\langle \Lambda_t^l(F^*)\zeta, \Lambda_t^m(G)\eta \rangle$ is equal to

$$\begin{aligned} &\Big\langle \int_0^t \Lambda_s^{l-1}(\widetilde{F^*}) \, d\Lambda(s) \, \zeta, \int_0^t \Lambda_s^{m-1}(\widetilde{G}) \, d\Lambda(s) \, \eta \Big\rangle \\ &\quad = \int_0^t ds \, \big\{ \langle \Lambda_s^{l-1}(\widetilde{F^*})\zeta(s), \Lambda_s^m(\widehat{G})\eta(s) \rangle + \langle \Lambda_s^l(\widehat{F^*})\zeta(s), \Lambda_s^{m-1}(\widetilde{G})\eta(s) \rangle \\ &\qquad\qquad + \langle \Lambda_s^{l-1}(\widetilde{F^*})\zeta(s), (\Delta_{\mathsf{h}} \otimes I_{\mathcal{F}}) \Lambda_s^{m-1}(\widetilde{G})\eta(s) \rangle \big\}, \end{aligned} \tag{2.15}$$

where $\zeta(s) = x\widehat{f}(s)\varepsilon(f)$ and $\eta(s) = y\widehat{g}(s)\varepsilon(g)$. Note that $\zeta(s), \eta(s) \in \mathsf{h}_1 \odot \mathcal{E}$ for each $s \ge 0$. Therefore, in view of the identity

$$(\Delta_{\mathsf{h}} \otimes I_{\mathcal{F}}) \Lambda_s^{m-1}(\widetilde{G}) = \Lambda_s^{m-1}(\widetilde{K}) \text{ where } K = \Delta[\mathsf{h}, \{1\}; m]G, \tag{2.16}$$

and the fact that $(l-1+m)$, $(l+m-1)$ and $(l-1+m-1)$ are all less than p, the inductive hypothesis applies to each of the three summands in the integrand (with h_1 in place of h).

It follows that $\Lambda_s^m(\widehat{G})\eta(s) \in \operatorname{Dom} \Lambda_s^{l-1}(\widetilde{F^*})^*$ and similarly for the other two terms and, using (2.4) and (2.16), it also follows that (2.15) equals $\langle \zeta, (I + I' + I'')\eta \rangle$ where

$$I = \int_0^t \Lambda_s^{l-1}(\widetilde{F^*})^* \Lambda_s^m(\widehat{G}) \, d\Lambda(s), \quad I' = \int_0^t \Lambda_s^l(\widehat{F^*})^* \Lambda_s^{m-1}(\widetilde{G}) \, d\Lambda(s), \text{ and}$$

$$I'' = \int_0^t \Lambda_s^{l-1}(\widetilde{F^*})^* \Lambda_s^{m-1}(\widetilde{K}) \, d\Lambda(s).$$

In particular $\Lambda_t^m(G)(\mathsf{h} \odot \mathcal{E}) \subset \operatorname{Dom} \Lambda_t^l(F^*)^*$. By the inductive hypothesis, the linearity of quantum stochastic integration and Lemma 2.1,

$$\begin{aligned} I &= \sum_{n \ge 0} \sum_{\lambda \cup \mu = \{1,\ldots,n\}} \int_0^t \Lambda_s^n \big(\widetilde{F}(\lambda; n) \Delta[\mathsf{h}_1, \lambda \cap \mu; n] \widehat{G}(\mu; n) \big) \, d\Lambda(s) \\ &= \sum_{n \ge 0} \sum_{\lambda \cup \mu = \{1,\ldots,n\}} \Lambda_t^{n+1} \Big(F(\overset{\bullet\to}{\lambda}; n+1) \Delta[\mathsf{h}, \overset{\bullet\to}{\lambda} \cap \overset{\circ\to}{\mu}; n+1] G(\overset{\circ\to}{\mu}; n+1) \Big), \end{aligned}$$

and by symmetry,

$$I' = \sum_{n \ge 0} \sum_{\lambda \cup \mu = \{1,\ldots,n\}} \Lambda_t^{n+1} \Big(F(\overset{\circ\to}{\lambda}; n+1) \Delta[\mathsf{h}, \overset{\circ\to}{\lambda} \cap \overset{\bullet\to}{\mu}; n+1] G(\overset{\bullet\to}{\mu}; n+1) \Big).$$

Also, by Lemma 2.1

$$\begin{aligned} I'' &= \sum\nolimits_{n\geq 0} \sum\nolimits_{\lambda\cup\mu=\{1,\dots,n\}} \int_0^t \Lambda_s^n\big(\widetilde{F}(\lambda;n)\Delta[\mathsf{h}_1,\lambda\cap\mu;n]\widetilde{K}(\mu;n)\big)\,d\Lambda(s) \\ &= \sum\nolimits_{n\geq 0} \sum\nolimits_{\lambda\cup\mu=\{1,\dots,n\}} \Lambda_t^{n+1}\Big(F(\overset{\bullet\to}{\lambda};n+1)\Delta[\mathsf{h},\overset{\bullet\to}{\lambda}\cap\overset{\bullet\to}{\mu};n+1]G(\overset{\bullet\to}{\mu};n+1)\Big). \end{aligned}$$

