CONSTANT MEAN CURVATURE SURFACES OF ANY POSITIVE GENUS

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Abstract. We show the existence of several new families of non-compact constant mean curvature surfaces: (i) singly-punctured surfaces of arbitrary genus $g \geq 1$, (ii) doubly-punctured tori, and (iii) doubly periodic surfaces with Delaunay ends.

1. Introduction

Since Wente's discovery [15] of tori there has been revived interest in the study of nonminimal constant mean curvature (CMC) surfaces. The investigation has been informed by analytical methods such as work by Kapouleas [7] and Korevar, Kusner and Solomon [11] as well as by techniques from integrable systems such as in Pinkall and Sterling [12] and Bobenko [2].

In the late 1990's Dorfmeister, Pedit and Wu [4] formulated a Weierstraß type representation for CMC surfaces, which involves solving a holomorphic complex linear 2×2 system of ordinary differential equations with values in a loop group, and subsequently factorising the solution into two factors, one of which turns out to be a moving frame of the Gauß map, from which the surface can be constructed. Unfortunately, the factorisation is not explicit and much qualitative information about the solution is obscured in the process. In representing surfaces with non-trivial topology the main challenge is to keep track of the monodromy representation of the moving frame.

Any CMC surface comes in an isometric $\mathbb{S}^1$ family, but the periods of a non-simply connected surface are generally only closed for one specific value $\lambda_0 \in \mathbb{S}^1$ of the *spectral parameter*. The Weierstraß data for a CMC surface consists of a Riemann surface Σ, a point z_0 on the universal cover $\widetilde{\Sigma}$, a *holomorphic potential* ξ on Σ and an initial condition Φ_0. Solving the initial value problem

$$d\Phi = \Phi\,\xi,\ \Phi(z_0) = \Phi_0 \tag{1}$$

yields a solution Φ and corresponding monodromy representation both of which depend on the spectral parameter, since generally both ξ and Φ_0 do. If the monodromy M satisfies for all deck transformations the following three conditions

$$M|_{\mathbb{S}^1} \in \mathrm{SU}_2, \tag{2}$$
$$M|_{\lambda_0} = \pm\mathrm{Id}, \tag{3}$$
$$d_\lambda M|_{\lambda_0} = 0, \tag{4}$$

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then the resulting associated family factors through the fundamental group at λ_0 and we thus have a CMC immersion $f : \Sigma \to \mathbb{R}^3$. In equation (4) and throughout this work we denote by d_- the derivative with respect to the subscript, which we omit in the case of the exterior derivative on the Riemann surface, as in (1). Condition (3) kills the rotational periods while (4) takes care of the translational periods, and both can be ensured by properties on ξ. The condition (2) is harder to satisfy and makes use of varying the initial condition Φ_0. These three conditions have been used in a number of papers, starting with the work of Dorfmeister and Haak [3] and by several of the authors while investigating CMC immersions of the n-punctured Riemann sphere, the so called n-Noids [8], [9] and [14]. Another approach to studying embedded CMC 3-Noids can be found in the work of Große-Brauckmann, Kusner and Sullivan [6].

The purpose of this paper is to show the existence of new CMC surfaces by exhibiting Weierstraß data $(\Sigma,\, \xi,\, \Phi_0,\, z_0)$ that fulfill the above requirements (2)–(4) and to initiate the study of higher genus surfaces via loop group techniques. Briefly summarising the contents of this paper, after providing some general sufficient conditions on Weierstraß data to satisfy the condition (2), (3) and (4) we apply these results to prove existence of new examples of CMC surfaces

(i) of any positive genus and a single end,
(ii) of genus 1 with two ends,
(iii) which are doubly-periodic with infinitely many ends asymptotic to Delaunay ends.

Although the last mentioned surfaces (iii) are immersions of genus zero domains with infinitely many punctures, they have natural quotient surfaces with positive genus.

2. Preliminary results

We denote an annular neighbourhood of the unit circle $\mathbb{S}^1$ for some real $r \in (0,\, 1]$ by $A_r = \{\lambda \in \mathbb{C} : r \leq |\lambda| \leq 1/r\}$. It is common abuse to call a map $M : A_r \to \mathrm{SL}_2(\mathbb{C})$ *unitary* if $M|_{\mathbb{S}^1} \in \mathrm{SU}_2$.

Definition 2.1. *We shall call a map $M : A_r \to \mathrm{SL}_2(\mathbb{C})$* unitarisable *on A_s for some $s \in [r,\, 1]$ if there exists a map $h : A_s \to \mathrm{GL}_2(\mathbb{C})$ for some $s \in [r,\, 1]$ such that $h\, M\, h^{-1} : A_s \to \mathrm{SL}_2(\mathbb{C})$ is unitary.*

We use the following notation for diagonal and off–diagonal 2×2 matrices:

$$\mathrm{diag}[u,\, v] = \left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right),\ \mathrm{off}[u,\, v] = \left(\begin{smallmatrix} 0 & u \\ v & 0 \end{smallmatrix}\right).$$

In preparation for Theorems 3.1 and 4.1 we first provide some technical results. The next lemma gives conditions on a matrix which ensure that after unitarisation, it satisfies the closing conditions at $\lambda_0 = e^{i\, x_0}$.

Lemma 2.2. *Let $J \subset \mathbb{R}$ be an open interval, and let $M : J \to \mathrm{SL}_2(\mathbb{C})$, $U : J \to \mathrm{SU}_2$ be smooth maps with $\mathrm{tr}\, M = \mathrm{tr}\, U$. If $M(x_0) = \pm \mathrm{Id}$ and $d_x M(x_0)$ is nilpotent for $x_0 \in J$, then $U(x_0) = \pm\mathrm{Id}$ and $d_x U(x_0) = 0$.*

Proof. Since $\mathrm{tr}\, U(x_0) = \pm 2$ and $U(x_0) \in \mathrm{SU}_2$, we have $U(x_0) = \pm\mathrm{Id}$. Let $\tau = \frac{1}{2}\,\mathrm{tr}\, M = \frac{1}{2}\,\mathrm{tr}\, U$. We differentiate the Cayley-Hamilton equations $M^2 - 2\tau M = U^2 - 2\tau U = -\mathrm{Id}$ twice and evaluate at x_0 to get $\pm d_x M(x_0)^2 = d_x^2 \tau(x_0)\mathrm{Id} = \pm d_x U(x_0)^2$. So $d_x M(x_0)$ nilpotent implies $d_x U(x_0)$ is also nilpotent. Since $U \in \mathrm{SU}_2$, we have $d_x U \in \mathfrak{su}_2$, and so nilpotency implies $d_x U(x_0) = 0$. $\square$

Lemma 2.3 computes the derivatives of a solution to a linear ODE with respect to a parameter. From this the series expansion of the trace of the monodromy with respect to the parameter can be computed, and hence the trace can be estimated in a small interval, as we will see in Lemma 2.5. See [3] for a related theorem.

