Swift-Hohenberg Equation for Lasers

J. Lega*
Institut Non Lineaire de Nice, UMR Centre National de la Recherche Scientifique 129, 1361 Route des Lucioles, 06560 Valbonne, France

J. V. Moloney†
Physics Department, University College Cork, Cork, Ireland

A. C. Newell*
Department of Mathematics, University of Arizona, Tucson, Arizona 85721
(Received 25 May 1994)

Pattern formation in large aspect ratio, single longitudinal mode, two-level lasers with flat end reflectors, operating near peak gain, is shown to be described by a complex Swift-Hohenberg equation for class A and C lasers and by a complex Swift-Hohenberg equation coupled to a mean flow for the case of a class B laser.

PACS numbers: 42.55.–f, 42.30.Sy, 42.65.–k

Large aspect ratio lasers offer an excellent paradigm for the study of extended structures in nonlinear systems [1]. Recent experiments in wide aperture lasers have demonstrated the existence of transverse patterns, often showing a variety of defects [2–5]. From a theoretical point of view, the universal nature of pattern formation near threshold is captured by generic order parameter equations whose form depends on symmetry properties of the lasing system [6–12].

In this Letter we demonstrate that a complex Swift-Hohenberg equation of the form

$$\frac{\partial\psi}{\partial t} = (\mu + i\nu)\psi + i\alpha\nabla^2\psi - (1 + i\beta)(\Omega + \nabla^2)^2\psi - (1 + i\gamma)|\psi|^2\psi \qquad (1)$$

provides the generic description of transverse pattern formation in wide aperture, single longitudinal mode, two-level lasers, when the laser is operating near peak gain (small detuning from maximum lasing emission). In Eq. (1), ψ is a complex field and μ, ν, α, β, and γ are real parameters. This equation, in its present form or with real coefficients and a real or complex order parameter, is a classic of pattern-forming models. First of all, it is a generalization to oscillatory systems [13,14] of the Swift-Hohenberg (SH) equation, which has been proposed as a model of stationary convection [15]. Second, it is also the generic equation for codimension-2 transitions [16] where the coefficient in front of the diffusive term is allowed to change sign. In terms of its solutions, we see that if Ω is positive, a traveling wave of the form $\psi = \sqrt{\mu}\exp\{i[\pm\sqrt{\Omega}\,x + (\nu - \gamma\mu - \alpha\Omega)t]\}$ will grow for positive μ, while when Ω is negative, a spatially homogeneous solution $\psi = \sqrt{\mu - \Omega^2}\exp[i(\nu - \gamma\mu + (\gamma - \beta)\Omega^2)t]$ will develop for $\mu > \Omega^2$. In the (μ,ν) plane, the transition point ($\mu = 0$, $\nu = 0$) where the trivial ($\psi = 0$), the homogeneous, and the traveling wave solutions coexist is the nonvariational analog of the Lifshitz point encountered in the phase diagrams of some magnetic systems [17]. In this latter case, however, the dynamics is variational, and the order parameter ψ is either real [18] or complex [19]. In terms of pattern formation, a Lifshitz point corresponds to the coexistence of three different patterns, as exemplified for instance in [20] for electrohydrodynamic convection in nematic liquid crystals. Recently, a SH equation has been considered for a laser with injected signal in Ref. [21]. Here, we will see that when the detuning is small, an equation similar to (1) comes naturally as a solvability condition for the existence of solutions to the Maxwell-Bloch (MB) laser equations in the form of asymptotic series in powers of the small detuning parameter. We will call this equation the laser Swift-Hohenberg equation and see that it captures the main features of the laser dynamics in so-called [22] class A and C lasers.

In commonly encountered class B lasers, such as commercially important gas CO_2 and semiconductor lasers, the polarization damping rate is many orders of magnitude faster than the cavity and population inversion decay rates. In this case, we will see that the population inversion acts like a mean flow, driving the active modes at finite wave number, and the resulting coupled complex order parameter equations provide a generalized rate equation description of a wide aperture laser. Some important conclusions to be drawn from the present study are that increased stiffness of the laser problem leads to a rapid shrinking of the stable domain of traveling wave lasing solutions and that the predictions of the complex order parameter equation description in terms of the SH equation in the nonstiff limit or the generalized coupled rate equations in the stiff limit provide remarkably good agreement with those of the full laser MB equations, even for lasing well beyond threshold.

