Math. Z. (2003) **Mathematische Zeitschrift**

DOI: 10.1007/s00209-003-0587-y

Dressing CMC n-Noids

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Received: 14 December 2002 / in final form: 3 April 2003 /
Published Online: ■■ –

Abstract. The purpose of this paper is to construct CMC n-noids with bubbletons. We recall a specific class of dressing matrices, which we will refer to as simple factor dressing matrices, which in many known cases add bubbletons to a CMC surface. Our new surfaces are obtained by dressing known n-noids with embedded Delaunay ends by well chosen simple factor dressing matrices in such a way that the dressed surface is also a CMC n-noid.

Mathematics Subject Classification (2000): 53A10.

Introduction

Constant mean curvature n-noids with Delaunay ends were first proven to exist by Kapouleas [9].

Of particular interest are those n-noids which have embedded Delaunay ends. Two families of examples exist in the literature. Firstly there is a construction and classification of all (Alexandrov) embedded trinoids by Grosse-Brauckmann, Kusner and Sullivan [7]. Secondly there are examples of n-noids due to Dorfmeister and Wu [6] and Schmitt [19, 18], which have embedded Delaunay ends. In the case of trinoids the computer generated images indicate that these two families are identical.

The term"bubbleton" arose from the study of Sievert's classical examples of constant Gauss curvature surfaces [20]. The constant mean curvature parallel surfaces to Sievert's are the single bubbletons. These first appeared in [16]. By repeatedly applying the Bianchi-Bäcklund transformation (an ancient form of dressing)

First author supported by CEC Contract HPRN-CT-2000-00101 EDGE. Second author supported by National Science Foundation grant DMS-00-76085.

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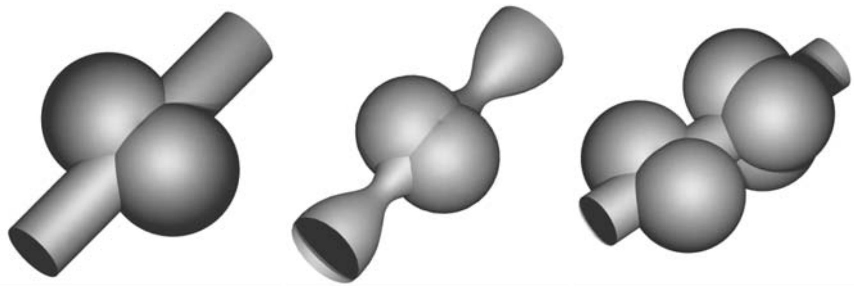

Fig. 1 Two-lobed Bubbletons with round cylindrical ends and Delaunay ends. Multibubbleton with 2 and 3 lobes with round cylindrical ends

to the round cylinder one produces the "multi-bubbletons" studied by Sterling and Wente [21] and in unpublished work [22]. Bubbletons have individual (renormalized) energies which are invariant under isospectral flows which appear in computer animations as bubbletons colliding through each other and spinning with respect to each other. Bubbletons can also be added to Delaunay surfaces (Wente [26]) and isospectrally deformed along them (Pinkall and Sterling [17]). More recently this has been investigated by Kilian [10] using the Dorfmeister-Pedit-Wu (DPW) method [5] and by Kobayashi [12] using a modification of the Terng-Uhlenbeck approach [23].

This paper initiates the investigation of the set of n-noids with bubbletons. We are able to add a bubbleton to known n-noids with embedded Delaunay ends and produce n-noids with a bubbleton. Moreover, we can deform some of the examples via a two-parameter family of deformations which imitates the isospectral orbits found in the finite-type setting (especially in the multi-bubbleton theory). Unlike the isospectral case, within these deformation families it appears that at certain values spheres bubble off. We hope to return to these issues in a later paper. Spheres bubbling off is a branching phenomenon of CMC deformations first discovered by Wente [24,25].

We begin by recalling the basic facts concerning loop groups, the DPW method and dressings in section 1. In particular we define the simple factor matrices which play a central role in our construction. In section 2 we study the period problem via monodromy and its relationship with dressing. In particular we relate the eigenlines of the monodromy with period closing dressings by simple factor matrices.

The first main result of the paper, which is for the case of trinoids, theorem 3.6, is obtained as follows. We start with known trinoid input data to the DPW method, gauging to a setting where the DPW integration produces loops which can be explicitly calculated via hypergeometric functions and whose monodromies are given in terms of exponentials and Γ functions. Then, as is done in [19,18,6], we find the dressing matrix which dresses this loop to a loop which produces a closed trinoid. The next step is to compute the set of points in the loop parameter at which all the end monodromies have a common eigenline. It was the discovery of such points which was the crucial step in the development of the proof of existence of these examples. For such points, simple factor matrices can be constructed which will dress the given closed trinoid to a closed trinoid with a bubbleton on it. It turns out that for some values of the loop parameters all the end monodromies are the identity matrix and this yields a two-parameter family of deformations of the bubbleton produced for that point.

Section 3.2. contains our second main result. Using a similar approach, but with a different proof (without hypergeometric functions) we dress some co-planar n-noids to n-noids with bubbletons.

A list of open problems and ideas for directions of future research is given in section 4.

We include a few graphics to indicate the kind of surfaces we construct. For many more examples and for mpegs showing the deformations please see the GANG Gallery of CMC surfaces, created by the second author, at `http://www.gang.umass.edu`.

The authors are grateful to Josef Dorfmeister and Hongyou Wu for making available their work [6] to us in preprint form.

1 Preliminaries

We begin by collecting well known results on loop groups, the DPW method and the dressing action.

1.1. Loop groups

For real $r \in (0, 1]$, let $\mathcal{C}_r = \{\lambda \in \mathbb{C} \,|\, |\lambda| = r\}$ and denote the analytic maps from $\mathcal{C}_r$ with values in $\mathrm{SL}_2(\mathbb{C})$ by $\Lambda_r\mathrm{SL}_2(\mathbb{C}) = \mathcal{O}(\mathcal{C}_r, \mathrm{SL}_2(\mathbb{C}))$. The Lie algebras of these groups, $\Lambda_r\mathrm{sl}_2(\mathbb{C})$, consist of analytic maps $g : \mathcal{C}_r \to \mathrm{sl}_2(\mathbb{C})$. We will use the following subgroups of $\Lambda_r\mathrm{SL}_2(\mathbb{C})$: let $I_r = \{\lambda \in \mathbb{C} \,|\, |\lambda| < r\}$,

$$\mathcal{B} = \left\{ \left(\begin{smallmatrix} a & c \\ 0 & 1/a \end{smallmatrix}\right) \,|\, a \in \mathbb{R}^*_+,\ c \in \mathbb{C} \right\}$$

and

$$\Lambda^+_r\mathrm{SL}_2(\mathbb{C}) = \{g \in \Lambda_r\mathrm{SL}_2(\mathbb{C}) \cap \mathcal{O}(I_r, \mathrm{SL}_2(\mathbb{C})) \,|\, g(0) \in \mathcal{B}\}\,.$$

Analogously, let $A_r = \{\lambda \in \mathbb{C} \,|\, r < |\lambda| < 1/r\}$ and set

$$\Lambda^*_r\mathrm{SL}_2(\mathbb{C}) = \left\{g \in \Lambda_r\mathrm{SL}_2(\mathbb{C}) \cap \mathcal{O}(A_r, \mathrm{SL}_2(\mathbb{C})) :\ g|_{S^1} \in \mathrm{SU}_2\right\}.$$

For $r = 1$ we will omit the subscript. Multiplication

$$\Lambda^*_r\mathrm{SL}_2(\mathbb{C}) \times \Lambda^+_r\mathrm{SL}_2(\mathbb{C}) \to \Lambda_r\mathrm{SL}_2(\mathbb{C}) \tag{1.1}$$

is a diffeomorphism onto [14] and consequently any $g \in \Lambda_r\mathrm{SL}_2(\mathbb{C})$ can be uniquely factored into $g = F\,B$ with $F \in \Lambda^*_r\mathrm{SL}_2(\mathbb{C})$ and $B \in \Lambda^+_r\mathrm{SL}_2(\mathbb{C})$. Moreover, given a map Φ with range $\Lambda_r\mathrm{SL}_2(\mathbb{C})$, we can factorize pointwise on the domain of Φ to obtain $\Phi = FB$ where F respectively B are maps of the same domain with values in $\Lambda^*_r\mathrm{SL}_2(\mathbb{C})$ respectively $\Lambda^+_r\mathrm{SL}_2(\mathbb{C})$. We denote the projection $\Phi \to F$ by $\mathsf{Uni}_r : \Lambda_r\mathrm{SL}_2(\mathbb{C}) \to \Lambda^*_r\mathrm{SL}_2(\mathbb{C})$.