Lemma 2.3. *Let Σ be a simply connected Riemann surface with coordinate z. Let $A(z,\,x) : \Sigma \times \mathbb{R} \to \mathfrak{gl}_2(\mathbb{C})$ be analytic in z and smooth in x and $B(x) : \mathbb{R} \to \mathfrak{gl}_2(\mathbb{C})$ be smooth. Let $X(z,\,x) : \Sigma \times \mathbb{R} \to \mathfrak{gl}_2(\mathbb{C})$ be the solution of the initial value problem*

$$(d_z X)\,(z,\,x) = X(z,\,x)A(z,\,x)\,, \quad X(z_0,\,x) = B(x). \tag{5}$$

Let $X_k(z) : \Sigma \to \mathfrak{gl}_2(\mathbb{C})$, for integers $k \geq 0$, be the solutions to the sequence of initial value problems

$$(d_z X_k)\,(z) = \sum_{i,j\geq 0, i+j=k} \frac{k!}{i!(k-i)!} X_i(z) A_j(z)\,, \quad X_k(z_0) = B_k,$$

where $A_j(z) := \left(d_x^j A\right)(z,\,x_0)$ and $B_k := \left(d_x^k B\right)(x_0)$. Then

$$\left(d_x^k X\right)(z,\,x_0) = X_k(z). \tag{6}$$

Proof. Differentiate (5) repeatedly with respect to x. $\square$

In the following, let Σ be a connected Riemann surface with universal cover $\widetilde{\Sigma}$ and Δ its group of deck transformations. We denote the holomorphic 1-forms on Σ by $\Omega'(\Sigma,\,\mathbb{C})$.

In the next Lemma we show that a certain class of potentials always ensures the closing conditions (3) and (4). Such potentials will be used in later examples (Theorems 3.1 and 4.1) to show the existence of new CMC surfaces.

Lemma 2.4. *Let $f,\,g \in \Omega'(\Sigma,\,\mathbb{C})$ and $t = \lambda^{-1}(\lambda-1)^2$ and*

$$A = \begin{pmatrix} 0 & f\,t \\ g & 0 \end{pmatrix}\,. \tag{7}$$

Let $w_0 \in \widetilde{\Sigma}$ and X be the solution to the initial value problem

$$dX = X\,A\,, \quad X(w_0,\,t) = \mathrm{Id}\,. \tag{8}$$

Let $\gamma \in \Delta$ and $M(t) := X(\gamma(w_0),\,t)$. Suppose that

$$\int_{w_0}^{\gamma(w_0)} g = 0. \tag{9}$$

Then $\tilde{M}(1) = \mathrm{Id}$ and $d_\lambda \tilde{M}(1) = 0$, where $\tilde{M}(\lambda) = M(t)$.

Proof. Note that $X(w,\,0) = \mathrm{Id} + \mathrm{off}[\,0,\,\int_{w_0}^{w} g\,]$. Hence $X(\gamma(w_0),\,0) = \mathrm{Id}$, and so $\tilde{M}(1) = \mathrm{Id}$. Then with $z_0 = w_0$, $e^{ix} = \lambda$, $e^{ix_0} = \lambda_0 = 1$, $B(x) = \mathrm{Id}$, A as in (7) and z fixed to $\gamma(w_0)$ in (6), it follows that $A_1(z)$ is identically zero, and Lemma 2.3 implies that $(d_\lambda \tilde{M})(1) = 0$. $\square$

The next lemma will be used in the proofs of Theorems 3.1 and 4.1 to show that certain monodromy groups can be unitarised.

Lemma 2.5. *Take the same notations and conditions as in Lemma 2.4, with t replaced by ct for some constant $c \in \mathbb{R} \setminus \{0\}$. Suppose that $\tau(ct) = \frac{1}{2} \operatorname{tr} M(ct)$ is real for all $t \in [-4, 0]$ and that*

$$I_0 I_2 + I_1^2 < 0 \,, \tag{10}$$

where

$$I_0 = \int_{w_0}^{\gamma(w_0)} f \,, \quad I_1 = \int_{w_0}^{\gamma(w_0)} \left(f \int g \right) \quad \text{and} \quad I_2 = \int_{w_0}^{\gamma(w_0)} \left(g \int \left(f \int g \right) \right) .$$

Then for $\tilde{\tau}(\lambda, c) = \tau(c\,t) = \tau(c\,\lambda^{-1}(\lambda - 1)^2)$ there exists a $c_0 > 0$ such that for all $|c| \in (0, c_0)$ we have

$$|\tilde{\tau}(\lambda, c)| < 1 \text{ for all } \lambda \in \mathbb{S}^1 \setminus \{1\} \,.$$

Proof. Let X be the solution of equation (8) with f replaced by cf, and write $X_{ij} = X_{ij}(w, t, c)$ for the entries of X. Defining $\hat{X}_{ij} = \frac{d}{d(ct)} X_{ij}(\gamma(w_0), 0, c)$, Lemma 2.3 implies

$$\hat{X}_{11} = \int_{w_0}^{\gamma(w_0)} \left(g \int f \right), \; \hat{X}_{12} = \int_{w_0}^{\gamma(w_0)} f$$
$$\hat{X}_{21} = \int_{w_0}^{\gamma(w_0)} \left(g \int f \left(\int g \right) \right) \text{ and } \hat{X}_{22} = \int_{w_0}^{\gamma(w_0)} \left(f \int g \right) .$$

Note that the $\hat{X}_{ij}$ are independent of c. Considering the first two derivatives of $\det X = 1$ with respect to ct and evaluating at $t = 0$ and $w = \gamma(w_0)$ yields

$$(d_{ct}\tau)(0, c) = 0 \,, \quad \left(d_{ct}^2 \tau\right)(0, c) = \hat{X}_{12}\hat{X}_{21} - \hat{X}_{11}\hat{X}_{22} = \hat{X}_{12}\hat{X}_{21} + \hat{X}_{22}^2 \,.$$

Note that the first of these two equations implies $\hat{X}_{11} = -\hat{X}_{22}$, which is used in the second of these two equations. Equation (10) implies that $\hat{X}_{12}\hat{X}_{21} + \hat{X}_{22}^2 < 0$, and so the second derivative with respect to ct of τ is negative, and τ attains a maximum of 1 at $t = 0$. Thus there exists a $k_0 > 0$ such that $|ct| \in (0, k_0]$ implies $|\tau(ct)| \in [0, 1)$. Let $c_0 = k_0/4$. Then for all c such that $|c| \in (0, c_0)$ and for all $t \in [-4, 0]$, we have $|\tau(ct)| \in [0, 1)$ and the lemma follows. □

Applying Lemma 2.3, similarly to the proofs of Lemmas 2.4 and 2.5, we obtain

Corollary 2.6. *With notations and conditions as in Lemmas 2.4 and 2.5, we have $d_\lambda^2 \tilde{M}(1) = 2(\operatorname{diag}[\,-I_1, I_1\,] + \operatorname{off}[\,I_0, I_2\,])$.*

3. Singly-punctured cmc surfaces of arbitrary genus

We construct a family of cmc immersions of a singly-punctured genus g Riemann surface into $\mathbb{R}^3$ with umbilics, for any positive g. The closing problem is solved by imposing symmetries so that the monodromy group can be shown to be unitarisable.