Now, for an ordered pair of subsets (λ',μ') of $\{1,\dots,n+1\}$ whose union is $\{1,\dots,n+1\}$ there are three mutually exclusive possibilities: $1\in\lambda'\setminus\mu', 1\in\mu'\setminus\lambda'$ and $1\in\lambda'\cap\mu'$ — in other words, (λ',μ') equals either, $(\overset{\bullet\to}{\lambda},\overset{\circ\to}{\mu})$, $(\overset{\circ\to}{\lambda},\overset{\bullet\to}{\mu})$ or $(\overset{\bullet\to}{\lambda},\overset{\bullet\to}{\mu})$ for a (unique) ordered pair of subsets (λ,μ) of $\{1,\dots,n\}$ satisfying $\lambda\cup\mu=\{1,\dots,n\}$. It follows that $I+I'+I''$ equals

$$\sum\nolimits_{n\geq 0}\sum\nolimits_{\lambda'\cup\mu'=\{1,\dots,n+1\}} \Lambda_t^{n+1}\big(F(\lambda';n+1)\Delta[\mathsf{h},\lambda'\cap\mu';n+1]G(\mu';n+1)\big),$$

and since $l,m\geq 1$ we may safely change variables to obtain

$$\sum\nolimits_{n'\geq 0}\sum\nolimits_{\lambda\cup\mu=\{1,\dots,n'\}} \Lambda_t^{n'}\big(F(\lambda;n')\Delta[\mathsf{h},\lambda\cap\mu;n']G(\mu;n')\big),$$

so that $\mathcal{P}_{l,m}$ holds when $l+m=p$ too. The proof therefore follows by induction. □

Proposition 2.3. *Let* $F_n\in B(\mathsf{h}_n)$ *for* $n=0,1,\dots$ *satisfy* $\|F_n\|\leq C_1C_2^n$ *for constants* C_1 *and* C_2. *Then*

(a) $\sum_{n\geq 0}\Lambda_t^n(F_n)$ *converges absolutely on* $\mathsf{h}\odot\mathcal{E}$, *and*
(b) *if* $\sum_{n\geq 0}\Lambda_t^n(F_n)\zeta=0$ *for all* $\zeta\in\mathsf{h}\odot\mathcal{E}$ *and* $t\geq 0$, *then* $F_n=0$ *for all* $n\geq 0$.

Proof. (a) follows immediately from (2.9). Suppose therefore that $\sum_{n\geq 0}\Lambda_t^n(F_n)$ converges to 0 on $\mathsf{h}\odot\mathcal{E}$ for each $t\geq 0$. Let $\mathcal{P}_p$ be the proposition

$$\langle\varphi\varepsilon(f),\sum\nolimits_{n\geq 0}\Lambda_t^n(F_{n+p})\psi\varepsilon(g)\rangle=0\quad\forall\varphi,\psi\in\mathsf{h}_p^0, f,g\in\mathbb{S},t\geq 0,$$

where the dense subspace h_p^0 of h_p is defined in (1.2). Note that $\mathcal{P}_0$ holds by assumption and (setting $t=0$) $\mathcal{P}_p$ implies that $F_p=0$.

Assume that $\mathcal{P}_p$ holds for $p=0,\dots,k$. By (2.8) and the Dominated Convergence Theorem, for all $\varphi,\psi\in\mathsf{h}_k^0$ and $f,g\in\mathbb{S}$,

$$\int_0^r ds\sum\nolimits_{n\geq 1}\int_0^s dt_1\dots\int_0^{t_{n-2}}dt_{n-1}\,\langle\varphi\widehat{f}(s)\widehat{f}^{\otimes(n-1)}(\mathbf{t}),F_{n+k}\psi\widehat{g}(s)\widehat{g}^{\otimes(n-1)}(\mathbf{t})\rangle$$

vanishes for all $r\geq 0$. Since the integrand function of s is a uniform limit of functions which are continuous on the common intervals of continuity (constancy) of f and g it must vanish identically. Choosing f and g such that $f(r)=c$ and $g(r)=d$ for r in an interval with left endpoint T, and setting $s=T$, shows that

$$\sum\nolimits_{n\geq 1}\int_0^T dt_1\cdots\int_0^{t_{n-2}}dt_{n-1}\,\langle\varphi\widehat{c}\widehat{f}^{\otimes(n-1)}(\mathbf{t}),F_{n+k}\psi\,\widehat{d}\,\widehat{g}^{\otimes(n-1)}(\mathbf{t})\rangle=0.$$

Since $f_{[0,T[}$ and $g_{[0,T[}$ were unconstrained, this identity therefore holds for all $c,d\in\mathsf{k},\varphi,\psi\in\mathsf{h}_k^0, f,g\in\mathbb{S}$ and $T>0$. Changing variable this implies that

$$\sum\nolimits_{m\geq 0}\int_0^t dt_1\cdots\int_0^{t_{m-1}}dt_m\,\langle\varphi\widehat{c}\widehat{f}^{\otimes m}(\mathbf{t}),\,F_{m+k+1}\psi\,\widehat{d}\,\widehat{g}^{\otimes m}(\mathbf{t})\rangle=0$$

and so, by (2.8) and the fact that $\widehat{\mathsf{k}}=\mathrm{Lin}\{\widehat{c}:c\in\mathsf{k}\}$, $\mathcal{P}_p$ holds for $p=k+1$. Therefore $\mathcal{P}_p$ holds for all p, so that $F_0,F_1,\dots$ are all zero. □

Remark. An alternative proof of (b) may be constructed from the identity

$$\sum_{n\geq 1} \Lambda_t^n(F_n) = \int_0^t \Big(\sum_{n\geq 1} \Lambda_s^{n-1}(\widetilde{F_n}) \Big)\, d\Lambda(s),$$

noting that the integrand process is continuous and appealing to the independence of quantum stochastic integrators ([LW1], Proposition 2.2).