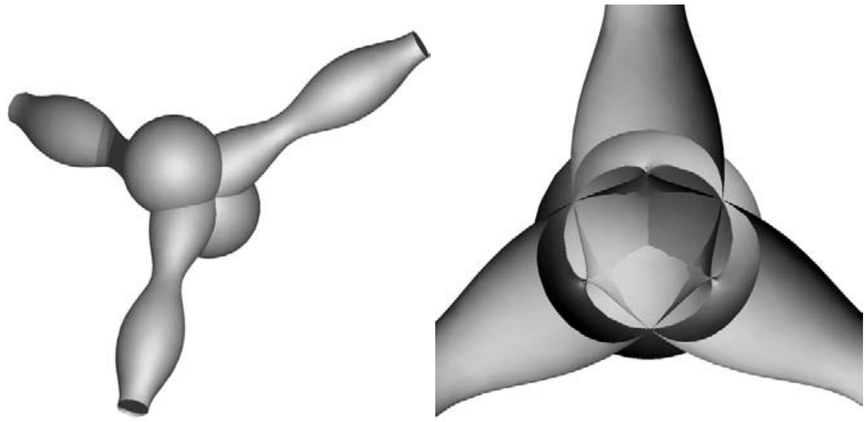

Fig. 2 Maximal equilateral trinoid with two-lobed bubble

1.2. DPW method

Let $\Lambda\Omega(\Sigma)$ denote the holomorphic 1-forms on a Riemann surface Σ with values in

$$\left\{\xi \in \mathcal{O}(\mathbb{C}^*, \mathrm{sl}_2(\mathbb{C})) : \xi(\lambda) = \sum \xi_j \lambda^j,\ j \geq -1,\ \xi_{-1} \in \mathbb{C}^* \otimes \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\right\}.$$

The DPW method [5] constructs all conformal CMC immersions of the universal cover $\widetilde{\Sigma}$ in three steps: Let $\xi \in \Lambda\Omega(\widetilde{\Sigma})$, $\tilde{z}_0 \in \widetilde{\Sigma}$ and $\Phi_0 \in \Lambda_r\mathrm{SL}_2(\mathbb{C})$.

1. Solve $d\Phi = \Phi\xi,\ \Phi(\tilde{z}_0) = \Phi_0$ to obtain a unique map $\Phi : \widetilde{\Sigma} \to \Lambda_r\mathrm{SL}_2(\mathbb{C})$.
2. Project Φ pointwise on $\widetilde{\Sigma}$ to obtain $F = \mathsf{Uni}_r(\Phi) : \widetilde{\Sigma} \to \Lambda_r^*\mathrm{SL}_2(\mathbb{C})$.
3. Evaluate the Sym-Bobenko formula

$$f_\mu = \mathsf{Sym}_\mu[F] = -2i\mu H^{-1}\ \frac{\partial F}{\partial \lambda}\Big|_\mu F^{-1} \tag{1.2}$$

to obtain for each $|\mu| = 1$ a conformal immersion (possibly branched) $\widetilde{\Sigma} \to \mathrm{su}_2 \cong \mathbb{R}^3$ with constant mean curvature $H \neq 0$. Maps $\widetilde{\Sigma} \to \Lambda_r^*\mathrm{SL}_2(\mathbb{C})$ generated in the above steps 1 and 2 are called r-unitary frames and will be denoted by $\mathcal{F}_r(\widetilde{\Sigma})$. If the triple $(\xi,\ \tilde{z}_0,\ \Phi_0)$ generates $F \in \mathcal{F}_r(\widetilde{\Sigma})$ and $h \in \Lambda_r^+\mathrm{SL}_2(\mathbb{C})$, then the triple $(\xi,\ \tilde{z}_0,\ h\Phi_0)$ generates the r-unitary frame $\mathsf{Uni}_r(hF)$. This variation of initial condition by left multiplication is the dressing action [2] in our context. For fixed $F \in \mathcal{F}_r(\widetilde{\Sigma})$, we denote the isotropy under dressing by

$$\mathrm{Iso}(F) = \left\{h \in \Lambda_r^+\mathrm{SL}_2(\mathbb{C}) \mid \mathsf{Uni}_r(hF) = F\right\}$$

and recall from [4] that if $\mathsf{Sym}_\mu[F]$ has umbilics, then $\mathrm{Iso}(F) = \{\mathrm{I}\}$.

1.3. Simple factors

Let $\pi_L : \mathbb{C}^2 \to L$ be the hermitian projection onto a line $L \in \mathbb{P}^1$. For $\lambda_0 \in \mathbb{C}$, simple factors [23] are loops of the form

$$\psi_{L,\lambda_0}(\lambda) = \pi_L + \frac{\lambda_0 - \lambda}{1 - \overline{\lambda_0}\lambda}\,\pi_L^{\perp}. \tag{1.3}$$

To obtain elements of $\Lambda_r^+\mathrm{SL}_2(\mathbb{C})$, write

$$(\det \psi_{L,\lambda_0}(0))^{-1/2}\,\psi_{L,\lambda_0}(0) = Q_{L,\lambda_0}\,R \tag{1.4}$$

with $Q_{L,\lambda_0} \in \mathrm{SU}_2$, $R \in \mathcal{B}$ and restrict to $0 < |\lambda_0| < 1$.

Definition 1.1. *Let $L \in \mathbb{P}^1$ and $\lambda_0 \in \mathbb{C}$ with $0 < |\lambda_0| < 1$. A simple factor of $\Lambda_r^+\mathrm{SL}_2(\mathbb{C})$ with $r < |\lambda_0|$, is a loop of the form*

$$h_{L,\lambda_0} = (\det \psi_{L,\lambda_0})^{-1/2}\,Q_{L,\lambda_0}^{-1}\psi_{L,\lambda_0} \tag{1.5}$$

with ψ_{L,λ_0} and Q_{L,λ_0} defined as in (1.3) *respectively* (1.4)*. The set $\mathcal{H}_{\lambda_0} = \{h_{L,\lambda_0} \mid L \in \mathbb{P}^1\}$ can be identified with $\mathbb{P}^1$.*

By construction $h_{L,\lambda_0} \in \Lambda_s^*\mathrm{SL}_2(\mathbb{C})$ for $s > |\lambda_0|$. By Proposition 4.2 in [23] dressing by simple factors is explicit. We prove this result in our setting.

Theorem 1.2. *Let $F(z,\lambda) \in \mathcal{F}_r(\widetilde{\Sigma})$ for $r \in (0,1)$. Let h_{L,λ_0} be simple with $\lambda_0 \in \mathbb{C}$, $r < |\lambda_0| < 1$ and $L \in \mathbb{P}^1$. Then*

$$\mathsf{Uni}_r(h_{L,\lambda_0}\,F) = h_{L,\lambda_0}\,F\,h_{L',\lambda_0}^{-1} \text{ with } L' = \overline{F(z,\lambda_0)}^{\,t}\,L. \tag{1.6}$$

Proof. Clearly $h_{L,\lambda_0}Fh_{L',\lambda_0}^{-1} : \widetilde{\Sigma} \to \Lambda_r^*\mathrm{SL}_2(\mathbb{C})$ away from $1/\overline{\lambda_0}$ and λ_0 where it respectively its inverse have a simple pole. Since

$$h_{L,\lambda_0}Fh_{L',\lambda_0}^{-1} = Q_{L,\lambda_0}^{-1}\psi_{L,\lambda_0}F\psi_{L',\lambda_0}^{-1}Q_{L',\lambda_0}$$

it suffices to show that the residues $\mathrm{res}_{\lambda_0}\,G^{-1} = \mathrm{res}_{1/\overline{\lambda_0}}\,G = 0$ for $G = \psi_{L,\lambda_0}F\psi_{L',\lambda_0}^{-1}$. From $\overline{F}^{t}L \perp F^{-1}L^{\perp}$ we obtain $L'^{\perp} = F(z,\lambda_0)^{-1}L^{\perp}$ and $\pi_{L'}^{\perp}F(z,\lambda_0)^{-1} = F(z,\lambda_0)^{-1}\pi_L^{\perp}$. Thus

$$\mathrm{res}_{\lambda_0}G^{-1} = \lim_{\lambda\to\lambda_0}(\lambda-\lambda_0)\psi_{L',\lambda_0}F^{-1}\psi_{L,\lambda_0}^{-1} = (1-|\lambda_0|^2)\pi_{L'}F(z,\lambda_0)^{-1}\pi_L^{\perp} = 0.$$