Theorem 3.1. *Let $n \geq 2$ be an even integer. Let Σ be the singly-punctured hyperelliptic genus $n/2$ Riemann surface defined by $\Sigma = \{(z, w) \in \mathbb{C}^2 \mid w^2 = z(1 - z^n)\}$. Let*

$$\xi = \begin{pmatrix} 0 & c\lambda^{-1}(\lambda - 1)^2 w^{-1} dz \\ d(z^{n-1}w) & 0 \end{pmatrix}, \quad c \in \mathbb{R}^*.$$

Then for c sufficiently close to zero, ξ induces a conformal cmc immersion $\Sigma \to \mathbb{R}^3$ with order $2n$ dihedral symmetry.

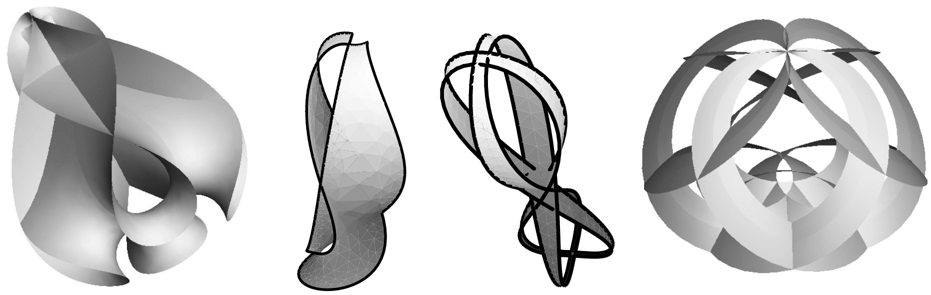

FIGURE 1. The three figures on the left are parts of a CMC singly-punctured torus, as in Theorem 3.1 with $n = 2$. The left-most image shows the torus with a neighborhood of the end removed. The surface has 90° rotation symmetry and reflection symmetry. The second image shows one-fourth of the surface, which extends to the full surface (again with a neighborhood of the end removed) by these symmetries. The third image is a skeletal portion of the surface. The Hopf differential has a pole of order 6 at the end. The right-most figure is a CMC doubly-punctured torus, as in Theorem 4.1, and the image here shows a skeleton of this torus (with a doubly-punctured disk containing the ends removed).

Proof. Choose a basepoint $(z_0,\, w_0) \in \widetilde{\Sigma}$ in the fibre of $(0,\, 0) \in \Sigma$ and let Φ be the solution to the initial value problem (1) with $\Phi_0 = \Phi(z_0,\, w_0) = \mathrm{Id}$. We must show that there exists a unitariser for the monodromy of Φ and verify the closing conditions.

With $\alpha = \exp(\pi i/n)$, for $k \in \{0, \dots, n-1\}$ let $\gamma_k : [0,\, 1] \to \Sigma$ be the curve from $(0,\, 0)$ to $(\alpha^{2k},\, 0)$ and back to $(0,\, 0)$ along the straight line in the z-plane from 0 to α^{2k}, defined by

$$\gamma_k(s) = \begin{cases} \left(2s\alpha^{2k}, (-\alpha)^k \left|\sqrt{2s(1-2^n s^n)}\right|\right), & 0 \le s \le 1/2 \\ \left(2(1-s)\alpha^{2k}, -(-\alpha)^k \left|\sqrt{2(1-s)(1-2^n(1-s)^n)}\right|\right) & 1/2 \le s \le 1. \end{cases}$$

Let $\hat\gamma_k$ be the lifted curves originating at $(z_0,\, w_0)$ and $M_k(\lambda) := \Phi(\hat\gamma_k(1),\, \lambda)$. Then $M_0, \dots, M_{n-1}$ generate the monodromy group of Φ, since Σ has only one puncture at $(z,\, w) = (\infty,\, \infty)$, and the monodromy about the puncture is $N = \prod_{k=0}^{n-1} M_k$. Lemma 2.4 then implies

$$M_k(1) = \mathrm{Id}, \quad d_\lambda M_k(1) = 0\ , \quad k \in \{0, \dots, n-1\}\ . \tag{11}$$

Hence we also have $N(1) = \mathrm{Id}$, $d_\lambda N(1) = 0$. It remains to show that there exists a unitariser for the monodromy group. For this, we compute the monodromy group's symmetries. We define the following maps on Σ:

$$\sigma(z,\, w) = (\alpha^2 z,\, \alpha w)\ ,\ \rho(z,\, w) = (z,\, -w)\ \text{ and } \theta(z,\, w) = (\overline{z},\, \overline{w})\ . \tag{12}$$

Then for $g_\sigma = \mathrm{diag}[\,\sqrt{\alpha}^{-1},\, \sqrt{\alpha}\,]$ and $g_\rho = \mathrm{diag}[\,-i,\, i\,]$ we have

$$\sigma^*\xi = g_\sigma^{-1}\, \xi\, g_\sigma,\ \rho^*\xi = g_\rho^{-1}\, \xi\, g_\rho \text{ and } \overline{\theta^*\xi(1/\bar\lambda)} = \xi(\lambda),$$

where expressions like $\sigma^*\xi$ denote $\sigma^*\xi((z,\, w),\, \lambda) = \xi(\sigma(z,\, w),\, \lambda)$. Since $(0,\, 0)$ is a fixed point of σ, ρ and θ, we define the lifts $\hat\sigma$, $\hat\rho$, $\hat\theta$ that map $(0,\, 0)$ to $(z_0,\, w_0)$.

Then using $\Phi(z_0,\, w_0) = \mathrm{Id}$, we obtain

$$\hat\sigma^*\Phi = g_\sigma^{-1}\Phi\, g_\sigma\ , \quad \hat\rho^*\Phi = g_\rho^{-1}\Phi\, g_\rho\ , \quad \overline{\hat\theta^*\Phi(1/\bar\lambda)} = \Phi(\lambda)\ .$$

Hence the monodromy group has the following symmetries:

$$\begin{aligned} M_k^{(-1)^k} &= g_\sigma^{-k} M_0\, g_\sigma^k\ , \quad k \in \{0, \dots, n-1\}\ , \\ M_0^{-1} &= g_\rho^{-1} M_0\, g_\rho\ , \\ \overline{M_0} &= M_0 \quad \text{for all } \lambda \in \mathbb{S}^1\ . \end{aligned} \tag{13}$$