The dynamics of the two-level laser in a section transverse to the main direction of propagation of the electromagnetic wave is described, in the simple case of a single

 0031-9007/94/73(22)/2978(4)$06.00

longitudinal mode and flat end mirrors, by the following set of MB equations [23], written here in complex Lorenz notation [1]:

$$\begin{aligned} e_t - ia\nabla^2 e &= -\sigma e + \sigma p, \\ p_t + (1 + i\Omega)p &= (r - n)e, \\ n_t + bn &= \tfrac{1}{2}(e^* p + e p^*). \end{aligned} \tag{2}$$

The complex variables e and p are the scaled envelopes of the electric and polarization fields, n is proportional to the difference between the atomic inversion and the initial inversion, σ and b are, respectively, the decay rates of the electric field and of the population inversion, both scaled to the decay rate of the polarization, and the detuning Ω is the difference between the atomic and the cavity frequencies, divided by the polarization decay rate. The solutions and properties of this model have already been described in the literature [1,11,24]. It is useful for the following to note that the lasing threshold is given by $r_c = 1 + (\Omega - ak_c^2)^2/(1+\sigma)^2$, with $k_c = 0$ if $\Omega < 0$, and $k_c^2 = \Omega/a$ if $\Omega > 0$. In other words, the nature of the bifurcation changes depending on the sign of the detuning Ω. In order to capture the behavior of the MB equations for both signs of the detuning, we will assume $\Omega = \epsilon\Omega_1$ small and look for solutions (e,p,n) in the form of a power series expansion in the small parameter ϵ. The laser variables will also depend on slow temporal and spatial scales, which have to be determined. For $\Omega = 0$, $k_c = 0$, and it is easily shown that, above threshold, a band of wave vectors k of width $R^{1/4} = (r-1)^{1/4}$ centered about $k_c = 0$ is experiencing growth. The right scaling for the spatial variables is then $X = (r-1)^{1/4}x$ and $Y = (r-1)^{1/4}y$. In order to have the terms in Ω of the same order as the spatial derivative term $ia\nabla^2$, we will assume $r = 1 + \epsilon^2$, $X = \sqrt{\epsilon}x$, and $Y = \sqrt{\epsilon}y$. We also need to introduce two slow time scales, namely $T_1 = \epsilon t$ and $T_2 = \epsilon^2 t$. Substituting these expressions into the MB equations and identifying the coefficients of powers of ϵ at each order, we obtain, by applying the solvability conditions at $O(\epsilon^2)$ and $O(\epsilon^3)$ and rescaling to the original time and space scales [25], the laser Swift-Hohenberg equation:

$$\begin{aligned} (\sigma + 1)\frac{\partial\psi}{\partial t} = \sigma(r-1)\psi - \frac{\sigma}{(1+\sigma)^2}(\Omega + a\nabla^2)^2\psi \\ + ia\nabla^2\psi - i\Omega\sigma\psi - \frac{\sigma}{b}|\psi|^2\psi, \end{aligned} \tag{3}$$

which is a particular version of Eq. (1). The original variables are related to ψ by the following formulas:

$$\begin{aligned} e &= \psi, \\ p &= \psi - \frac{ia}{1+\sigma}\nabla^2\psi - \frac{i\Omega}{1+\sigma}\psi + \frac{r-1}{1+\sigma}\psi \\ &\quad - \frac{1}{b(1+\sigma)}|\psi|^2\psi - \frac{1}{(1+\sigma)^3}(\Omega + a\nabla^2)^2\psi, \\ n &= \frac{1}{b}|\psi|^2 + \frac{ia}{1+\sigma}\left(\frac{1}{2b} + \frac{1}{b^2}\right)(\psi\nabla^2\bar{\psi} - \bar{\psi}\nabla^2\psi). \end{aligned}$$