Similarly, $L' = \overline{F(z,\lambda_0)}^{\,t}L$ yields $F(z,1/\overline{\lambda_0})\pi_{L'} = \pi_L F(z,1/\overline{\lambda_0})$ and hence

$$\mathrm{res}_{\overline{\lambda_0}^{-1}}G = \lim_{\lambda\to 1/\overline{\lambda_0}}(1-\overline{\lambda_0}\lambda)\psi_{L,\lambda_0}F\psi_{L',\lambda_0}^{-1} = (1/\overline{\lambda_0}-\lambda_0)\pi_L^{\perp}F(z,1/\overline{\lambda_0})\pi_{L'} = 0.$$

As $h_{L',\lambda_0} : \widetilde{\Sigma} \to \Lambda_r^+\mathrm{SL}_2(\mathbb{C})$, equation (1.6) provides the unique pointwise factorization of $h_{L,\lambda_0}\,F$ with respect to the splitting (1.1). □

2 The period problem

Let Σ be a connected Riemann surface with universal cover $\widetilde{\Sigma}$ and let Δ denote the group of deck transformations. Let $\xi \in \Lambda\Omega(\Sigma)$ and $\Phi : \widetilde{\Sigma} \to \Lambda_r \mathrm{SL}_2(\mathbb{C})$ be a solution of the differential equation $d\Phi = \Phi\xi$. Writing $\tau^*\Phi = \Phi \circ \tau$ for $\tau \in \Delta$, define the monodromy of Φ with respect to τ by $\mathcal{M}_\Phi(\tau) = (\tau^*\Phi)\Phi^{-1}$. A choice of base point $\tilde{z}_0 \in \widetilde{\Sigma}$ and initial condition $\Phi_0 \in \Lambda_r \mathrm{SL}_2(\mathbb{C})$ gives the monodromy representation $\mathcal{M}_\Phi : \Delta \to \Lambda_r \mathrm{SL}_2(\mathbb{C})$. Henceforth, when we speak of the monodromy representation, or simply monodromy, we tacitly assume that it is induced by such an underlying triple $(\xi, \tilde{z}_0, \Phi_0)$. If $F = \mathsf{Uni}_r(\Phi)$, we are not assured that $\mathcal{M}_F(\tau) = (\tau^*F)F^{-1}$ is z-independent for all $\tau \in \Delta$. To circumvent this issue, we shall ensure that $\mathcal{M}_\Phi$ is $\Lambda_r^*\mathrm{SL}_2(\mathbb{C})$-valued, since then $\mathcal{M}_F = \mathcal{M}_\Phi$ by uniqueness of the Iwasawa decomposition [11]. Further, if $(\xi, \tilde{z}_0, \Phi_0)$ generates f_μ and $\mathcal{M}_\Phi(\tau) \in \Lambda_r^*\mathrm{SL}_2(\mathbb{C})$ for all $\tau \in \Delta$, then the following are equivalent: (i) there exists a $\mu_0 \in S^1$ such that $\tau^* f_{\mu_0} = f_{\mu_0}$ for all $\tau \in \Delta$, and (ii) $\mathcal{M}_\Phi$ satisfies

$$\mathcal{M}_\Phi(\tau, \mu_0) = \pm \mathrm{I} \text{ and } \partial_\lambda \mathcal{M}_\Phi(\tau, \lambda)|_{\mu_0} = 0 \tag{2.1}$$

for all $\tau \in \Delta$. For a proof of this result and related ideas see [3] and [10].

Theorem 2.1. *Let $F \in \mathcal{F}_r(\widetilde{\Sigma})$ with monodromy $\mathcal{M}_F$. Let $f_\mu = \mathsf{Sym}_\mu[F]$ and assume that $\tau^* f_{\mu_0} = f_{\mu_0}$ for $\mu_0 \in S^1$ and for all $\tau \in \Delta$.*

(i) Assume there exists $\lambda_0 \in \mathbb{C}$, $r < |\lambda_0| < 1$ such that $\mathcal{M}_F(\tau, \lambda_0)$ is reducible. Let L be an eigenline for $\overline{\mathcal{M}_F(\tau, \lambda_0)}^t$ for all $\tau \in \Delta$ and h_{L,λ_0} simple. Then $\tau^* \hat{f}_{\mu_0} = \hat{f}_{\mu_0}$ for all $\tau \in \Delta$ for $\hat{f}_\mu = Sym_\mu[\mathsf{Uni}_r(h_{L,\lambda_0} F)]$.

(ii) Assume there exists $\lambda_0 \in \mathbb{C}$, $r < |\lambda_0| < 1$ such that $\mathcal{M}_F(\tau, \lambda_0) \in \{\pm \mathrm{I}\}$ for all $\tau \in \Delta$. Let h_{L,λ_0} simple for arbitrary $L \in \mathbb{P}^1$. Then $\tau^* \hat{f}_{\mu_0} = \hat{f}_{\mu_0}$ for all $\tau \in \Delta$ for $\hat{f}_\mu = \mathsf{Sym}_\mu[\mathsf{Uni}_r(h_{L,\lambda_0} F)]$.

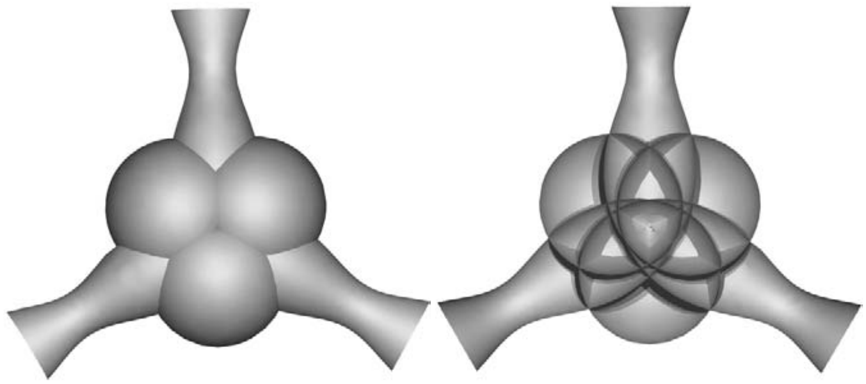

Fig. 3 Maximal equilateral trinoid with three-lobed bubble

Proof. Let $L \in \mathbb{P}^1$ with $\overline{\mathcal{M}_F(\tau,\lambda_0)}^t L = L$ for all $\tau \in \Delta$ and h_{L,λ_0} simple. By (1.6), $\widehat{F} = \mathsf{Uni}_r(h_{L,\lambda_0}F) = h_{L,\lambda_0} F h^{-1}_{L',\lambda_0}$ with $L' = \overline{F(z,\lambda_0)}^t L$. For all $\tau \in \Delta$ we have

$$\tau^* h_{L',\lambda_0} = h_{\tau^* L',\lambda_0} = h_{\tau^* \overline{F(z,\lambda_0)}^t L,\lambda_0} = h_{\overline{F(z,\lambda_0)}^t \overline{\mathcal{M}_F(\tau,\lambda_0)}^t L,\lambda_0} = h_{L',\lambda_0}.$$

Hence $\mathcal{M}_{\widehat{F}}(\tau,\lambda) = h_{L,\lambda_0}\mathcal{M}_F(\tau,\lambda)h^{-1}_{L,\lambda_0}$ for all $\tau \in \Delta$. Since the conditions (2.1) are invariant under conjugation, this concludes the proof. □

3 Dressed n-noids

The remainder of the paper is devoted to the construction of dressed trinoids (theorem 3.6) and dressed equilateral coplanar n-noids (theorem 3.12). The construction starts with an n-noid ([19, 18]) with asymptotically Delaunay ends and dresses it with simple factors. Computer experiments show that these dressings add multilobed bubbles situated near the center of the surface while preserving the asymptotically Delaunay end behavior.

The proof in each case exhibits a discrete subset Λ along which the end monodromies have a common eigenline. By theorem 2.1, there exists a simple factor matrix dressing matrix which produces a closed CMC surface with bubbles.

Computer experiments show that the set Λ parametrizes the number of lobes of the added bubble. Some values in Λ produce isolated examples, while others produce families parametrized by $\mathbb{P}^1$. Computer experiments show that lobes of the bubbles move around as one moves in this family. For example, on an equilateral trinoid with a 4-lobed bubble, it is possible to rotate three of lobes around an end while fixing the fourth lobe. It is also possible, as the fourth lobe remains fixed, that the three lobes become spherical and "bubble off" as mentioned in the introduction; the resulting surface has only one lobe.

By iterating the above construction, multiple bubbles with different lobe counts can be added to a n-noid.