Note that the third of these symmetries also follows from the facts that $\lambda^{-1}(\lambda-1)^2 \in \mathbb{R}$ for all $\lambda \in \mathbb{S}^1$ and $\gamma_0(s) \in \mathbb{R}^2$ for all $s \in [0,1]$. Denoting the entries of M_0 by M_{ij}, the third symmetry in (13) implies that the M_{ij} are all real, and the second symmetry in (13) implies that $M_{11} = M_{22}$, for all $\lambda \in \mathbb{S}^1$. In particular, $\mathrm{tr}(M_0)$ is real for all $\lambda \in \mathbb{S}^1$. The integrals I_j in equation (10) are then

$$I_0 = 2c \int_{\hat\gamma} |w|^{-1} dz\ , \quad I_1 = 0\ , \quad \text{ and } I_2 = \frac{2c}{n} \int_{\hat\gamma} z^n d(z^{n-1}|w|)\ ,$$

for the curve $\hat\gamma(s) = (s, |\sqrt{s(1-s^n)}|) \in \Sigma$, $s \in [0,1]$. Using the formula, valid for $\mathrm{Re}\, n > 0$, $\mathrm{Re}\, r > 0$ and $\mathrm{Re}\, s > 0$,

$$\int_{\hat\gamma} z^{r-1}(1-z^n)^{s-1}\, dz = \frac{\Gamma(\frac{r}{n})\,\Gamma(s)}{n\,\Gamma(\frac{r}{n}+s)}\ ,$$

where $\Gamma(\ell) = \int_0^\infty y^{\ell-1}e^{-y}dy$ is the Euler gamma function, we get

$$I_0 I_2 + I_1^2 = -\frac{8\pi c^2(2n-1)\cot(\frac{\pi}{2n})}{(n-1)(3n-1)(5n-1)} < 0.$$

Hence by Lemma 2.5, with $\tau(\lambda,\, c) = \frac{1}{2}\,\mathrm{tr}(M_0(\lambda))$, there exists a $c_0 > 0$ such that for all c satisfying $|c| \in (0,\, c_0)$,

$$|\tau(\lambda,\, c)| < 1 \text{ for all } \lambda \in \mathbb{S}^1 \setminus \{1\}\ , \tag{14}$$

and $\tau(1,\, c) = 1$. Then $\tau = M_{11} = M_{22} \in \mathbb{R}$ has modulus at most 1 for all $\lambda \in \mathbb{S}^1$. Thus $-M_{12}M_{21} = 1 - M_{11}^2 \geq 0$ on $\mathbb{S}^1$, so

$$v := -\frac{M_{21}}{M_{12}} \geq 0 \text{ on } \mathbb{S}^1\ .$$

Furthermore, v is finite and strictly positive on $\mathbb{S}^1 \setminus \{1\}$, by (14).

Let us now consider the behavior of v at $\lambda = 1$. By (11), we know that

$$M_{12}|_{\lambda=1} = M_{21}|_{\lambda=1} = d_\lambda M_{12}|_{\lambda=1} = d_\lambda M_{21}|_{\lambda=1} = 0\ .$$

Applying Corollary 2.6, we have $d_\lambda^2 M_{12}|_{\lambda=1} = 2I_0$ and $d_\lambda^2 M_{21}|_{\lambda=1} = 2I_2$. Since $I_1 = 0$ and $I_0I_2 + I_1^2 < 0$, we conclude that I_0 and I_2 are both nonzero, so M_{12} and M_{21} both have zeroes of order exactly two at $\lambda = 1$. Hence v is nonzero and finite at $\lambda = 1$. Thus v is a strictly positive finite function on all of $\mathbb{S}^1$, and therefore $\sqrt[4]{v}$ can be globally and smoothly defined on $\mathbb{S}^1$. Then by the first symmetry of (13), the diagonal unitariser is given by

$$h = \begin{pmatrix} \sqrt[4]{v} & 0 \\ 0 & (\sqrt[4]{v})^{-1} \end{pmatrix},$$

which simultaneously unitarizes $M_0, \dots, M_{n-1}$ on $\mathbb{S}^1$, i.e. $hM_jh^{-1} \in \mathrm{SU}_2$ for all $\lambda \in \mathbb{S}^1$. Therefore the monodromy group of $h\Phi$ is unitarized on all of $\mathbb{S}^1$.

By Equation (11) and Lemma 2.2, the monodromy group hM_jh^{-1} still satisfies the closing conditions (3) and (4) at $\lambda = 1$. Hence the resulting CMC immersion is well-defined on Σ.

Since the coefficient $cw^{-1}dz$ of the λ^{-1} term of the upper-right entry of the potential ξ has no zeros or poles on the singly-punctured Riemann surface Σ, the CMC immersion is unbranched, see [3], Theorem 3.1.

We now consider the symmetries of the CMC immersion resulting from $h\Phi$. Since $(0,\,0)$ is fixed by the map σ and h is independent of (z, w) and $[h,\, g_\sigma] = 0$ and also $\hat{\sigma}^*\Phi = g_\sigma^{-1}\Phi g_\sigma$, we have that $(\hat{\sigma}^k)^*(h\Phi) = g_\sigma^{-k}(h\Phi)g_\sigma^k$, where $\hat{\sigma}^k$ is the composition of $\hat{\sigma}$ with itself k times.

Let $h\Phi = FB$ be the Iwasawa decomposition with respect to $\mathbb{S}^1$ (Theorem 8.1.1 [13]), pointwise on $\widetilde{\Sigma}$. Since $g_\sigma^{-k}Fg_\sigma^k$ is unitary and $g_\sigma^{-k}Bg_\sigma^k$ positive, the unitary part of $(\hat{\sigma}^k)^*(h_1\Phi)$ is $g_\sigma^{-k}Fg_\sigma^k$. The symmetry $(\hat{\sigma}^k)^*F = g_\sigma^{-k}Fg_\sigma^k$ descends to the immersion via the Sym-Bobenko formula [2], see also [8] section 4, and results in a rotation of angle $k\pi/n$ about an axis independent of k. Hence the surface has an order $2n$ rotational symmetry.

To show dihedral symmetry, we now need only show that the surface has at least one reflective symmetry across a plane parallel to the common axis of the rotational symmetries. We will show that the map $\theta(z, w) = (\bar{z}, \bar{w})$ is such a reflective symmetry, by showing that the immersion generated by F via the Sym-Bobenko formula [2], and denoted by f, satisfies

$$\theta^* f = -\bar{f}\ .$$

Because $\xi|_\lambda = \xi|_{\lambda^{-1}}$, we have $\Phi(z, w, \lambda) = \Phi(z, w, \lambda^{-1})$ and consequently

$$\overline{\Phi(\bar{z}, \bar{w}, \bar{\lambda})} = \overline{\Phi(\bar{z}, \bar{w}, \bar{\lambda}^{-1})} = \overline{\hat{\theta}^*\Phi(\bar{\lambda}^{-1})} = \Phi(z, w, \lambda).$$