3. Structure relations

The product formula and convergence and injectivity criteria of the previous section lead easily to necessary and sufficient conditions for a k-bounded operator to be the stochastic generator of a homomorphic Markovian cocycle on $\mathcal{A}$, in terms of a sequence of identities.

The Feller property. For a classical Markov semigroup, on a locally compact Hausdorff space with countable base, the *Feller property* is invariance of the algebra of continuous functions vanishing at infinity, together with strong continuity on this (C^*-)algebra ([Sha]); a *Feller process* is a Markov process whose semigroup is Feller. (Here we pass on Rogers and Williams' warning: "Every author has his or her own definition ... " — these authors face this by referring to *Feller-Dynkin* semigroups ([RoW])). A significant implication of this assumption is that every Feller process has a modification whose sample paths are càdlàg (right continuous with left limits). For many Markov processes of interest for applications the invariance condition entails the strong continuity property ([Lig]). All Lévy processes, such as the Poisson process and Brownian motion in $\mathbb{R}^d$ are Feller processes ([Ber]); a wide class of processes governed by a stochastic differential equation driven by a Lévy process are Feller too ([App]). As a noncommutative example, the Dirac Laplacian on the Clifford C^*-algebra of a complete Riemannian manifold without boundary generates a Feller semigroup if and only if the curvature operator of the manifold is nonnegative ([CiS]). In noncommutative topology a C^*-algebra is thought of as a *noncommutative space*, in view of the Gelfand-Naimark Theorem characterising commutative C^*-algebras. The above C^*-algebra actually consists of the continuous sections of the Clifford bundle over the manifold which vanish at infinity.

Feller cocycles. Consider the quantum stochastic differential equation

$$dk_t = k_t \circ \theta^\alpha_\beta \, d\Lambda^\beta_\alpha(t), \quad k_0(a) = a \otimes I_{\mathcal{F}} \tag{3.1}$$

([Par],[LW1]), in which $[\Lambda^\alpha_\beta]$ is the matrix of fundamental processes (2.2), defined with respect to a basis $\eta = (e_i)_{i\geq 1}$ for k and $[\theta^\alpha_\beta]$ is a matrix of bounded operators on $\mathcal{A}$. There is at most one (weakly regular, weak) solution of (3.1) ([LW1], Theorem 3.1). When a solution k exists we refer to it as the *Markovian cocycle generated* by $[\theta^\alpha_\beta]$ in the basis η (see [LW2]), and $[\theta^\alpha_\beta]$ as the *stochastic generator* of k in the basis η. With noncommutative topology in mind, we use the term *Feller cocycle* when $\mathcal{A}$ is *not* a von Neumann algebra (cf. Remark (iii) below).

The following existence theorem is from [LW3].

Theorem 3.1. *Let θ be a k-bounded operator $\mathcal{A} \to \mathrm{M}(\widehat{\mathsf{k}};\mathcal{A})_\mathrm{b}$. Then there is a (strongly regular) process k on $\mathcal{A}$ which (strongly) satisfies* (3.1) *for $\theta^\alpha_\beta := E^{e(\alpha)}\theta(\cdot)E_{e(\beta)}$ and any basis of the form $\widehat{\eta} = (e(\alpha))_{\alpha\geq 0}$ for $\widehat{\mathsf{k}}$.*

Remarks. (i) In [LW3] the theorem is stated for a unital C^*-algebra, but the proof is valid for any operator space — in particular for a nonunital C^*-algebra.

(ii) For each $t \geq 0$ and $a \in \mathcal{A}$, $\operatorname{Dom} k_t(a)^* \supset \mathfrak{h} \odot \mathcal{E}$ and $(k_t^\dagger : a \mapsto k_t(a^*)^*|_{\mathfrak{h}\odot\mathcal{E}})_{t\geq 0}$ is the Markovian cocycle generated by the k-bounded operator $\theta^\dagger : a \mapsto \theta(a^*)^*$.

(iii) For each $\varepsilon, \varepsilon' \in \mathcal{E}$, the maps $E^{\varepsilon} k_t(\cdot) E_{\varepsilon'}$ $(t \geq 0)$ leave $\mathcal{A}$ invariant, moreover $t \mapsto E^{\varepsilon} k_t(\cdot) E_{\varepsilon'}$ is norm continuous. In particular this is true of the Markovian semigroup associated with k, namely $(\mathbb{E}_0 \circ k_t = E^{\varepsilon(0)} k_t(\cdot) E_{\varepsilon(0)})_{t \geq 0}$, which has the k-bounded operator θ^0_0 as its generator.

The next result demonstrates the importance of complete boundedness for stochastic generators.