3.1. Dressed Trinoids

The construction of dressed trinoids (theorem 3.6) is outlined as follows:

1. Write down a family of DPW potentials ξ to be used to produce trinoids (definition 3.8).
2. Solve the ODE $d\Phi = \Phi\xi$ explicitly in terms of hypergeometric functions.
3. Compute the monodromy representation for Φ explicitly in terms of the gamma function and find a set Λ of values of the loop parameter at which the monodromy is reducible (lemma 3.2).
4. Fix $\lambda_0 \in \Lambda$. In the case that the monodromy representation is reducible at λ_0, there exists a simple factor matrix $h \in \mathcal{H}_{\lambda_0}$ which dresses the trinoid to a closed surface. In the case that the monodromy representation takes values in $\{\pm \mathrm{I}\}$ at λ_0, every $g \in \mathcal{H}_{\lambda_0} \cong \mathbb{P}^1$ dresses the trinoid to a closed surface (theorem 2.1).

In preparation to the construction of CMC trinoids, lemma 3.2 discusses the situation in which the relevant monodromy is reducible. The potential in this lemma is a pointwise version of the potential used to construct the trinoids, and is related to it by the simple gauge (3.11). First some notation and discussion of the hypergeometric function and gamma function.

Notation 3.1.

$$\mathbb{Z}^{\leq 0} = \{k \in \mathbb{Z} \,|\, k \leq 0\} \qquad \mathbb{Z}^{<0} = \{k \in \mathbb{Z} \,|\, k < 0\}$$
$$\mathbb{Z}^{\geq 0} = \{k \in \mathbb{Z} \,|\, k \geq 0\} \qquad \mathbb{Z}^{>0} = \{k \in \mathbb{Z} \,|\, k > 0\}$$
$$\tfrac{1}{2}\mathbb{Z} = \{\tfrac{1}{2}k \,|\, k \in \mathbb{Z}\} \qquad \tfrac{1}{2} + \mathbb{Z} = \{\tfrac{1}{2} + k \,|\, k \in \mathbb{Z}\}.$$

We will require the hypergeometric function, defined as follows:

Remark 3.1. Let $a, b, c \in \mathbb{C}$ satisfy one of the following conditions:

(i) $c \notin \mathbb{Z}^{\leq 0}$;
(ii) $c \in \mathbb{Z}^{\leq 0}$ and either a or b is in $\mathbb{Z}^{\leq 0} \cap (c, 0]$.

Let $\Sigma_{\mathrm{H}} = \mathbb{P}^1 \setminus \{1, \infty\}$ and let $\Pi_{\mathrm{H}} : \widetilde{\Sigma_{\mathrm{H}}} \to \Sigma_{\mathrm{H}}$ be its universal cover. Let $p \in \Pi^{-1}(0)$. The *hypergeometric function* $F(a, b, c, \cdot) : \widetilde{\Sigma_{\mathrm{H}}} \to \mathbb{C}$ is the unique solution to the initial value problem

$$\begin{aligned}&z(1-z)F'' + (c - (a+b+1)z)F' - abF = 0,\\ &F(p) = 1, \quad F'(p) = ab/c,\end{aligned}$$

where differentiation is with respect to the last variable.

Remark 3.2. Let $\Sigma = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and let $\Pi : \widetilde{\Sigma} \to \Sigma$ be its universal cover. Then there is a covering map $\widetilde{\Sigma} \to \widetilde{\Sigma}_{\mathrm{H}} \setminus \Pi_{\mathrm{H}}^{-1}(0)$, and $F(a, b, c, \cdot)$ can be lifted to $\widetilde{\Sigma}$ via this covering map.

We will also use the following fact about the gamma function:

Remark 3.3. Γ is a meromorphic function on $\mathbb{C}$ with no zeros. The set of poles of Γ is $\mathbb{Z}^{\leq 0}$ and each of these poles is simple.

Lemma 3.2. *Let $\Sigma = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and let $\Pi : \widetilde{\Sigma} \to \Sigma$ be its universal cover. Let $m_0, m_1, m_\infty \in \mathbb{C}$ and let η be the holomorphic $\mathrm{sl}_2(\mathbb{C})$-valued 1-form on Σ defined by*

$$\eta = \begin{pmatrix} 0 & \dfrac{(m_\infty^2 - \frac{1}{4})z^2 - (m_0^2 - m_1^2 + m_\infty^2 - \frac{1}{4})z + (m_0^2 - \frac{1}{4})}{z^2(z-1)^2} \\ 1 & 0 \end{pmatrix} dz. \tag{3.1}$$

Let $p \in \widetilde{\Sigma}$, and let X be the solution of the ODE defined by the triple (η, p, I). Let $\mathcal{M}_X$ be the monodromy representation for X. Suppose that for some sign choice,

$$\tfrac{1}{2} \pm m_0 \pm m_1 \pm m_\infty \in \mathbb{Z}^{\leq 0}. \tag{3.2}$$

Then

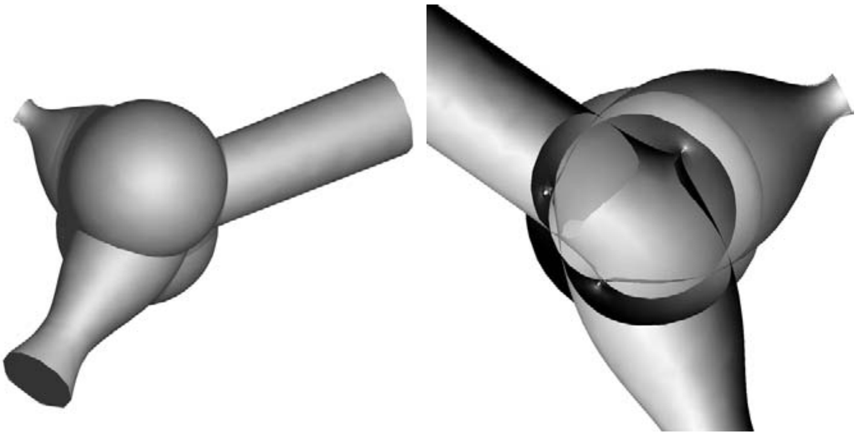

Fig. 4 Scalene trinoid with two-lobed bubble

(i) If at most one of $m_0,\ m_1,\ m_\infty$ *is in* $\frac{1}{2}\mathbb{Z}$*, then* $\mathcal{M}_X$ *is reducible.*
(ii) If $m = m_0 = m_1 = m_\infty \in \frac{1}{2} + \mathbb{Z}$*, then* $\mathcal{M}_X$ *takes values in* $\{\pm\,\mathrm{I}\}$.

Proof of (i). Assume first that $m_0,\ m_1 \notin \frac{1}{2}\mathbb{Z}$.

Let Δ be the group of deck transformations for the cover. For each $k = 1,\ 2$, let $\gamma_k : [0,\ 1] \to \Sigma$ be a curve which winds once counterclockwise around $z = k$ and does not wind around the other two punctures, let $\tilde{\gamma}_k$ be a lift of γ_k to $\widetilde{\Sigma}$, and let $\tau_k \in \Delta$ be the deck transformation satisfying $\tau_k(\tilde{\gamma}_k(0)) = \tilde{\gamma}_k(1)$. Then $\tau_0,\ \tau_1$ generate Δ.

Define the functions $f,\ g$ and $\Phi,\ \Psi$ on $\widetilde{\Sigma}$ as follows:

$$
\begin{aligned}
f(m_0,\ m_1,\ m_\infty,\ z) &= z^{\frac{1}{2}+m_0}(1-z)^{\frac{1}{2}+m_1}\\
&\quad\times F\left(\tfrac{1}{2}+m_0+m_1+m_\infty,\ \tfrac{1}{2}+m_0+m_1-m_\infty,\ 1+2m_0,\ z\right)\\
g(m_0,\ m_1,\ m_\infty,\ z) &= z^{\frac{1}{2}+m_0}(1-z)^{\frac{1}{2}+m_1}\\
&\quad\times F\left(\tfrac{1}{2}+m_0+m_1+m_\infty,\ \tfrac{1}{2}+m_0+m_1-m_\infty,\ 1+2m_1,\ 1-z\right).
\end{aligned}
$$

and

$$
\Phi = \begin{pmatrix} f(m_0,\ m_1,\ m_\infty,\ z) & f'(m_0,\ m_1,\ m_\infty,\ z)\\ f(-m_0,\ m_1,\ m_\infty,\ z) & f'(-m_0,\ m_1,\ m_\infty,\ z)\end{pmatrix}
$$

$$
\Psi = \begin{pmatrix} g(m_0,\ m_1,\ m_\infty,\ z) & g'(m_0,\ m_1,\ m_\infty,\ z)\\ g(m_0,\ -m_1,\ m_\infty,\ z) & g'(m_0,\ -m_1,\ m_\infty,\ z)\end{pmatrix},
$$

where the prime denotes differentiation with respect to z. A calculation shows that $\det\Phi = -2m_0$ and $\det\Psi = 2m_1$. Since we are assuming that $m_0,\ m_1 \notin \frac{1}{2}\mathbb{Z}$, then Φ and Ψ take values in $\mathrm{GL}_2(\mathbb{C})$.