This further implies that $\overline{\Phi(\gamma_0(s), \bar{\lambda})} = \Phi(\gamma_0(s), \lambda)$ and so $\overline{M(\bar{\lambda})} = M(\lambda)$, and in turn $\overline{h(\bar{\lambda})} = h(\lambda)$, since $\theta^*\gamma_0(s) = \gamma_0(s)$. Thus

$$\overline{h(\bar{\lambda})\,\Phi(\bar{z}, \bar{w}, \bar{\lambda})} = h(\lambda)\Phi(z, w, \lambda)$$

and consequently $\overline{F(\bar{z}, \bar{w}, \bar{\lambda})}\,\overline{B(\bar{z}, \bar{w}, \bar{\lambda})} = F(z, w, \lambda)\,B(z, w, \lambda)$. Uniqueness of the Iwasawa decomposition yields $\overline{F(\bar{z}, \bar{w}, \bar{\lambda})} = F(z, w, \lambda)$ and implies $\theta^*f = -\bar{f}$. □

Remark 3.2. *Note that the end of any surface in Theorem 3.1 is not asymptotically Delaunay, because the order of the Hopf differential there is strictly less than -2. This is also implied by* [11]*, since Delaunay ends have non-zero weight, but the balancing formula implies that the single end of any surface in Theorem 3.1 must have zero weight.*

4. CMC IMMERSIONS OF A DOUBLY-PUNCTURED TORUS

In this section, we construct immersions of a doubly-punctured genus 1 Riemann surface into $\mathbb{R}^3$ with umbilics.

Theorem 4.1. *Let $\mathcal{T} = \{[z] \in \mathbb{C}/\Gamma \,|\, z \in \mathbb{C}\}$ be the square torus, where Γ is the 2-dimensional lattice generated by $2\omega_1 \in \mathbb{R}^+$ and $2\omega_2 = 2i\omega_1$. Let $\omega_3 = \omega_1 + \omega_2$. On the twice-punctured torus $\Sigma = \mathcal{T} \setminus \{[\omega_3/2],\, [-\omega_3/2]\}$, let ξ be the potential*

$$\xi = \begin{pmatrix} 0 & c\lambda^{-1}(\lambda-1)^2 \\ \wp''''(z+\omega_3/2) + \wp''''(z-\omega_3/2) & 0 \end{pmatrix} dz\ , \quad c \in \mathbb{R}^*,$$

where $\wp$ *is the Weierstrass* $\wp$*-function with respect to* $\mathcal{T}$ *satisfying* $(\wp')^2 = 4\wp(\wp^2-1)$ *and* $'$ *denotes the derivative with respect to* z*. Then for c sufficiently close to zero,* ξ *induces a conformal* CMC *immersion* $\Sigma \to \mathbb{R}^3$ *with order* 4 *dihedral symmetry.*

Remark 4.2. *Note that* $\wp'''' = 120\wp^3 - 72\wp$*. Then, since* $\wp(-z) = \wp(z)$ *and* $\wp(iz) = -\wp(z)$*, it follows that also* $\wp''''(-z) = \wp''''(z)$ *and* $\wp''''(iz) = -\wp''''(z)$*. These properties will be used in the following proof. One other particular property that we will need is, defining*

$$\begin{aligned}\mathcal{I}(z) &= \wp'''(z+\omega_3/2) + \wp'''(z-\omega_3/2) - \wp'''(\omega_3/2) - \wp'''(-\omega_3/2)\\ &= \wp'''(z+\omega_3/2) + \wp'''(z-\omega_3/2),\end{aligned}$$

that the integral $\int_0^{2\omega_1}(\mathcal{I}(z))^2 dz > 0$ *along the real axis from* 0 *to* $2\omega_1$ *is positive. This integral is real because of the relations* $(\mathcal{I}(\omega_1 \pm \bar{z}))^2 = \overline{(\mathcal{I}(z))}^2$*, and then one can check that it is positive for any choice of* $\omega_1 > 0$*.*

Proof. Choose a basepoint $w_0 \in \widetilde{\Sigma}$ in the fibre of $z_0 = 0 \in \mathbb{C}$, and let Φ be the solution to the initial value problem $d\Phi = \Phi\xi$, $\Phi(w_0) = \mathrm{Id}$. Let $\gamma_k = \gamma_k(s) \in \mathbb{C}$ be the straight-line curve from z_0 to $2\omega_k$ ($k \in \{1, 2\}$) defined by $\gamma_k(s) = 2s\omega_k$ for $s \in [0,1]$. Let $\delta_1 = \delta_1(s) \in \mathbb{C}$ for $s \in [0,1]$ be a curve from z_0 around $\omega_3/2$ in the counterclockwise direction and back to z_0 lying in a small neighborhood of the straight line from z_0 to $\omega_3/2$, and let $\delta_2 = \delta_2(s) = -\delta_1(s)$ be the curve from z_0 around $-\omega_3/2$ in the counterclockwise direction and back to z_0 that is the reflection of δ_1 through the point z_0.

Let $M_k = M_k(\lambda)$ be the respective global monodromies of Φ over the torus along γ_k, and let $A_k = A_k(\lambda)$ be the monodromies of Φ about the two punctures of Σ along δ_k ($k \in \{1, 2\}$). Then M_1, M_2, A_1, A_2 generate the monodromy group of Φ.

This proof follows the same strategy as the proof of Theorem 3.1. First, we note that all the generating elements M_1, M_2, A_1, A_2 of the monodromy group of Φ satisfy the closing conditions (3) and (4). This follows from Lemma 2.4, since the lower-left entry in ξ is the derivative with respect to z of a function that is well-defined on Σ and hence Equation (9) will be satisfied. Our main effort again goes into showing that there exists an initial condition that unitarises the monodromy group of Φ. To accomplish this, we first compute the symmetries of the monodromy group and define the following transformations of Σ:

$$\sigma(z) = z + \omega_3\ , \quad \rho(z) = iz + \omega_1\ , \quad \theta(z) = \bar{z} + \omega_1\ .$$

Then with $g = \mathrm{diag}[\,1/\sqrt{i},\ \sqrt{i}\,]$, the potential ξ has the symmetries

$$\sigma^*\xi = \xi\ , \quad \rho^*\xi = g^{-1}\,\xi\,g\ , \text{ and } \overline{\theta^*\xi(1/\bar{\lambda})} = \xi\ .$$

Hence $\hat{\sigma}^*\Phi = V_\sigma\Phi$, $\hat{\rho}^*\Phi = V_\rho\Phi g$ and $\overline{\hat{\theta}^*\Phi(1/\bar{\lambda})} = V_\theta\Phi$ for some z-independent V_σ, V_ρ and V_θ. Since $z_0 = 0$ is a fixed point of the two maps

$$\sigma^{-1}\rho^2 : z \mapsto -z\ , \quad \sigma^{-1}\rho\,\theta : z \mapsto i\bar{z}$$

(we interpret these compositions as being applied in order from rightmost first to leftmost last), and since $\Phi(z_0) = \mathrm{Id}$, we have $V_\rho^2 V_\sigma^{-1} = g^{-2}$ and $V_\theta V_\rho V_\sigma^{-1} = g^{-1}$. It follows that

$$\begin{aligned} M_1^{-1} &= g^{-2}M_1g^2\ , \quad A_2 = g^{-2}A_1g^2\ ,\\ \overline{M_2(1/\bar{\lambda})} &= g^{-1}M_1(\lambda)g\ , \quad \overline{A_1(1/\bar{\lambda})}^{-1} = g^{-1}A_1(\lambda)g\ . \end{aligned} \tag{15}$$

The first and third equations in (15) imply that also $M_2^{-1} = g^{-2} M_2 g^2$.