Proposition 3.2. *Let k be a completely positive contraction process on $\mathcal{A}$ which (weakly) satisfies* (3.1) *for some matrix $[\theta^\alpha_\beta]$ of bounded operators on $\mathcal{A}$ and basis η for k. Then, in terms of the basis $\widehat{\eta} = (e(\alpha))_{\alpha \geq 0}$ for $\widehat{\mathsf{k}}$, $\theta^\alpha_\beta = E^{e(\alpha)} \theta(\cdot) E_{e(\beta)}$ for a completely bounded operator $\theta : \mathcal{A} \to \mathrm{M}(\widehat{\mathsf{k}}; \mathcal{A})_{\mathrm{b}}$.*

Proof. If $\mathcal{A}$ is nonunital then defining ${}^u\mathcal{A}$, ${}^u k_t$ and $[{}^u\theta^\alpha_\beta]$ respectively by ${}^u\mathcal{A} = C^*(\mathcal{A}, I_{\mathfrak{h}})$, ${}^u k_t(a + \lambda I_{\mathfrak{h}}) = k_t(a) + \lambda I_{\mathfrak{h} \otimes \mathcal{F}}$ and ${}^u\theta^\alpha_\beta(a + \lambda I_{\mathfrak{h}}) = \theta^\alpha_\beta(a)$ reduces the proposition to the unital case where Proposition 5.1 and Theorem 5.2 of [LW1] together imply the conclusion. □

Before deriving the necessary and sufficient conditions on θ for the (possibly unbounded) cocycle k to be multiplicative, we note that the quantum stochastic differential equation (3.1) and the cocycle property can be cast into more satisfactory forms when k enjoys some regularity.

Firstly, if θ is a k-bounded map $\mathcal{A} \to \mathrm{M}(\widehat{\mathsf{k}}; \mathcal{A})_{\mathrm{b}}$ and the solution k of (3.1) (in which $[\theta^\alpha_\beta]$ and $[\Lambda^\alpha_\beta]$ are determined by *some* basis for k) is itself k-bounded — as it is in most cases of interest — then k satisfies the following basis independent stochastic integral equation:

$$k_t(a) = a \otimes I_{\mathcal{F}} + \int_0^t \widehat{k_s}(\theta(a))\, d\Lambda(s),$$

where $\widehat{k_s}$ denotes $(k_s)^{\widehat{\mathsf{k}}}$ in the notation (1.4). Continuity of the integrand process on $\mathfrak{h}^0_1 \odot \mathcal{E}$ is ensured by (3.9) of [LW1]. In turn k satisfies (3.1) with $[\theta^\alpha_\beta]$ and $[\Lambda^\alpha_\beta]$ now determined (from θ) by *any* basis for k.

Secondly, in [LW2] the cocycle property is encoded using the following family of maps on $\mathcal{A}$: $\{E^{\varepsilon(f)} k_t(\cdot) E_{\varepsilon(g)} : f, g \in \mathbb{S}, t \geq 0\}$. When the process k is completely bounded this formulation simplifies to the recognisable cocycle identity

$$k_{s+t} = \widehat{k}_s \circ \sigma_s \circ k_t,$$

where σ is the semigroup of (right) shifts on $\mathrm{M}(\mathcal{F}; \mathcal{A})_{\mathrm{b}}$, and, in the notation (1.3),

$$\widehat{k}_s := \Psi_s \circ \widetilde{k}_s^{(\mathcal{F}^s)} \circ \Phi_s.$$

Here Φ_s and Ψ_s are the natural isomorphisms

$$\sigma_s(\mathrm{M}(\mathcal{F}; \mathcal{A})_{\mathrm{b}}) \xrightarrow{\Phi_s} \mathrm{M}(\mathcal{F}^s; \mathcal{A})_{\mathrm{b}} \quad \text{and} \quad \mathrm{M}(\mathcal{F}^s; \mathrm{M}(\mathcal{F}_s; \mathcal{A})_{\mathrm{b}})_{\mathrm{b}} \xrightarrow{\Psi_s} \mathrm{M}(\mathcal{F}; \mathcal{A})_{\mathrm{b}},$$

$\mathcal{F}_s$ and $\mathcal{F}^s$ denote the symmetric Fock spaces over $L^2([0, s]; \mathsf{k})$ and $L^2([s, \infty[; \mathsf{k})$ respectively, and $\widetilde{k}_s : \mathcal{A} \to \mathrm{M}(\mathcal{F}_s; \mathcal{A})_{\mathrm{b}}$ is defined via $k_s(a) = \Psi_s(\widetilde{k}_s(a) \otimes I^s)$, I^s being the identity operator on $\mathcal{F}^s$.

Homomorphic cocycles. By Proposition 3.2 a necessary condition for a matrix $[\theta^\alpha_\beta]$ of maps in $B(\mathcal{A})$ to stochastically generate a *-homomorphic Markovian cocycle on $\mathcal{A}$ (in a basis η) is that it is the component matrix of a completely bounded map θ. By Theorem 6.5 of [LW1] a further necessary condition is that θ satisfy

$$\theta(ab) = \theta(a)\iota(b) + \iota(a)\theta(b) + \theta(a)\Delta\theta(b). \tag{3.2}$$

Since the cocycle is real (that is $k^\dagger = k$) if and only if its stochastic generator is real ([LW1], Proposition 3.2) the next result entails necessary and sufficient conditions.