Let $\Phi_0 = \Phi(p)$ and $\Psi_0 = \Psi(p)$. Then $X = \Phi_0^{-1}\Phi = \Psi_0^{-1}\Psi$ is the solution to the ODE defined by the triple $(\eta,\ p,\ \mathrm{I})$. Moreover (by equation (15.3.6) in [1])

Φ and Ψ are related by

$$C = \Phi\Psi^{-1} = \begin{pmatrix} \frac{\Gamma(1+2m_0)\Gamma(-2m_1)}{\Gamma(\frac{1}{2}+m_0-m_1-m_\infty)\Gamma(\frac{1}{2}+m_0-m_1+m_\infty)} & \frac{\Gamma(1+2m_0)\Gamma(2m_1)}{\Gamma(\frac{1}{2}+m_0+m_1-m_\infty)\Gamma(\frac{1}{2}+m_0+m_1+m_\infty)} \\ \frac{\Gamma(1-2m_0)\Gamma(-2m_1)}{\Gamma(\frac{1}{2}-m_0-m_1-m_\infty)\Gamma(\frac{1}{2}-m_0-m_1+m_\infty)} & \frac{\Gamma(1-2m_0)\Gamma(2m_1)}{\Gamma(\frac{1}{2}-m_0+m_1-m_\infty)\Gamma(\frac{1}{2}-m_0+m_1+m_\infty)} \end{pmatrix} \tag{3.3}$$

where $1/\Gamma(x)$ is taken to be 0 if x is a pole of Γ (i.e. if $x \in \mathbb{Z}^{\leq 0}$). Since Φ and Ψ take values in $\mathrm{GL}_2(\mathbb{C})$, then $C \in \mathrm{GL}_2(\mathbb{C})$.

Let $\mathcal{M}_\Phi$, $\mathcal{M}_\Psi$ be the respective monodromy representations of Φ and Ψ. A calculation shows that

$$\begin{aligned} L_0 = \mathcal{M}_\Phi(\tau_0) &= -\begin{pmatrix} \exp(2\pi i m_0) & 0 \\ 0 & \exp(-2\pi i m_0) \end{pmatrix} \\ L_1 = \mathcal{M}_\Psi(\tau_1) &= -\begin{pmatrix} \exp(2\pi i m_1) & 0 \\ 0 & \exp(-2\pi i m_1) \end{pmatrix}. \end{aligned} \tag{3.4}$$

Then $\mathcal{M}_X$ is given by the generators

$$\begin{aligned} M_0 = \mathcal{M}_X(\tau_0) &= \Phi_0^{-1} L_0 \Phi_0 \\ M_1 = \mathcal{M}_X(\tau_1) &= \Phi_0^{-1} C L_1 C^{-1} \Phi_0 \end{aligned} \tag{3.5}$$

The hypothesis $m_0,\ m_1 \notin \frac{1}{2}\mathbb{Z}$ implies that none of the numerators of the entries of C have poles. The hypothesis (3.2) implies that one of denominators of the entries of C has a pole. Hence one of the entries of C is 0. It follows that CL_1C^{-1} is upper or lower triangular, so $[0:1]$ or $[1:0]$ is a common eigenline of L_0 and CL_1C^{-1}. Since $M_0,\ M_1$ are respectively conjugate to these by by an element of $\mathrm{GL}_2(\mathbb{C})$, then $M_0,\ M_1$ have a common eigenline. It follows that $\mathcal{M}_X$ is reducible.

The remaining cases $m_0,\ m_\infty \notin \frac{1}{2}\mathbb{Z}$ and $m_1,\ m_\infty \notin \frac{1}{2}\mathbb{Z}$ can be shown in two ways, either by writing down a solution to $dX = X\eta$ involving $F(\cdot,\ \cdot,\ \cdot,\ z^{-1})$, or by moving the poles $(0,\ \infty)$ or $(1,\ \infty)$ respectively to $(0,\ 1)$ via a coordinate change and a gauge.

Proof of (ii). Assume $m = m_0 = m_1 = m_\infty \in \frac{1}{2} + \mathbb{Z}$. Then

$$\begin{aligned} f(m,\ m,\ m,\ z) &= z^{\frac{1}{2}+m}(1-z)^{\frac{1}{2}+m} F(\tfrac{1}{2}+3m,\ \tfrac{1}{2}+m,\ 1+2m,\ z) \\ f(-m,\ m,\ m,\ z) &= z^{\frac{1}{2}-m}(1-z)^{\frac{1}{2}+m} F(\tfrac{1}{2}+m,\ \tfrac{1}{2}-m,\ 1-2m,\ z). \end{aligned}$$

If $m > 0$, then $1+2m \notin \mathbb{Z}^{\leq 0}$, so by remark 3.1, condition (i), $f(m,\ m,\ m,\ z)$ is defined on $\widetilde{\Sigma}$. Since $1-2m \in \mathbb{Z}^{\leq 0}$, and $\frac{1}{2}-m \in \mathbb{Z}^{\leq 0}$ and $\frac{1}{2}-m \geq 1-2m$, by remark 3.1, condition (ii), $f(-m,\ m,\ m,\ z)$ is defined on $\widetilde{\Sigma}$. Hence Φ is defined on $\widetilde{\Sigma}$. Similar arguments show that Φ and Ψ are defined on $\widetilde{\Sigma}$ for all $m \in \frac{1}{2}+\mathbb{Z}$.

Then we have that $L_0 = L_1 = (-1)^{1+2m}\,\mathrm{I}$ (equation (3.4)). Since $M_0,\ M_1$ are obtained from $L_0,\ L_1$ respectively by conjugation by an element of $\mathrm{GL}_2(\mathbb{C})$, then $M_0 = M_1 = (-1)^{1+2m}\,\mathrm{I}$ (equation (3.5)). It follows that $\mathcal{M}_X$ takes values in $\{\pm\,\mathrm{I}\}$. □

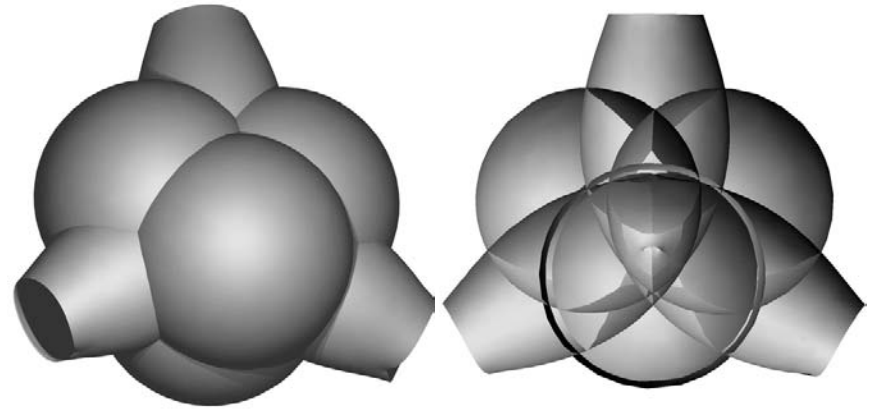

Fig. 5 Maximal equilateral trinoid with four-lobed bubble

The following definition (see [19]) sets out a family of trinoid potentials parametrized by the three end weights.