Because the potential is real-valued along the curve γ_1 when $\lambda \in \mathbb{S}^1$, we conclude that M_1 is a real-valued matrix for all $\lambda \in \mathbb{S}^1$. This fact, combined with the first equation in (15), implies that M_1 has the form

$$M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & a_1 \end{pmatrix} ,$$

where $a_1 = a_1(\lambda) = \overline{a_1(1/\bar{\lambda})}$, $b_1 = b_1(\lambda) = \overline{b_1(1/\bar{\lambda})}$ and $c_1 = c_1(\lambda) = \overline{c_1(1/\bar{\lambda})}$. Furthermore, the fourth equation in (15) implies

$$A_1 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} , \quad \text{where}$$

$$\overline{a_2(1/\bar{\lambda})} = d_2(\lambda) \, , \; \overline{b_2(1/\bar{\lambda})} = -ib_2(\lambda) \, , \; \overline{c_2(1/\bar{\lambda})} = ic_2(\lambda) \, . \tag{16}$$

From this it is clear that $\tau_1 = \frac{1}{2} \operatorname{tr} M_1$ and $\tau_2 = \frac{1}{2} \operatorname{tr} A_1$ are real for all $\lambda \in \mathbb{S}^1$.

We will now show that M_1 and A_1 are simultaneously unitarizable for small $|c|$. Toward this goal, we first apply Lemma 2.5 to show that for small $|c|$ we have $|\tau_1(\lambda)| < 1$ for all $\lambda \in \mathbb{S}^1 \setminus \{1\}$: We take Σ and ξ as in Theorem 4.1 and take $f = dz$ and $g = (\wp''''(z+\omega_3/2) + \wp''''(z-\omega_3/2))dz$ and the curve $\gamma = \gamma_1$. Then, in Lemma 2.5, we have $I_0 = 2\omega_1 > 0$ and $I_1 = 0$. To compute I_2, integration by parts yields

$$I_2 = -\int_{\gamma_1} \left(f \left(\int g \right)^2 \right) = -\int_{\gamma_1} (\mathcal{I}(z))^2 dz \, ,$$

where $\mathcal{I}(z)$ is as defined in Remark 4.2. Then by Remark 4.2, we have $I_2 < 0$. Thus the conditions of Lemma 2.5 hold and we conclude that for all $c \in \mathbb{R}$ sufficiently close to 0, we have

$$|\tau_1(\lambda)| < 1 \quad \text{for all } \lambda \in \mathbb{S}^1 \setminus \{1\} \, . \tag{17}$$

From (17) and the fact that $b_1 c_1 = a_1^2 - 1 \in \mathbb{R}$ on $\mathbb{S}^1$, we have

$$(b_1 c_1)|_{\lambda=1} = 0 \;\text{ and }\; (b_1 c_1)|_{\mathbb{S}^1 \setminus \{1\}} < 0 \, . \tag{18}$$

Thus we can define a function

$$v = -\frac{c_1}{b_1}$$

that is finite and nonzero on $\mathbb{S}^1 \setminus \{1\}$. Furthermore, by (18) and the fact that $b_1 \in \mathbb{R}$ on $\mathbb{S}^1$, we conclude that

$$0 < v < \infty \tag{19}$$

for all $\lambda \in \mathbb{S}^1 \setminus \{1\}$. Similar to the arguments in proving Theorem 3.1, Corollary 2.6 implies that b_1 and c_1 both have zeroes of order exactly two, hence v is nonzero and finite at $\lambda = 1$ as well. It follows that $\sqrt[4]{v} > 0$, representing the positive fourth root of v, is globally and smoothly defined on $\mathbb{S}^1$. We define

$$h = \begin{pmatrix} \sqrt[4]{v} & 0 \\ 0 & (\sqrt[4]{v})^{-1} \end{pmatrix} . \tag{20}$$

Because $b_1 c_1 \leq 0$ and $\sqrt{v} \geq 0$, the conjugate $hM_1h^{-1} \in \mathrm{SU}_2$ for $|\lambda| = 1$.

The image of the path $\gamma_2\delta_1$ under the map ρ is homotopic to the path $\gamma_1^{-1}\delta_1$. Hence $A_1 M_2$ and $A_1 M_1^{-1}$ are conjugate and so have the same trace. Hence

$$\operatorname{tr}(A_1(M_2 - M_1^{-1})) = b_2 c_1 (1+i) + c_2 b_1 (1-i) = 0 \, .$$

It follows that

$$c_2 b_1 + i c_1 b_2 = 0\ . \tag{21}$$

In (21), either both b_2 and c_2 are identically zero, or neither of them are identically zero. If b_2 and c_2 are identically zero, then A_1 is diagonal and $A_1 \in \mathrm{SU}_2$ for all $\lambda \in \mathbb{S}^1$. Hence we have succeeded in simultaneously unitarizing both M_1 and A_1 on $\mathbb{S}^1$ by conjugating by h. We may then proceed to the final paragraph of this proof, which gives the concluding argument for proving Theorem 4.1. Therefore, without loss of generality, let us assume that neither b_2 nor c_2 is identically zero.

Under the assumption that b_2 and c_2 are not identically zero, by (21) we also have

$$v = -\frac{\overline{c_1(1/\bar\lambda)}}{b_1(\lambda)} = -\frac{\overline{c_2(1/\bar\lambda)}}{b_2(\lambda)}\ .$$

Furthermore, (16) and (19) then imply that $b_2 = r_1(1+i)$ and $c_2 = r_2(1-i)$ with $r_1, r_2 \in \mathbb{R}$ and $r_1^{-1} r_2 \leq 0$, on $\mathbb{S}^1$. These facts together show that also the conjugate $hA_1h^{-1} \in \mathrm{SU}_2$ for $|\lambda| = 1$.

Thus we have simultaneously unitarised M_1 and A_1 on $\mathbb{S}^1$. Since $g \in \mathrm{SU}_2$ and commutes with h, conjugation by h also unitarizes M_2 and A_2, so the full monodromy group is unitarized on $\mathbb{S}^1$. Now, like in the proof of Theorem 3.1, using Iwasawa splitting on $\mathbb{S}^1$ and noting that the monodromy of $h\,\Phi$ still satisfies (3) and (4), we conclude that the resulting CMC surface given by the Sym-Bobenko formula is defined on Σ. Finally, analogous to the arguments at the end of the proof of Theorem 3.1, the order 4 dihedral symmetry of the resulting CMC immersions can be shown, and since the coefficient cdz of the λ^{-1} term of the upper-right entry of the potential ξ has no zeros or poles on the twice-punctured Riemann surface Σ, the resulting CMC immersion is unbranched [3]. This completes the proof. □

5. Doubly-periodic CMC surfaces in $\mathbb{R}^3$ with ends that are asymptotically Delaunay

In this section, we provide a third class of Weierstraß data for which the monodromy can be unitarised. The potential is of interest to us because, although the monodromy can be unitarised, the monodromy does not satisfy (3) at $\lambda_0 = 1$. The relaxing of this closing condition is what allows the resulting CMC immersions to extend to doubly-periodic surfaces (when $n = 3, 4, 6$ in the theorem).