First some notation. For a k-bounded operator $\theta : \mathcal{A} \to \mathrm{M}(\widehat{\mathsf{k}};\mathcal{A})_{\mathrm{b}}$ define $\theta_n \in \mathsf{k}\text{-}B(\mathcal{A};\mathrm{M}(\widehat{\mathsf{k}}^{\otimes n};\mathcal{A})_{\mathrm{b}})$ by

$$\theta_0 := \mathrm{id}_{\mathcal{A}}, \quad \text{and} \quad \theta_n := \theta^n \circ \cdots \circ \theta^1 \quad \text{for } n \geq 1$$

where $\theta^n := \theta^{\mathsf{K}}$ for $\mathsf{K} = \widehat{\mathsf{k}}^{\otimes(n-1)}$, and θ^{K} is defined in (1.4). These satisfy $\|\theta_n\| \leq (C_\theta)^n$ where

$$C_\theta = \begin{cases} (1+\dim \mathsf{k})\,\|\theta\| & \text{if } \dim \mathsf{k} < \infty \\ \|\theta\|_{\mathrm{cb}} & \text{otherwise .} \end{cases} \tag{3.3}$$

In the notations

$$\theta^\chi_\delta := E^\chi \theta(\cdot)E_\delta, \quad \theta^\chi := E^\chi\theta(\cdot) \text{ and } \theta_\delta := \theta(\cdot)E_\delta,$$

the following identity is relevant

$$E^{\chi(1)\cdots\chi(n)}\theta_n(a)E_{\delta(1)\cdots\delta(n)} = \theta^{\chi(n)}_{\delta(n)} \circ \cdots \theta^{\chi(1)}_{\delta(1)}(a). \tag{3.4}$$

Recall the shuffle notation (2.10).

Theorem 3.3. *Let θ be a k-bounded operator $\mathcal{A} \to \mathrm{M}(\widehat{\mathsf{k}};\mathcal{A})_{\mathrm{b}}$ and let k be the Markovian cocycle on $\mathcal{A}$ generated by θ. Then the following are equivalent*:

(i) *k is weakly multiplicative, that is*

$$k^\dagger_t(a^*)^*k_t(b) = k_t(ab).$$

(ii) *For all $n \geq 1$, θ satisfies*

$$\theta_n(ab) = \sum_{\lambda\cup\mu=\{1,\dots,n\}} \theta_{\#\lambda}(a)(\lambda;n)\Delta[\lambda\cap\mu;n]\theta_{\#\mu}(b)(\mu;n). \tag{3.5}$$

Proof. Let $a,b \in \mathcal{A}$ and set $F_n = \theta_n(a,b) - \theta_n(ab)$, where $\theta_n(a,b)$ is the right-hand side of (3.5). The estimate preceding (3.3), implies that $\|F_n\| \leq 2\|a\|(2+C_\theta)^{2n}\|b\|$, therefore by Proposition 2.3, $\sum_{n\geq 0}\Lambda^n_t(F_n)$ converges absolutely on $\mathfrak{h} \odot \mathcal{E}$. Now $\sum_{n=0}^N \Lambda^n_t(\theta_n(c))$ amounts to a basis independent expression for the N^{th} Picard approximant to $k_t(c)$ ([LW3],[LW1], cf. [HLP]), and similarly for $k^\dagger_t$. Therefore, by (2.8) and Theorem 2.2,

$$\begin{aligned}\langle k^\dagger_t(a^*)\zeta, k_t(b)\eta\rangle &- \langle\zeta, k_t(ab)\eta\rangle \\ &= \sum\nolimits_{l,m\geq 0}\langle\Lambda^l_t(\theta^\dagger_l(a^*))\zeta, \Lambda^m_t(\theta_m(b))\eta\rangle - \sum\nolimits_{n\geq 0}\langle\zeta, \Lambda^n_t(\theta_n(ab))\eta\rangle \\ &= \langle\zeta, \sum\nolimits_{n\geq 0}\Lambda^n_t\big(\theta_n(a,b) - \theta_n(ab)\big)\eta\rangle,\end{aligned}$$

for $\zeta,\eta \in \mathfrak{h} \odot \mathcal{E}$. The result therefore follows from Proposition 2.3. □

Remark. There is no claim here that the individual summands in (3.5) lie in $\mathrm{M}(\widehat{\mathsf{k}}^{\otimes n};\mathcal{A})_{\mathrm{b}}$.

4. Sufficiency and Dilation

Identity (3.2) coincides with the case $n = 1$ of the structure relations (3.5). In turn formal iteration of (3.2) leads to (3.5). The question now is: when is this justified? The next result supplies two answers, the second of which covers all previously considered cases.

Theorem 4.1. *The Markovian cocycle generated by a k-bounded operator $\theta : \mathcal{A} \to \mathrm{M}(\widehat{\mathsf{k}};\mathcal{A})_{\mathrm{b}}$ is weakly multiplicative under either of the following conditions*:

(a) *For each $n \geq 0$, $a, b \in \mathcal{A}$ and $\lambda, \mu \subset \mathbb{N}$ satisfying $\lambda \cup \mu = \{1, \dots, n\}$, the operators $A = \theta_{\#\lambda}(a)(\lambda; n)\Delta[\lambda \cap \mu; n]$ and $B = \theta_{\#\mu}(b)(\mu; n)$ satisfy*

$$AB \in \mathrm{M}(\widehat{\mathsf{k}}^{\otimes n}; \mathcal{A})_{\mathrm{b}}, \text{ and}$$
$$\theta^{n+1}(AB) = \theta^{n+1}(A)\iota(B) + \iota(A)\theta^{n+1}(B) + \theta^{n+1}(A)\Delta_{\mathfrak{h}_n}\theta^{n+1}(B).$$

(b) *θ satisfies (3.2) and, for some orthonormal basis $(e(i))_{i \geq 1}$ for $\mathsf{k} \subset \widehat{\mathsf{k}}$,*

$$\theta_n(\theta^\chi(a)\Delta\theta_\delta(b)) = \text{w.o.}\sum_{i \geq 1} \theta_n(\theta^\chi_{e(i)}(a)\theta^{e(i)}_\delta(b))$$

for all $n \geq 1$, $a, b \in \mathcal{A}$ and $\chi, \delta \in \widehat{\mathsf{k}}$.