Definition 3.3. *[19] The family of trinoid potentials* $\mathcal{T}_{\text{trinoid}}$ *is the set of potentials* ξ_{w_0,w_1,w_∞} *defined as follows. Let* $w_0,\ w_1,\ w_\infty \in (-\infty,\ 1] \setminus \{0\}$, *let* $n_k = \frac{1}{2}\sqrt{1-w_k} \in (-\infty,\ \frac{1}{2}] \setminus \{0\} (k \in \{0,\ 1,\ \infty\})$ *and suppose the following inequalities are satisfied:*

$$\begin{aligned} &|n_0| + |n_1| + |n_\infty| \le 1 \\ &|n_i| \le |n_j| + |n_k|, \quad \{i,\ j,\ k\} = \{0,\ 1,\ \infty\} \end{aligned} \tag{3.6}$$

$$|w_i| \le |w_j| + |w_k|, \quad \{i,\ j,\ k\} = \{0,\ 1,\ \infty\}. \tag{3.7}$$

Let $\Sigma = \mathbb{P}^1 \setminus \{0,\ 1,\ \infty\}$ *and let* $\xi_{w_0,w_1,w_\infty} \in \Lambda\Omega(\Sigma)$ *be defined by*

$$\xi_{w_0,w_1,w_\infty} = \begin{pmatrix} 0 & \lambda^{-1} \\ \dfrac{w_\infty z^2 - (w_0 - w_1 + w_\infty)z + w_0}{16z^2(z-1)^2}(1-\lambda)^2 & 0 \end{pmatrix} dz. \tag{3.8}$$

The neck inequalities 3.6 are the spherical triangle inequalities and the weight inequalities 3.6 follow from the balancing formula on the weighted end axes. In the case of three positive weights, the neck inequalities imply the weight inequalities. Together, these inequalities are necessary and sufficient conditions for the unitarization of the monodromy representation on $\mathbb{S}^1$ (see [19]).

The existence of a dressing which closes the trinoids is given in [19]:

Theorem 3.4. *[19] Let* $\Sigma = \mathbb{P}^1 \setminus \{0,\ 1,\ \infty\}$, *let* $\Pi : \widetilde{\Sigma} \to \Sigma$ *be its universal cover, and let* Δ *be the group of deck transformations for this cover. Let* $\xi_{w_0,w_1,w_\infty} \in \mathcal{T}_{\text{trinoid}}$ *(definition 3.3). Let* $p \in \widetilde{\Sigma}$ *and let* $\Phi : \widetilde{\Sigma} \to \Lambda_1\mathrm{SL}_2(\mathbb{C})$ *be a solution to the ODE defined by the triple* $(\xi,\ p,\ \mathrm{I})$. *Then there exists analytic map* $h_{\text{close}} : \{0 \le |\lambda| < 1\} \to \mathrm{GL}_2(\mathbb{C})$ *such that for any* $r \in (0,\ 1)$, *the CMC immersion* $f = \mathsf{Sym}_1[\mathsf{Uni}_r(h_{\text{close}}\Phi)]$ *satisfies* $\tau^* f = f$ *for all* $\tau \in \Delta$ *and has three asymptotically Delaunay ends with weights* $w_0,\ w_1,\ w_\infty$.

The following definition gives a set along which the trinoid monodromy will be shown to be reducible.

Definition 3.5. *Define* μ_w *by*

$$\mu_w = \frac{1}{2}\sqrt{1 + w\frac{(1-\lambda)^2}{4\lambda}}. \tag{3.9}$$

For $k \in \{0,\ 1,\ \infty\}$*, let* $w_k \subset (-\infty, 1] \setminus \{0\}$*, and write* $\mu_k = \mu_{w_k}$*. Define the set*

$$\Lambda_{w_0,w_1,w_\infty} = \left\{\lambda \in \{0 < |\lambda| < 1\} \mid \tfrac{1}{2} \pm \mu_0 \pm \mu_1 \pm \mu_\infty \in \mathbb{Z}^{\leq 0}\right\}, \tag{3.10}$$

meaning that $\frac{1}{2} \pm \mu_0 \pm \mu_1 \pm \mu_\infty \in \mathbb{Z}^{\leq 0}$ *holds for some choice of signs.*

Remark 3.4. For any $w_0,\ w_1,\ w_\infty \subset (-\infty, 1] \setminus \{0\}$, $\Lambda_{w_0,w_1,w_\infty}$ is an infinite discrete set for which 0 is an accumulation point.

In the case of three equal weights, $\Lambda_{w,w,w} \subseteq \Lambda_{3,w}$, given explicitly by (3.13).

Notation. Let Σ be a Riemann surface, $\xi \in \Lambda\Omega(\Sigma)$ and $g : \Sigma \longrightarrow \mathrm{GL}_2(\mathbb{C})$. The gauged potential $\xi . g$ is

$$\xi . g = g^{-1}\xi g + g^{-1}dg.$$

The following theorem constructs dressed trinoids.

Theorem 3.6. *Let* $\Pi : \widetilde{\Sigma} \to \Sigma$ *and* Δ *be as in theorem 3.4. Let* ξ_{w_0,w_1,w_∞} *be a trinoid potential (definition 3.3). Let* Φ *and* h_{close} *be as in theorem 3.4. Let* $\Lambda_{w_0,w_1,w_\infty}$ *as in definition 3.5. Then*

(i) For every $\lambda_0 \in \Lambda_{w_0,w_1,w_\infty}$*, the monodromy* $\mathcal{M}_{h_{\mathrm{close}}\Phi}(\lambda_0)$ *is reducible. Hence there exists a simple factor matrix* $h \in \mathcal{H}_{\lambda_0}$ *such that for any* $r \in (0,\ |\lambda_0|)$*, the CMC immersion* $f = \mathsf{Sym}_1[\mathsf{Uni}_r(hh_{\mathrm{close}}\Phi)]$ *satisfies* $\tau^* f = f$ *for all* $\tau \in \Delta$*.*
(ii) If $\mu = \mu_0 = \mu_1 = \mu_\infty \in \frac{1}{2} + \mathbb{Z}$*, then* $\mathcal{M}_{h_{\mathrm{close}}\Phi}(\lambda_0)$ *takes values in* $\{\pm \mathrm{I}\}$*. Hence for every simple factor matrix* $h \in \mathcal{H}_{\lambda_0}$*, for any* $r \in (0,\ |\lambda_0|)$*, the CMC immersion* $f = \mathsf{Sym}_1[\mathsf{Uni}_r(hh_{\mathrm{close}}\Phi)]$ *satisfies* $\tau^* f = f$ *for all* $\tau \in \Delta$*.*

Proof. Use the notation $\mathcal{M}_X$ for the monodromy representation of X. Let

$$g_{\mathrm{par}} = \begin{pmatrix} 0 & i\lambda^{-1/2} \\ i\lambda^{1/2} & 0 \end{pmatrix}. \tag{3.11}$$

Then $\xi . g_{\mathrm{par}}$ has the form of η in lemma 3.2, with $\mu_k(\lambda)$ playing the role of m_k, $k \in \{0, 1, \infty\}$. With $g_{\mathrm{par}}^{-1}\Phi g_{\mathrm{par}}$ playing the role of X, by lemma 3.2, $\mathcal{M}_{g_{\mathrm{par}}^{-1}\Phi g_{\mathrm{par}}}(\lambda_0) = \mathcal{M}_{g_{\mathrm{par}}^{-1}\Phi}(\lambda_0)$ is reducible if $\lambda_0 \in \Lambda$, and takes values in $\{\pm \mathrm{I}\}$ if $\mu = \mu_0 = \mu_1 = \mu_\infty \in \frac{1}{2} + \mathbb{Z}$.

But $\mathcal{M}_{h_{\mathrm{close}}\Phi} = \mathrm{Ad}_{h_{\mathrm{close}}g_{\mathrm{par}}}\mathcal{M}_{g_{\mathrm{par}}^{-1}\Phi}$. It follows that $\mathcal{M}_{h_{\mathrm{close}}\Phi}(\lambda_0)$ is reducible if $\lambda_0 \in \Lambda$, and takes values in $\{\pm \mathrm{I}\}$ if $\mu = \mu_0 = \mu_1 = \mu_\infty \in \frac{1}{2} + \mathbb{Z}$. The result then follows by theorem 2.1. □

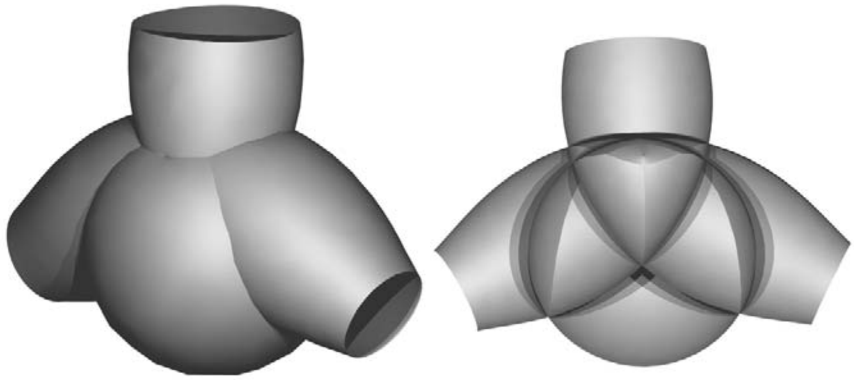

Fig. 6 Maximal equilateral trinoid with one-lobed bubble

Remark 3.5. Computer experiments show that dressing a trinoid by a simple factor matrix as specified in theorem 3.6 produces again a trinoid with asymptotically Delaunay ends with the same respective weights, but with a "bubble" appearing near the center of the trinoid. This bubble is analogous to the bubble added to round cylinders and Delaunay surfaces.