One can check that the nonlasing solution $\psi = 0$, which corresponds to $(e,p,n) = (0,0,0)$, becomes unstable with respect to spatial perturbations of wave vector k if

$$\frac{\sigma}{1+\sigma}(r-1) > \frac{\sigma}{(1+\sigma)^3}(\Omega - ak^2)^2,$$

which is the same condition as for the two-level laser [1,7]. Above threshold, Eq. (3) admits a traveling wave solution of the form $\psi = R\exp[i(kx + \omega t)]$, where ω and R^2 read

$$\begin{aligned} \omega &= -(\sigma\Omega + ak^2)/(1+\sigma), \\ R^2 &= b\left[r - 1 - \left(\frac{\Omega - ak^2}{1+\sigma}\right)^2\right]. \end{aligned} \tag{4}$$

Again, this formula is to be compared with the traveling wave solution of the MB equations. Using the expressions of e, p, and n in terms of ψ obtained above, we have

$$e = R\exp[i(kx + \omega t)], \qquad p = e\left[1 + i\frac{ak^2 - \Omega}{1+\sigma}\right],$$
$$n = |e|^2/b,$$

which is exactly the traveling wave solution of the MB equations [1,11].

In the following, we investigate the stability of the above traveling wave solutions and compare the results obtained from the SH equation to those given by the full system. It is convenient to summarize the results by drawing the domains of existence and stability in the (k,r) plane. By analogy with convective systems, the stability domain is called the *Busse balloon* [26].

Figure 1 gives an example of stability diagrams obtained from the MB equations, for different values of the stiffness parameter b. The curve labeled $b = 0.8$ corresponds to the nonstiff limit, where the single Swift-Hohenberg equation (3) is expected to be valid. Figure 2 shows the same plots obtained from the SH equations. In the case of the single SH equation, the stability domain is bounded by the Eckhaus boundary [27], given by

$$-2a\frac{\sigma}{(1+\sigma)^3}(\Omega - 3ak^2) - \frac{8a^2k^2b\sigma}{(1+\sigma)^5R^2}(\Omega - ak^2)^2 = 0,$$

which does not depend on b. This curve (labeled E in Fig. 2) is almost identical to the boundary obtained from the MB equations for a stiffness parameter of order 1 (curve labeled $b = 0.8$ in Fig. 1). It is easy to show, by rescaling the SH equation (3) above, that the coefficient b can be removed completely, which means that the single SH equation is insensitive to the stiffness of the problem. However, it accurately captures the laser dynamics in the nonstiff limit. It is worth noticing that what was a higher order phase instability for the full problem appears here as a regular (Eckhaus) phase instability.

We are now interested in the stiff limit ($b \to 0$) of the MB equations, which describes a class B laser. In this case, the Busse balloon shrinks to a very narrow band centered about k_c, as shown in Fig. 1. The laser output in the unstable domain is then highly disorganized and shows many defects, which are advected away by the unstable carrying traveling wave. Such a limit is

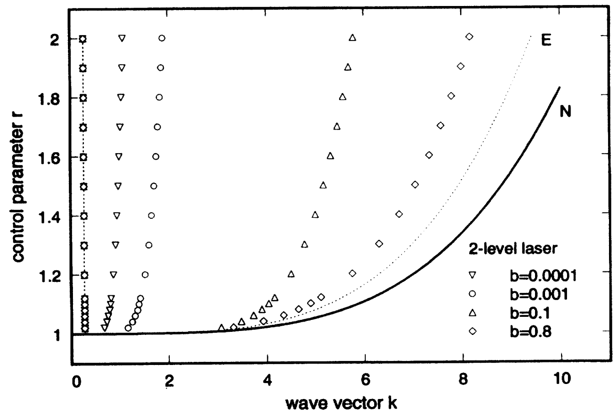

FIG. 1. Existence and stability domains of the traveling wave solutions to the MB equations, for different values of b. The other parameters are $a = 0.01$, $\sigma = 0.1$, and $\Omega = 0.001$. The left boundary of the Busse balloon does not depend on b. Note that, although the Eckhaus boundary (E) coincides with the right boundary of the Busse balloon close to threshold, higher order phase instabilities always come in far from threshold.