Proof. By Theorem 3.3 it suffices to prove that, under either of the conditions (a) or (b), $\mathcal{I}_n$ holds for all n, where $\mathcal{I}_n$ is the identity

$$\theta_n(ab) = \sum_{\lambda \cup \mu = \{1, \dots, n\}} \theta[a, \lambda; n]\Delta[\lambda \cap \mu; n]\theta[b, \mu; n], \quad a, b \in \mathcal{A}$$

in the notation $\theta[a, \lambda; n] := \theta_{\#\lambda}(a)(\lambda; n)$. Note that $\mathcal{I}_n$ reduces to (3.2) in case $n = 1$; assume that $\mathcal{I}_n$ holds for $n \leq p$.

(a) First note that for $\lambda \subset \{1, \dots, p\}$,

$$\theta^{p+1}\big(\theta[a, \lambda; p]\big) = \theta[a, \overset{\leftarrow\bullet}{\lambda}; p+1] \text{ and } \iota\big(\theta[a, \lambda; p]\big) = \theta[a, \overset{\leftarrow\circ}{\lambda}; p+1]$$

in the notation (2.11), and so, by (1.5), if also $\nu \subset \{1, \dots, p\}$ then

$$\theta^{p+1}\big(\theta[a, \lambda; p]\Delta[\nu; p]\big) = \theta[a, \overset{\leftarrow\bullet}{\lambda}; p+1]\Delta[\overset{\leftarrow\circ}{\nu}; p+1].$$

Thus, for $a, b \in \mathcal{A}$, $\theta_{p+1}(ab) = \theta^{p+1}(\theta_p(ab))$ which equals

$$\begin{aligned}
&\sum_{\lambda \cup \mu = \{1, \dots, p\}} \theta^{p+1}\big(\theta[a, \lambda; p]\Delta[\lambda \cap \mu; p]\theta[b, \mu; p]\big) \\
&\quad = \sum_{\lambda \cup \mu = \{1, \dots, p\}} \Big\{\theta^{p+1}\big(\theta[a, \lambda; p]\Delta[\lambda \cap \mu; p]\big)\iota\big(\theta[b, \mu; p]\big) \\
&\qquad\qquad + \iota\big(\theta[a, \lambda; p]\Delta[\lambda \cap \mu; p]\big)\theta^{p+1}\big(\theta[b, \mu; p]\big) \\
&\qquad\qquad + \theta^{p+1}\big(\theta[a, \lambda; p]\Delta[\lambda \cap \mu; p]\big)\Delta_{\mathfrak{h}_p}\theta^{p+1}\big(\theta[b, \mu; p]\big)\Big\} \\
&\quad = \sum_{\lambda \cup \mu = \{1, \dots, p\}} \Big\{\theta[a, \overset{\leftarrow\bullet}{\lambda}; p+1]\Delta[\overset{\leftarrow\bullet}{\lambda} \cap \overset{\leftarrow\circ}{\mu}; p+1]\theta[b, \overset{\leftarrow\circ}{\mu}; p+1] \\
&\qquad\qquad + \theta[a, \overset{\leftarrow\circ}{\lambda}; p+1]\Delta[\overset{\leftarrow\circ}{\lambda} \cap \overset{\leftarrow\bullet}{\mu}; p+1]\theta[b, \overset{\leftarrow\bullet}{\mu}; p+1] \\
&\qquad\qquad + \theta[a, \overset{\leftarrow\bullet}{\lambda}; p+1]\Delta[\overset{\leftarrow\bullet}{\lambda} \cap \overset{\leftarrow\bullet}{\mu}; p+1]\theta[b, \overset{\leftarrow\bullet}{\mu}; p+1]\Big\} \\
&\quad = \sum_{\lambda' \cup \mu' = \{1, \dots, p+1\}} \theta[a, \lambda'; p+1]\Delta[\lambda' \cap \mu'; p+1]\theta[b, \mu'; p+1],
\end{aligned}$$

where the last step is achieved by a change of variable (cf. the proof of Theorem 2.2). Thus by induction $\mathcal{I}_n$ holds for all n.