In the case of the equilateral trinoids, in general the number of lobes of the bubble is $l = k + 1$, where k is the parameter defining the set $\Lambda_{3,w}$ in equation (3.13). The cases $l = 2$ and $l = 3$, dressing the maximal equilateral trinoid, are illustrated in figures 2 and 3 respectively. The value $l = 4$ is the first value of l which supports a continuous two-parameter family of deformations. Figure 5 shows a trinoid with a 4-lobed bubble, while figure 6 is a degenerate example in the 4-lobed family showing only one remaining lobe after the other three appear to become spheres and "bubble off."

Figure 4 shows a scalene trinoid with a 2-lobed bubble.

3.2. Dressed equilateral coplanar n-noids

The construction of dressed n-noids (theorem 3.12) is outlined as follows:

1. Write down a family of DPW potentials $\xi_{n,w}$ which can be dressed to produce n-noids with asymptotically Delaunay ends (definition 3.7).
2. Find a set of points $\Lambda_{n,w}$ at which the monodromy representation is reducible.
3. For each $\lambda \in \Lambda_{n,w}$ there exists a simple factor matrix $h \in \mathcal{H}_\lambda$ which dresses the n-noid to a closed surface (theorem 2.1).

The following definition (see [18]) sets out a family of equilateral n-noid potentials parametrized by the number of ends and end weight.

Definition 3.7. *[18] The family of n-noid potentials $\mathcal{T}_{\text{noid}}$ is the set of DPW potentials $\xi_{n,w}$ defined as follows. Let $n \geq 3$ and $w \in (0, (\frac{4}{n})(1 - \frac{1}{n})]$. Let $\Sigma = \mathbb{P}^1 \setminus$*

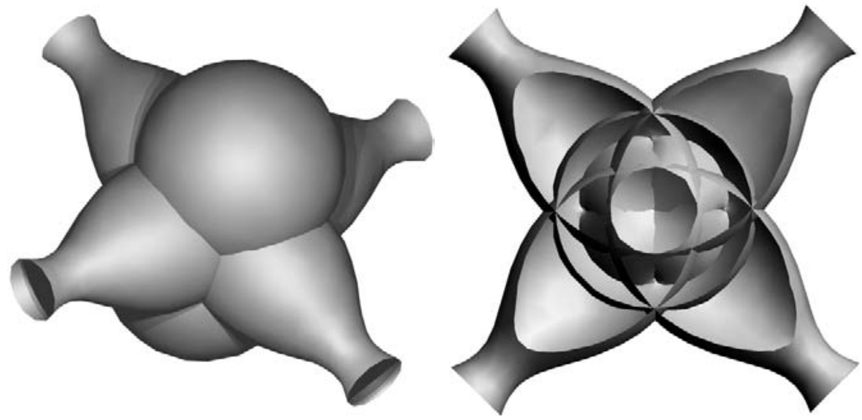

Fig. 7 Fournoid with two-lobed bubble

$\{z \in \mathbb{C} \,|\, z^n = 1\}$. *Let* $\xi_{n,w} \in \Lambda\Omega(\Sigma)$ *be defined by*

$$\xi_{n,w} = \begin{pmatrix} 0 & \lambda^{-1} \\ \dfrac{n^2 w z^{n-2}}{16(z^n-1)^2}(1-\lambda)^2 & 0 \end{pmatrix} dz. \tag{3.12}$$

Remark 3.6. The potential $\xi_{3,w}$ is equivalent to the trinoid potential $\xi_{w,w,w}$ (definition 3.3) via a coordinate change and a gauge.

The existence of a dressing which closes the n-noids is given in [18]:

Theorem 3.8. *[18] Let* $n \geq 2$. *Let* $\Sigma = \mathbb{P}^1 \setminus \{z \in \mathbb{C} \,|\, z^n = 1\}$, *let* $\Pi : \widetilde{\Sigma} \to \Sigma$ *be its universal cover, and let* Δ *be the group of deck transformations for this cover. Let* $\xi_{n,w} \in \mathcal{T}_{\text{noid}}$ *(definition 3.7). Let* $p \in \Pi^{-1}(0)$ *and let* $\Phi : \widetilde{\Sigma} \to \Lambda_1 \mathrm{SL}_2(\mathbb{C})$ *be a solution to the ODE defined by the triple* $(\xi,\ p,\ \mathrm{I})$. *Then there exists an analytic map* $h_{\text{close}} : \{0 \leq |\lambda| < 1\} \to \mathrm{GL}_2(\mathbb{C})$ *such that for any* $r \in (0, 1)$, *the CMC immersion* $f = \mathsf{Sym}_1[\mathsf{Uni}_r(h_{\text{close}}\Phi)]$ *extends smoothly to* ∞, *satisfies* $\tau^* f = f$ *for all* $\tau \in \Delta$, *and has n asymptotically Delaunay ends with weight* w.

Before constructing dressed n-noids (theorem 3.12) we give two technical lemmas and a definition.

Lemma 3.9. *Let* $A \in \mathrm{SL}_2(\mathbb{C})$ *and let* $a,\ a^{-1}$ *be the eigenvalues of* A. *Let* $n \in \mathbb{Z}$. *Then (i)* $A^n = \mathrm{I}$ *iff* $a^n = 1$, *and (ii)* $A^n = -\mathrm{I}$ *iff* $a^n = -1$.

Lemma 3.10. *Let* $A,\ B \in \mathrm{GL}_2(\mathbb{C})$ *with eigenvalues* $\{a_1,\ a_2\}$ *and* $\{b_1,\ b_2\}$ *respectively. Then* A *and* B *have a common eigenline iff the eigenvalues of* AB *are* $\{a_1b_1,\ a_2b_2\}$ *or* $\{a_1b_2,\ a_2b_1\}$.

The following definition gives a set along which the n-noid monodromy will be shown to be reducible.

Definition 3.11. *With n and w as in definition 3.7, define the set* $\Lambda_{n,w} \subset (0,\ 1)$ *by*

$$\Lambda_{n,w} = \left\{ \frac{1}{w}\left(\sqrt{4m^2-1+w} - \sqrt{4m^2-1}\right)^2 \,\middle|\, m = \tfrac{1}{2} + \tfrac{k}{n},\ k \in \mathbb{Z}^{>0},\ (k \bmod n) \in \{-1,\ 0,\ 1\} \right\}. \tag{3.13}$$

$\Lambda_{n,\,w}$ is an infinite discrete subset of (0, 1) for which 0 is an accumulation point.

The following theorem constructs dressed n-noids.

Theorem 3.12. *Let* $\Pi : \widetilde{\Sigma} \to \Sigma$ *and* Δ *be as in theorem 3.8. Let* $\xi_{n,w} \in \mathcal{T}_{\text{noid}}$ *be a n-noid potential (definition 3.7). Let* $\Lambda_{n,w}$ *as in definition 3.11. Let* Φ *and* h_{close} *be as in theorem 3.8. Then for every* $\lambda_0 \in \Lambda_{n,w}$ *there exists a simple factor matrix* $h \in \mathcal{H}_{\lambda_0}$ *such that for any* $r \in (0,\ |\lambda_0|)$*, the CMC immersion* $f = \mathsf{Sym}_1[\mathsf{Uni}_r(hh_{\text{close}}\Phi)]$ *satisfies* $\tau^* f = f$ *for all* $\tau \in \Delta$.

Proof. Let $\mathcal{M}_{h_{\text{close}}\Phi}$ be the monodromy representation for $h_{\text{close}}\Phi$. For each $k = 0, \ldots, n-1$, let $\gamma_k : [0,\ 1] \to \Sigma$ be a curve which wind once counterclockwise around $\exp(2\pi i k/n)$ and does not wind around any of the other $n-1$ punctures, let $\tilde{\gamma}_k$ be a lift of γ_k to $\widetilde{\Sigma}$, and let $\tau_k \in \Delta$ be the deck transformation satisfying $\tau_k(\tilde{\gamma}_k(0)) = \tilde{\gamma}_k(1)$. Then $\tau_0, \ldots, \tau_{n-1}$ generate Δ.

Let $M_k = \mathcal{M}_{h_{\text{close}}\Phi}(\tau_k)$ $(k = 0, \ldots, n-1)$.