a relevant model for the description of wide aperture CO_2 and semiconductor lasers. It can easily be seen that the laser Swift-Hohenberg equation will not be a good model for the laser dynamics. Indeed, Fig. 3(a) shows the behavior of the real parts of the three biggest eigenvalues of the linearized system obtained from the MB equations about the traveling wave solution with wave vector $k = 0.9$, for $r = 1.2$ and $b = 0.01$. These eigenvalues are plotted as functions of the perturbation wave vector q along x. A similar plot is given in Fig. 3(b), for a nonstiff laser, in this case for $b = 0.2$. One immediately sees that two eigenvalues are close to zero in the nonstiff case, which makes reasonable the elimination of the three most damped scalar variables and then justifies the use of the laser Swift-Hohenberg equation. However, having b small in the stiff limit makes the corresponding eigenvalue close to zero. One can then at best eliminate two scalar quantities, which means that two coupled equations are in order to accurately describe the laser dynamics in this case.

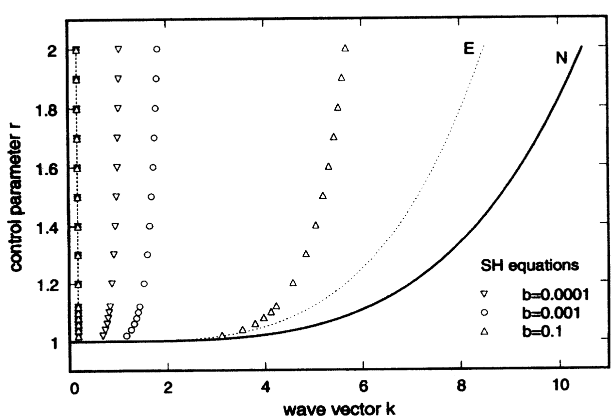

FIG. 2. Same plot as for Fig. 1, but for the reduced SH equations (5). The Busse balloon given by the single SH equation (3) is delimited by the Eckhaus boundary (E).

Since the traveling wave solution to the full laser problem is

$$n = \frac{|e|^2}{b} = r - 1 - \left(\frac{\Omega - ak^2}{\sigma + 1}\right)^2 = O(\epsilon^2),$$

we see that if b is, say, of order ϵ^2, n and $|e| = |\psi|$ are also of order ϵ^2. We will then look for expressions of e, p, and n starting at order ϵ^2. The derivation of the coupled equations follows the above procedure and is presented in detail in Ref. [25]. They read at order 4 in ϵ

$$\begin{aligned}(\sigma + 1)\frac{\partial \psi}{\partial t} &= \sigma(r - 1)\psi + ia\nabla^2\psi - i\sigma\Omega\psi \\ &\quad - \frac{\sigma}{(1 + \sigma)^2}(\Omega_1 + a\nabla^2)^2\psi - \sigma n\psi, \qquad (5) \\ \frac{\partial n}{\partial t} &= -bn + |\psi|^2.\end{aligned}$$

The instability threshold of the trivial solution $\psi = 0$, $n = 0$ is the same as for the laser Swift-Hohenberg equation, and the traveling wave solution above threshold is also unchanged. The phase instability boundaries are also the same as for the laser Swift-Hohenberg equation [25]. The only difference is that because of the coupling of ψ and n, higher order instabilities occur, and the Busse balloon shrinks. The stability diagram obtained from these two coupled equations for the same laser parameters as in