(b) For this case note that, in the bra- and -ket notation, (3.4) gives the identity

$$\theta[\theta^\chi_\delta(a), \lambda; p] = \langle \mathfrak{h}, \chi, \widehat{\mathsf{k}}^{\otimes p}|\, \theta[a, \overset{\bullet\rightarrow}{\lambda}; p+1]\, |\mathfrak{h}, \delta, \widehat{\mathsf{k}}^{\otimes p}\rangle. \tag{4.1}$$

Letting $(e(i))_{i \geq 1}$ be an orthonormal basis for k for which condition (b) holds,

$$\langle \mathfrak{h}, \chi, \widehat{\mathsf{k}}^{\otimes p}|\theta_{p+1}(ab)|\mathfrak{h}, \delta, \widehat{\mathsf{k}}^{\otimes p}\rangle = \theta_p(\theta^\chi_\delta(ab)) = \theta_p\big(\theta^\chi_\delta(a)b + a\theta^\chi_\delta(b) + \theta^\chi(a)\Delta\theta_\delta(b)\big)$$

which, by the inductive assumption, equals

$$\sum_{\lambda\cup\mu=\{1,\ldots,p\}} \Big\{\theta[\theta^\chi_\delta(a),\lambda;p]\Delta[\lambda\cap\mu;p]\theta[b,\mu;p] + \theta[a,\lambda;p]\Delta[\lambda\cap\mu;p]\theta[\theta^\chi_\delta(b),\mu;p]$$

$$+ \text{w.o.}\sum\nolimits_{i\geq 1}\theta[\theta^\chi_{e(i)}(a),\lambda;p]\Delta[\lambda\cap\mu;p]\theta[\theta^{e(i)}_\delta(b),\mu;p]\Big\}.$$

Applying (4.1) and then summing over i yields

$$\sum_{\lambda\cup\mu=\{1,\ldots,p\}} \langle\mathfrak{h},\chi,\widehat{\mathsf{k}}^{\otimes p}|\,T_{\lambda,\mu}\,|\widehat{\mathsf{k}}^{\otimes p},\delta,\mathfrak{h}\rangle,$$

where

$$\begin{aligned} T_{\lambda,\mu} &= \theta[a,\overset{\bullet\to}{\lambda};p+1]\Delta[\overset{\bullet\to}{\lambda}\cap\overset{\circ\to}{\mu};p+1]\theta[b,\overset{\circ\to}{\mu};p+1] \\ &\quad+\theta[a,\overset{\circ\to}{\lambda};p+1]\Delta[\overset{\circ\to}{\lambda}\cap\overset{\bullet\to}{\mu};p+1]\theta[b,\overset{\bullet\to}{\mu};p+1] \\ &\quad+\theta[a,\overset{\bullet\to}{\lambda};p+1]\Delta[\overset{\bullet\to}{\lambda}\cap\overset{\bullet\to}{\mu};p+1]\theta[b,\overset{\bullet\to}{\mu};p+1]. \end{aligned}$$

The proof is once again completed by a change of variable. □

Corollary 4.2. *The Markovian cocycle generated by a* k*-bounded operator* $\theta:\mathcal{A}\to \mathrm{M}(\widehat{\mathsf{k}};\mathcal{A})_{\mathrm{b}}$ *satisfying* (3.2) *is weakly multiplicative in each of the following cases*:

(α_1) $\theta(a)E_\chi\in\mathcal{A}\otimes_{\mathrm{sp}}|\widehat{\mathsf{k}}\rangle$ *for all* $a\in\mathcal{A},\chi\in\widehat{\mathsf{k}}$.
(α_2) $\theta(\mathcal{A})\subset\mathcal{A}\otimes_{\mathrm{sp}}B(\widehat{\mathsf{k}})$.
(β_1) *Each* θ_n *is sequentially strong operator to weak operator continuous.*
(β_2) $\mathcal{A}$ *is a von Neumann algebra and* θ *is ultraweakly continuous.*

Proof. Note that

$$\theta^\chi(a)\Delta\theta_\delta(b) = \text{s.o.}\sum\nolimits_{i\geq1}\theta^\chi_{e(i)}(a)\theta^{e(i)}_\delta(b) \tag{4.2}$$

for any orthonormal basis $(e(i))_{i\geq1}$ for $\mathsf{k}\subset\widehat{\mathsf{k}}$.

(α_1) In this case the convergence (4.2) is in norm, and so the (norm) continuity of θ_n ensures that Condition (b) holds. In fact Condition (a) holds too.

(α_2) This is clearly a special case of (α_1).

(β_1) Condition (b) is obviously satisfied, in view of (4.2).

(β_2) Since ultraweak continuity is inherited by θ_n, this is a special case of (β_1). Since identity (3.2) is also inherited, it is a special case of Condition (a) as well. □

When $\mathcal{A}$ is unital and θ is real (α_1) is equivalent to $\theta(\mathcal{A})\subset\mathcal{M}(\mathcal{A}\otimes\mathcal{K}(\widehat{\mathsf{k}}))$, the multiplier algebra of the tensor product of $\mathcal{A}$ with the compact operators on $\widehat{\mathsf{k}}$.

Remarks. (i) The cases (α_1) and (β_2) succumb to a short direct proof ([LW4]).

(ii) The case of finite dimensional noise considered by Evans is covered by (α_2).

(iii) The case considered by Mohari and Sinha is covered by (β_1) since they dealt with regular *-homomorphic cocycles — whose stochastic generators are necessarily completely bounded (by Proposition 3.2) — and their regularity condition applied to a completely bounded operator θ implies strong operator continuity of each θ_n on bounded sets (by [LW1], (1.20)).

(iv) Both (α_2) and (β_2) cover the case of a finite dimensional algebra. Note that k-boundedness follows from boundedness in this case ([EfR], Corollary 2.2.4)

(v) More generally if $\mathcal{A}=\overline{\bigcup_n\mathcal{A}_n}$ and $\theta(\mathcal{A}_n)\subset\mathcal{A}\otimes_{\mathrm{sp}}B(\widehat{\mathsf{k}})$ for each n then (α_2) holds. Notwithstanding the examples below, this may arise for an AF algebra.