Theorem 5.1. *Let $n \geq 3$ be an integer, and define the Riemann surface $\Sigma = (\mathbb{C} \setminus \mathcal{P}) \cup \{\infty\}$ with $\mathcal{P} = \{z \in \mathbb{C} \,|\, z^n = 1\}$. Let*

$$\xi = \begin{pmatrix} 0 & \lambda^{-1} dz \\ v(\lambda) \frac{n^2 z^{n-2}}{(z^n-1)^2} dz & 0 \end{pmatrix}\ , \tag{22}$$

where

$$v(\lambda) = \frac{(n-2)^2 w}{16 n^2}(1-\lambda)^2 + \frac{1-n}{n^2}\lambda\ , \quad w \in \left[\frac{-8n}{(n-2)^2}, 0\right)\ . \tag{23}$$

Let $w_0 \in \widetilde{\Sigma}$ be in the fibre of $z_0 = 0 \in \Sigma$ and let Φ be the solution of $d\Phi = \Phi\xi$, $\Phi(w_0) = \mathrm{Id}$. Then there exists an initial condition that unitarises the monodromy of Φ.

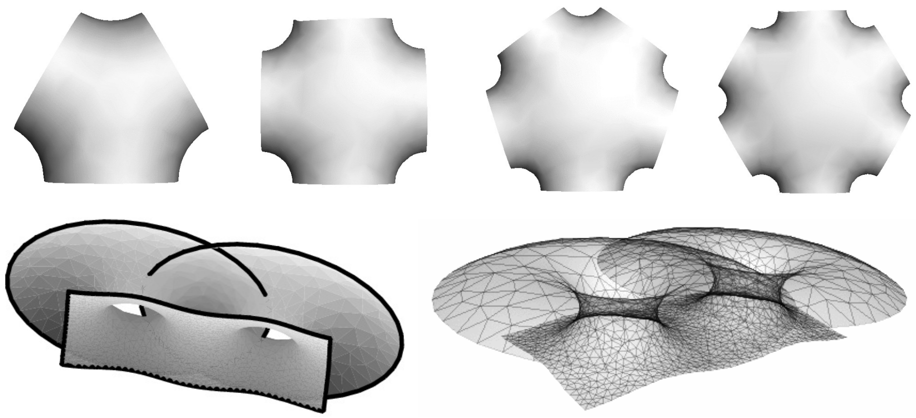

FIGURE 2. CMC surfaces with 3-, 4-, 5- and 6-fold symmetry (the CMC immersions f in (26) produced from Theorem 5.1) in the upper row. In the case of 3-, 4- and 6-fold symmetry, doubly periodic CMC surfaces can be constructed by reflection in planes perpendicular to the plane of this page. The annulur ends of each surface are nodoidal with equal weights and parallel same-directed axes. Two views of a portion of one of the doubly-periodic surfaces with 4-fold symmetry are shown in the lower images.

Proof. Let $\alpha = \exp(\pi i/n)$, and define the closed polygonal loop $\gamma_0 : [0,1] \to \Sigma$ as follows:

$$\gamma_0(t) = \begin{cases} 4t\alpha^{-1} & \text{for} \quad 0 \le t \le 1/4, \\ (2-4t)\alpha^{-1} + 8t - 2 & \text{for} \quad 1/4 \le t \le 1/2, \\ 6 - 8t + (4t-2)\alpha & \text{for} \quad 1/2 \le t \le 3/4, \\ (4-4t)\alpha & \text{for} \quad 3/4 \le t \le 1. \end{cases}$$

Then define the loops

$$\gamma_j(t) = \alpha^{2j}\gamma_0(t)\ , \quad j = 1, 2, ..., n-1\ .$$

Let M_j be the monodromy of Φ along γ_j. Then M_0, M_1, ..., M_{n-1} generate the monodromy group of Φ. Under the transformation $\rho : z \to \alpha^2 z$ of Σ, we have $\rho^*\xi = g^{-1}\,\xi\, g$, where $g = \mathrm{diag}[\,\alpha^{-1},\, \alpha\,]$. Because $\rho(z_0) = z_0$ and $\Phi(z_0) = \mathrm{Id}$, we have $M_j = g^{-j} M_0 g^j$.

Changing variables to $\tilde z = 1/z$ and gauging ($\xi \mapsto \xi.g = g^{-1}\,\xi\, g + g^{-1}\,dg$) by $\tilde g = \mathrm{diag}[\,\tilde z^{-1},\, \tilde z\,]$, we have

$$\xi.\tilde g = \begin{pmatrix} -\tilde z^{-1} & -\lambda^{-1} \\ \frac{-v(\lambda) n^2 \tilde z^{n-2}}{(\tilde z^n - 1)^2} & \tilde z^{-1} \end{pmatrix} d\tilde z\ .$$

Then one solution of $d\tilde\Phi = \tilde\Phi \cdot (\xi.\tilde g)$ is $\tilde\Phi = \exp\left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \log \tilde z\right) \tilde P(\tilde z, \lambda)$, where $\tilde P(\tilde z, \lambda)$ is well-defined and holomorphic with respect to $\tilde z$ and is nonsingular at $\tilde z = 0$. Furthermore, $\tilde P(\tilde z, \lambda)$ is defined for all $\lambda \in \mathbb{S}^1$. (This follows from a well-known result in the theory of ordinary differential equations, see [8] section 8.) It follows that the monodromy of Φ along the loop $\gamma_{n-1} ... \gamma_1 \gamma_0$ (here again composition of these loops is from rightmost first to leftmost last) encircling $z = \infty$ is

$(M_0)(g^{-1}M_0g)...(g^{1-n}M_0g^{n-1}) = \mathrm{Id}$. Thus $(M_0g^{-1})^n = -\mathrm{Id}$. Hence the eigenvalues of M_0g^{-1} are constant and are n-th roots of -1. Since

$$\Phi|_{\lambda=1} = \begin{pmatrix} d_zB & B \\ d_zD & D \end{pmatrix} ,$$

with

$$B = \alpha \sqrt[n]{z^n - 1} \int_0^z \left(\sqrt[n]{\zeta^n - 1} \right)^{-2} d\zeta , \quad D = \alpha^{-1} \sqrt[n]{z^n - 1} ,$$

we have that M_0 is upper-triangular at $\lambda = 1$ and the upper-left (resp. lower-right) entry of its diagonal is α^{-2} (resp. α^2). So the eigenvalues of M_0g^{-1} are the same as the eigenvalues of g^{-1}:

$$\text{(eigenvalues of } g^{-1}) = \text{(eigenvalues of } M_0g^{-1}) = \alpha^{\pm 1} . \tag{24}$$