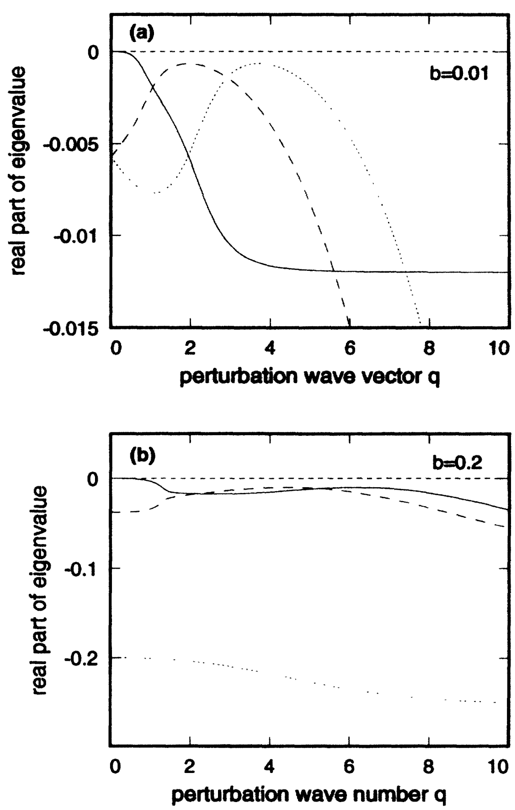

FIG. 3. Real parts of the three biggest eigenvalues obtained from the linearized MB equations about the traveling wave solution at $k = 0.9$ and $r = 1.2$. (a) Stiff limit ($b = 0.01$). (b) Nonstiff limit ($b = 0.2$). The other parameters are unchanged. The two eigenvalues which are not shown have a real part close to -1.

Fig. 1 are given in Fig. 2. The Busse balloon is correctly reduced to a narrow band about $k = k_c$, and the dynamics of the MB equations is now satisfactorily captured by the reduced equations. Again, we stress the fact that the Busse balloon of the laser Swift-Hohenberg equation alone, which is delimited by the long wavelength phase instability boundary (curve labeled E on Fig. 2), is not accurate in the stiff limit.

These coupled order parameter equations provide the key result of this Letter. We have shown that these universal amplitude equations, strictly valid in the neighborhood of the critical point, also hold true well beyond the onset of lasing. In addition, the traditional single SH equation is insensitive to the degree of stiffness of the original physical problem. As a consequence of this, a mean flow must be coupled to the SH equation, which is consistent with the observation that the population inversion variable n in the MB laser equations acts as a weakly damped mode when the problem becomes stiff [28]. The effect of this additional degree of freedom is to destabilize the laser further through a short wavelength higher order phase instability which cannot be captured by the usual phase (Cross and Newell [29]) evolution equation. Both experiments and numerical simulations on broad area semiconductor lasers display strong filamentation instabilities, which manifest themselves through an approximately parabolic dispersion curve [(k, ω) spectrum]. Because of the shrinking of the Busse balloon, numerical simulations of the MB equations in the stiff limit show qualitatively similar features, which justifies taking the coupled SH equations (5) as a robust generalized rate equation model. Unlike conventional rate equation models, which display spurious, nonphysical, high-wave-number instabilities [11], Eqs. (5) naturally incorporate diffusion at short spatial scales, thereby regularizing the dynamics.

The authors wish to thank the Arizona Center for Mathematical Sciences (ACMS) for support. ACMS is sponsored by AFOSR Contract No. F49620-94-1-0144DEF. The authors acknowledge support from University College Cork. J. V. Moloney and J. Lega were supported in part under a European Economic Community Twinning Grant No. SCI-0325-C(SMA).

---

*Part of this work was carried out by the author at Unversity College Cork, Cork, Ireland.

†Permanent address: Department of Mathematics, University of Arizona, Tucson, Arizona 85721.

[1] A. C. Newell and J. V. Moloney, *Nonlinear Optics* (Addison Wesley Publishing Co., Redwood City, CA, 1992).

[2] J. R. Tredicce, E. J. Quel, A. Ghazzawi, C. Green, M. A. Pernigo, L. M. Narducci, and L. A. Lugiato, Phys. Rev. Lett. **62**, 1274 (1989); C. Green, G. B. Mindlin, E. J. D'Angelo, H. G. Solari, and J. R. Tredicce, Phys. Rev. Lett. **65**, 3124 (1990).

[3] D. Dangoisse, D. Hennequin, C. Lepers, E. Louvergneaux, and P. Glorieux, Phys. Rev. A **46**, 5955 (1992).