With σ defined by $\sigma(z) = \exp(2\pi i/n)z$, a calculation shows that $\xi_{n,w}$ has the symmetry $\sigma^*\xi_{n,w} = \xi_{n,w}\cdot g$, where

$$g = \begin{pmatrix} \exp(-\pi i/n) & 0 \\ 0 & \exp(\pi i/n) \end{pmatrix}, \tag{3.14}$$

whence (see [18])

$$M_j = h^{-j} M_0 h^j, \quad j = 1, \ldots, n-1. \tag{3.15}$$

Since

$$g^n = -\mathrm{I}, \tag{3.16}$$

and

$$\textstyle\prod_{j=0}^{n-1} M_j = \mathrm{I}, \tag{3.17}$$

equations (3.17), (3.15) and (3.16) imply that

$$(g^{-1}M_0)^n = -\mathrm{I}\,. \tag{3.18}$$

By lemma 3.9 the eigenvalues of $g^{-1}M_0$ are n'th roots of -1.

Since the eigenvalues depend analytically on λ, and the n'th roots of -1 are discrete, the eigenvalues of $g^{-1}M_0$ must be constant. Since $M_0(1) = \mathrm{I}$, the eigenvalues of $g^{-1}M_0$ must be $\exp(\pm\pi i/n)$.

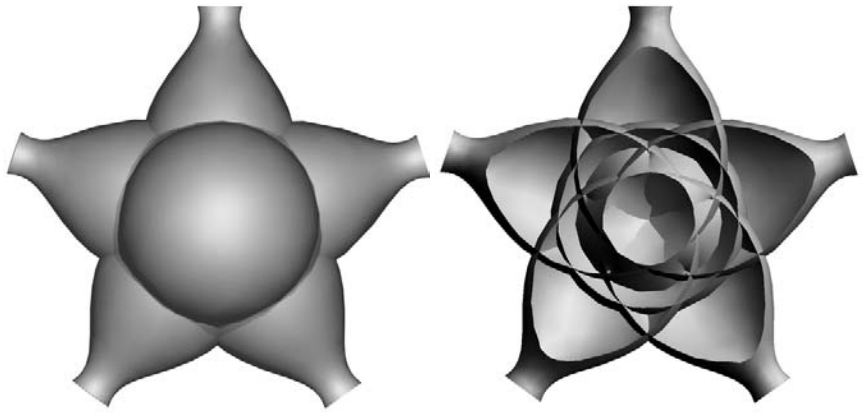

Fig. 8 Fivenoid with two-lobed bubble

Let $\mu_w(\lambda)$ be defined as in equation (3.9). Then the eigenvalues of $M_0(\lambda)$ are $-\exp(\pm 2\pi i \mu_w(\lambda))$. Fix $\lambda_0 \in \Lambda_{n,w}$. Then $\mu_w(\lambda_0) = 1/2 + k/n$ for some $k \in \mathbb{Z}^{>0}$ with $k \bmod n \in \{-1, 0, 1\}$. Then the eigenvalues of $M_0(\lambda_0)$ are $\exp(\pm 2\pi i k/n)$ for some $k \in \{-1, 0, 1\}$. The relevant matrices and their eigenvalues are summarized in the following table:

matrix	eigenvalues
g^{-1}	$\exp(\pm\pi i/n)$
$M_0(\lambda_0)$	$\exp(\pm 2\pi i k/n),\ k \in \{-1, 0, 1\}$
$g^{-1}M_0$	$\exp(\pm\pi i/n)$

Then g^{-1} and $M_0(\lambda_0)$ satisfy the hypotheses of lemma 3.10, so by that lemma, g^{-1} and $M_0(\lambda_0)$ have a common eigenline. It follows from equation 3.15 that all the end monodromies $M_0(\lambda_0), \dots, M_{n-1}(\lambda_0)$ have a common eigenline, and hence $\mathcal{M}_\Phi(\lambda_0)$ is reducible. Since $\mathcal{M}_{h_{\mathrm{close}}\Phi} = \mathrm{Ad}_{h_{\mathrm{close}}} \mathcal{M}_\Phi$, then $\mathcal{M}_{h_{\mathrm{close}}\Phi}(\lambda_0)$ is likewise reducible.

By theorem 2.1, there exists a simple factor matrix $h \in \mathcal{H}_{\lambda_0}$ such that for any $r \in (0, |\lambda_0|)$, the CMC immersion $f = \mathsf{Sym}_1[\mathsf{Uni}_r(hh_{\mathrm{close}}\Phi)]$ satisfies $\tau^* f = f$ for all $\tau \in \Delta$. □

Remark 3.7. Computer experiments show that dressing an n-noid by a simple factor matrix as specified in theorem 3.12 results in an n-noid with asymptotically Delaunay ends of the same necksize with a bubble added near the center of the n-noid. Figures 7 and 8 show a 4-noid and a 5-noid, each with a 2-lobed bubble. Figure 9 shows the surface with constant Gaussian curvature which is parallel to the CMC trinoid shown in figure 2.

4 Open Problems and possible directions of future research

1. Do the examples in this paper have embedded Delaunay ends? Graphics indicate that they do.

2. If, as expected, they do have embedded Delaunay ends, then are they (almost) embedded? Graphics indicate they are not. The existence of a two-parameter family of CMC trinoids with embedded Delaunay ends of the same neck sizes lends further evidence that they cannot be embedded because the classification of (almost) embedded trinoids [7] mentioned earlier shows that there are at most two (almost) embedded CMC trinoids with embedded Delaunay ends with the same neck sizes. The problem is that we cannot a priori rule out the possibility that they are not the same surfaces.
3. The examples and constructions of trinoids with bubbletons in this paper are very special, but nevertheless there are no other known examples. It is natural to ask: Have we construted all CMC trinoids with embedded Delaunay ends? This can be further refined into three questions:
   - For our starting potentials $\xi \in \mathcal{T}$ is there any other way to use simple matrices to dress to new closed trinoids with Delaunay ends other than the "common eigenline" approach?
   - For our starting potentials $\xi \in \mathcal{T}$ is there a way to use non-simple dressing matrices to dress to new closed trinoids with Delaunay ends? Especially matrices which are not compositions of simple ones. One possibility would be to consider using special "adding a wave" dressings (see, for example [15]) via a similar trick involving our method of simultaneously reducible monodromies.
   - Are there other (non-gauge equivalent) starting potentials which might be dressed to produce new closed trinoids with Delaunay ends. To what extent does the condition that an end be Delaunay determine the behavior of the potential near that end?
4. Of particular interest is whether or not it is possible to add bubbletons on the "leg" of a CMC trinoid far from the umbilics.
5. Define renormalized energy for CMC trinoids and investigate if it is invariant under the deformations produced in this paper. See [22].
6. Investigate the bubbling off phenomenon mentioned in section 3.
7. All CMC trinoids discussed in this paper yield, in the standard way, new infinite type solutions to the sinh-Gordon equation. Is there some modification of the spectral curve approach which applies here? In particular does the notion of isospectral orbits carry over? Could one define the "type" of a trinoid. Or at least the "type" of elements in the dressing orbit of a given Alexandrov embedded trinoid so that it plays a 0-type role similar to the cylinder in its dressing orbit.
8. The simple factor matrices used in this paper correspond, in the case of the cylinder, with the Bianchi-Bäcklund transformation [13]. In the case of the multi-bubbletons on a cylinder the "center-point of each bubbleton" is given explicitly in Bianchi's algorithm and is clearly visable in the graphics as the bubbletons slide along or spin around the cylinder. Is it possible to explicitly incorporate the "center-point" into the DPW representation?
9. Hoffman's work [8] on discrete CMC surfaces appears able to produce discrete analogs of the n-noids with bubbletons discribed in this paper. It would be interesting to know to what extent the theories are similar. In particular, in the discrete

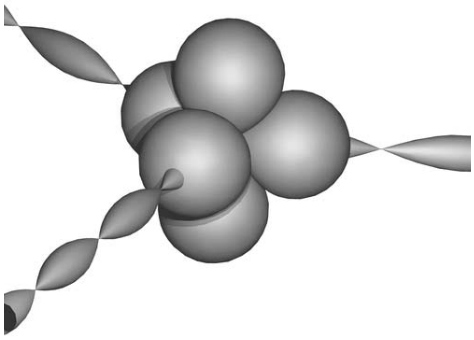

Fig. 9 Constant Gaussian curvature surface which is parallel to the CMC trinoid with a two-lobed bubble

case, are the deformations possible only for the same number of lobes as in our cases?

10. Several other soliton geometries admit "tons". For example Kuen's surfaces are the psuedo-spherical single-tons. For psuedo-spherical surfaces, Willmore surfaces, Hasimoto surfaces and other soliton geometries it would be interesting to explore to what extent the work of this paper carries over. Furthermore, are there physical interpretations in any of these settings?

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