Now we determine the eigenvalues of M_0 for general λ: For $\hat{g} = \mathrm{diag}[\,\sqrt{z-1},\ \frac{1}{\sqrt{z-1}}\,]$ we have

$$\xi.\hat{g} = A\frac{dz}{z-1} + O((z-1)^0) , \quad A = \begin{pmatrix} \frac{1}{2} & \lambda^{-1} \\ v(\lambda) & \frac{-1}{2} \end{pmatrix} ,$$

in a neighborhood of $z = 1$. Applying Lemma 9.1 in [8], we have that one solution of $d\hat{\Phi} = \hat{\Phi} \cdot (\xi.\hat{g})$ is $\hat{\Phi} = \exp\left(A \log(z-1)\right) \cdot \hat{P}(z, \lambda)$, where $\hat{P}(z, \lambda)$ is holomorphic and well-defined at $z = 1$. Furthermore, $\hat{P}(z, \lambda)$ is defined for any $\lambda \in \mathbb{S}^1$ at which the difference of the eigenvalues of A is not an integer. Hence $\hat{P}(z, \lambda)$ is defined on $\mathbb{S}^1$ minus a finite set of points.

Hence one solution of $d\check{\Phi} = \check{\Phi}\xi$ is $\check{\Phi} = \hat{\Phi}\,\hat{g}^{-1}$. Therefore any solution of $d\check{\Phi} = \check{\Phi}\xi$ has monodromy along γ_0 that is conjugate to $-\exp(2\pi i A)$. In particular, the eigenvalues of M_0 are $-\exp(\pm i\pi\sqrt{1 + 4\lambda^{-1}v(\lambda)})$, and so

$$\text{(eigenvalues of } M_0) = -\exp\left(\pm \tfrac{\pi i(n-2)}{n} \sqrt{1 + \tfrac{w}{4}\tfrac{(\lambda-1)^2}{\lambda}} \right) . \tag{25}$$

We now show that M_0 and g can be simultaneously unitarised at every point in $\mathbb{S}^1$ where $\hat{P}(z, \lambda)$ is defined: We define the half-traces $t_1 = (1/2)\,\mathrm{tr}(g^{-1})$, $t_2 = (1/2)\,\mathrm{tr}(M_0g^{-1})$ and $t_3 = (1/2)\,\mathrm{tr}(M_0)$. Then, since $M_0\, g^{-1}\, (M_0g^{-1})^{-1} = \mathrm{Id}$, the condition for simultaneous unitarizability [5], see also [1], of M_0 and g^{-1} and $(M_0g^{-1})^{-1}$ is

$$1 - t_1^2 - t_2^2 - t_3^2 + 2t_1t_2t_3 \geq 0 .$$

By Equations (24) and (25), this condition holds for all $\lambda \in \mathbb{S}^1$ (where $\hat{P}(z, \lambda)$ is defined) if and only if

$$-\cos\left(\tfrac{\pi(n-2)}{n} \sqrt{1 + \tfrac{w}{4}\tfrac{(\lambda-1)^2}{\lambda}} \right) \in \left[\cos\left(\tfrac{2\pi}{n}\right), 1\right] ,$$

and this in turn holds if and only if $w \in [\frac{-8n}{(n-2)^2}, 0]$, as in Equation (23).

It follows that the full monodromy group can be unitarized at all but a finite number of points in $\mathbb{S}^1$.

Note that if M_0 and g^{-1} commute for all $\lambda \in \mathbb{S}^1$, then M_0 must be diagonal, and hence $M_0(\lambda) \in \mathrm{SU}_2$ for all $\lambda \in \mathbb{S}^1$. In this case, Lemma 5.1 is then clearly true, so without loss of generality we may assume that $[M_0, g^{-1}] \neq 0$. Thus we can apply the gluing theorem [14] (see also [8]) to conclude there exists an initial condition h such that the monodromy group of $h\Phi$ is unitary. $\square$

Example 5.2. *Now let n, $\mathcal{P}$, Σ, ξ and w be as in Theorem 5.1. Let $\mathcal{D} = \{z \in \mathbb{C} \,|\, |z| \leq 1\}$ be the closed unit disk in $\mathbb{C}$. Let $h = h(\lambda)$ be the unitariser of the monodromy of the solution Φ of $d\,\Phi = \Phi\,\xi$ given by Theorem 5.1. Let*

$$f : \mathcal{D} \setminus \mathcal{P} \to \mathbb{R}^3 \tag{26}$$

be the CMC *immersion generated by the data $(\,\Sigma,\, \xi,\, h,\, 0\,)$. By the gluing theorem* [14]*, the immersion f via the Sym-Bobenko formula* [2]*, in* (26) *is defined when using r-Iwasawa splitting* [10] *for $r < 1$ and r sufficiently close to 1. Then, up to a rigid motion and homothety of $\mathbb{R}^3$, we find numerically that f has the following properties (see Figure 2):*

- *the image of f has order n dihedral symmetry,*
- *the boundary of the image of f consists of n complete planar geodesics that are congruent to each other, each lieing in a different plane*
$$\left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; \cos\left(\tfrac{2\pi j}{n}\right) x_1 + \sin\left(\tfrac{2\pi j}{n}\right) x_2 = 1\,\right\}$$
for $j = 0, 1, ..., n-1$,
- *f has n ends at the punctures in $\mathcal{P}$, and the image of each end is asymptotic to a $(\pi(n-2)/n)$-angle arc of a Delaunay nodoid,*
- *the axes of the asymptotically Delaunay ends are all vertical (i.e. parallel to the line $\{(0, 0, x_3) \in \mathbb{R}^3\}$) and the third coordinate x_3 of f satisfies $\lim_{z\in\mathcal{D}, z\to p} x_3 = +\infty$ for all $p \in \mathcal{P}$.*
- *If $n \in \{3,\, 4,\, 6\}$, the complete surface built by reflection across boundary planar geodesics is doubly periodic; in particular, it is invariant with respect to two independent translations of $\mathbb{R}^3$ parallel to the plane $\{(x_1, x_2, 0) \in \mathbb{R}^3\}$.*

Remark 5.3. *In the cases $n = 3, 4, 6$, the image $f(\mathcal{D} \setminus \mathcal{P})$ can be repeatedly reflected to produce a doubly-periodic surface with closed ends. By the asymptotics theorem* [14]*, the annular ends are asymptotically Delaunay with negative weight w.*

6. Open problems

(i) Can one prove that the surfaces in Theorems 3.1 and 4.1 are complete and properly immersed? Could one further prove the asymptotic behavior of their ends? In particular, are the ends of the examples in Theorem 3.1 asymptotic to ends of $2n$-legged Smyth surfaces?

(ii) By techniques like those used here, can one prove existence of a CMC surface with finite topology and asymptotically Delaunay ends and positive genus?

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