Examples. Let $\mathcal{A}=c$, the (unital) C^*-algebra of convergent sequences — acting on $\mathfrak{h}=l^2$, let $\mathsf{k}=l^2$, identify $\mathfrak{h}\otimes\widehat{\mathsf{k}}$ with $\bigoplus_{\alpha\geq0}\mathfrak{h}$ and let $d_k\in c$ be the sequence $(0,\ldots,0,1,0,\ldots)$ with 1 in the k^{th} place.

(a) Our first example illustrates the distinction between conditions (α_1) and (α_2). Let $Q \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ be the orthogonal projection with range $0 \oplus (d_1\mathfrak{h})^\perp \oplus (d_2\mathfrak{h})^\perp \oplus \cdots$. Then $Q \in \mathrm{M}(\widehat{\mathsf{k}}; \mathcal{A})_\mathrm{b}$ and so $a \mapsto -(a \otimes 1)Q$ defines a completely bounded map $\theta : \mathcal{A} \to \mathrm{M}(\widehat{\mathsf{k}}; \mathcal{A})_\mathrm{b}$. It is easily checked that θ satisfies (3.2), and also that $QE_\chi \in \mathcal{A} \otimes_\mathrm{sp} |\widehat{\mathsf{k}}\rangle$, so that (α_1) holds. However it is not hard to see that $Q \notin \mathcal{A} \otimes_\mathrm{sp} B(\widehat{\mathsf{k}})$, and so (α_2) fails.

(b) Our second example shows how (α_1) can fail. Define $\theta : \mathcal{A} \to \mathrm{M}(\widehat{\mathsf{k}}; \mathcal{A})_\mathrm{b}$ by $\theta(a) = \left[\begin{smallmatrix} -a & al \\ l^*a & -a \otimes I_\mathsf{k} \end{smallmatrix}\right]$ where $l = [d_1 \; d_2 \; d_3 \; \cdots]$. Then θ is completely bounded and satisfies (3.2), and both Conditions (a) and (b) of Theorem 4.1. However (α_1) is violated since $\theta(1)E_{(0)} = \left[\begin{smallmatrix} 1 \\ l^* \end{smallmatrix}\right] \notin \mathcal{A} \otimes_\mathrm{sp} |\widehat{\mathsf{k}}\rangle$.

In both cases the Feller cocycle generated by θ is *-homomorphic.

Cocycle as C^*-dilation. The sufficient criterion for weak multiplicativity contained in (α_1) allows a direct proof ([LW4]) of the following C^*-dilation result, obviating the need to first pass through the universal enveloping algebra, and then establish "invariance" of the C^*-algebra.

Theorem 4.3 ([GPS]). *Let $(\mathcal{P}_t)_{t \geq 0}$ be a completely positive contraction semigroup on $\mathcal{A}$ with bounded generator. If $\mathcal{A}$ is separable then there is a separable Hilbert space k and a *-homomorphic cocycle j on ${}^u\mathcal{A} := C^*(\mathcal{A}, I_\mathfrak{h})$ satisfying*

$$\mathcal{P}_t = \mathbb{E}_0 \circ j_t|_\mathcal{A}$$

where $\mathbb{E}_0 : \mathrm{M}(\mathcal{F}; {}^u\mathcal{A})_\mathrm{b} \to {}^u\mathcal{A}$ is the expectation map $A \mapsto E^{\varepsilon(0)} A E_{\varepsilon(0)}$.

Remark. Separability for a commutative C^*-algebra is equivalent to the locally compact Hausdorff space that is its spectrum being second countable, and thus σ-compact and Polish. This is a favoured category of state space in the general theory of classical Markov processes ([Sha]).

In conclusion we note that the arguments of this paper may be modified slightly to obtain the following result which is a strengthening of Theorem 4.1 when the generator of the cocycle is not k-bounded.

Theorem 4.4. *Let $[\theta^\alpha_\beta]$ be a regular mapping matrix of bounded operators on $\mathcal{A}$ and let k be the Markovian cocycle generated by $[\theta^\alpha_\beta]$ in any basis $(e_\alpha)_{\alpha \geq 0}$ for $\widehat{\mathsf{k}}$ with $e_0 = 1 \in \mathbb{C}$. If the mapping matrix $[(\theta^\beta_\alpha)^\dagger]$ is also regular and $[\theta^\alpha_\beta]$ satisfies*

$$\theta^{\alpha_1}_{\beta_1} \circ \cdots \circ \theta^{\alpha_{n-1}}_{\beta_{n-1}}\big(\theta^{\alpha_n}_{\beta_n}(ab) - \theta^{\alpha_n}_{\beta_n}(a)b - a\theta^{\alpha_n}_{\beta_n}(b)\big) = \text{w.o.}\sum_{i \geq 1} \theta^{\alpha_1}_{\beta_1} \circ \cdots \circ \theta^{\alpha_{n-1}}_{\beta_{n-1}}\big(\theta^{\alpha_n}_i(a)\theta^i_{\beta_n}(b)\big),$$

then k is weakly multiplicative.

Remark. This extends Theorem 6.2 of [LW1] since Mohari-Sinha regularity implies strong operator continuity on bounded sets for each component map θ^α_β.

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