[4] E. J. D'Angelo, E. Izaguirre, G. B. Mindlin, G. Huyet, L. Gil,and J. R. Tredicce, Phys. Rev. Lett. **68**,, 3702 (1992).

[5] F. T. Arecchi, G. Giacomelli, P. L. Ramazza, and S. Residori, Phys. Rev. Lett. **67**, 3749 (1991); F. T. Arecchi, S. Boccaletti, and P. L. Ramazza, Phys. Rev. Lett. **70**, 2277 (1993).

[6] For a review on order parameter equations, see, for instance, A. C. Newell, T. Passot, and J. Lega, Annu. Rev. Fluid Mech. **25**, 399 (1993), or M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).

[7] P. Coullet, L. Gil, and F. Rocca, Opt. Commun. **73**, 403 (1989).

[8] L. A. Lugiato, G.-L. Oppo, J. R. Tredicce, L. M. Narducci, and M. A. Pernigo, J. Opt. Soc. Am. B **7**, 1019 (1990).

[9] H. G. Solari and R. Gilmore, J. Opt. Soc. Am. B **7**, 828 (1990).

[10] L. Gil, K. Emilson, and G.-L. Oppo, Phys. Rev. A **45**, 567 (1992).

[11] P. K. Jakobsen, J. V. Moloney, A. C. Newell, and R. Indik, Phys. Rev. A **45**, 8129 (1992).

[12] R. López Ruiz, G. B. Mindlin, C. Pérez-Garcia, and J. R. Tredicce, Phys. Rev. A **47**, 500 (1993).

[13] B. A. Malomed, Z. Phys. B **55**, 241 (1984); **55**, 249 (1984).

[14] M. Bestehorn, R. Friedrich, and H. Haken, Z. Phys. B **75**, 265 (1989); **77**, 151 (1989); M. Bestehorn and H. Haken, Phys. Rev. A **42**, 7195 (1990).

[15] J. Swift and P. C. Hohenberg, Phys. Rev. A **15**, 319 (1977).

[16] P. Coullet and D. Repaux, *Instabilities and Nonequilibrium Structures,* edited by E. Tirapegui and D. Villaroel (D. Reidel Publishing Company, Dordrecht, 1987), pp. 179–195.

[17] R. M. Hornreich, M. Luban, and S. Shtrikman, Phys. Rev. Lett. **35**, 1678 (1975).

[18] A. Michelson, Phys. Rev. B **16**, 577 (1977).

[19] A. Michelson, Phys. Rev. B **16**, 585 (1977).

[20] E. Bodenshatz, W. Zimmermann, and L. Kramer, J. Phys. (Paris) **49**, 1875 (1988).

[21] M. Tlidi, M. Giorgiou, and P. Mandel, Phys. Rev. A **48**, 4605 (1993).

[22] J. R. Tredicce, F. T. Arecchi, G. L. Lippi, and G. P. Puccioni, J. Opt. Soc. Am. B **2**, 173 (1985).

[23] L. A. Lugiato, C. Oldano, and L. M. Narducci, J. Opt. Soc. Am. B **5**, 879 (1988).

[24] Q. Feng, J. V. Moloney, and A. C. Newell, Phys. Rev. Lett. **71**, 1705 (1993).

[25] J. Lega, J. V. Moloney, and A. C. Newell (to be published).

[26] F. H. Busse, Rep. Prog. Phys. **41**, 1929 (1978).

[27] In two dimensions, the left boundary of the Busse balloon corresponds to a zigzag instability and is given by $-2a\sigma/(1+\sigma)^3(\Omega - ak^2) = 0$.

[28] Our experience from numerical study of the eigenvalues of the linearization of the two-level laser suggests that an appropriate criterion for the validity of the single SH equation is that the damping coefficient b be bigger in magnitude than the cavity damping coefficient σ. When this condition is violated, the coupled equations still give a good description, even well beyond critical.

[29] M. C. Cross and A. C. Newell, Physica (Amsterdam) **10D**, 299 (1984).