An exact solution for equatorial geophysical water waves with an underlying current

David Henry
Department of Applied Mathematics,
University College Cork, Cork, Ireland
d.henry@ucc.ie

Abstract

In this paper we present an exact solution to the governing equations for equatorial geophysical water waves which admits an underlying current.

Keywords: Exact solution; Lagrangian variables; Coriolis force.
MSC (2010): 76B15; 74G05; 37N10.

1 Introduction

In this paper we present an exact solution to the $\beta-$plane governing equations for geophysical water waves which admit an underlying current. Geophysical ocean waves are those which take into account the Coriolis effects on the fluid body which are induced by the earth's rotation, and the $\beta-$plane approximation to the full governing equations applies in regions which are within 5° latitude of the equator [11, 13]. The wave solution which we construct in this paper correspond to steady zonal waves, travelling in the longitudinal direction with a constant speed of propagation $c > 0$, and which experience the presence of a constant underlying current of strength c_0.

Currents, such as the equatorial undercurrent (EUC), feature significantly in the geophysical dynamics of the equatorial region [7, 11]. For instance, the El Niño phenomena has recently been ascribed to the interplay between currents in the ocean and atmosphere [20], and the model we present in this paper is a first approach in incorporating the effects of a current into an exact solution for the governing equations. Additionally, the equator has the remarkable property of acting like a natural waveguide [12]. Accordingly, waves tend to be trapped in the equatorial region, and the waves which we present below inherit this feature— the amplitude of the waves decay rapidly in the meridional direction.

The approach we use to construct these waves is in the spirit of Gerstner's solution for the governing equations of two-dimensional gravity water waves, with significant modifications to incorporate geophysical effects along the lines of [6] . In 1802 Gerstner [14] found an explicit solution in Lagrangian variables for the full water wave equations (the form of this solution was later independently discovered by Rankine). Gerstner's wave is truly remarkable in the mathematical sense that it is one of only a handful of explicit solutions to the full governing equation which have constructed [5]. Gerstner's wave is a periodic travelling wave with a specific vorticity distribution (see [2, 5, 17] for a modern treatment of Gerstner's wave). Although the prescription of the flow is quite specific and rigid, remarkably this flow has been recently adapted to describe a wide-variety of interesting, and physically varied, water waves (cf. [5, 3, 23, 25], and particularly [6], where an exact solution to the geophysical governing equations was first derived). We note that all fluid particles follow closed trajectories in Gerstner's wave, something which is precluded for regular irrotational waves [4, 9, 10, 15, 16, 18, 19, 21, 22] and which must be due to the underlying vorticity distribution.

The introduction of a current-like term into Gerstner's formulation was performed by Mollo-Christensen [24] in the study of billows between two fluids. Here, we expand this formulation to admit the Coriolis effects of the rotating earth— these effects feature significantly for such large scale phenomena as currents. In particular, we find that the introduction of a steady underlying current in the geophysical context has interesting implications for the fluid motion, particularly in relation to the dispersion relation.

2 Governing equations

We take the earth to be a perfect sphere of radius $R = 6378km$, which has a constant rotational speed of $\Omega = 73.10^{-6} rad/s$. Then $g = 9.8ms^{-2}$ is the standard gravitational acceleration at the earth's surface, and $\beta = 2\Omega/R = 2.28 \cdot 10^{-11} m^{-1}s^{-1}$ is a parameter which will arise in subsequent considerations [11, 13]. From the viewpoint of a rotating reference frame with it's origin at the earth's surface, so that the $\{x, y, z\}$-coordinate frame is chosen with z as the vertical variable, x as the longitudinal variable (in the direction due east), and y is the latitudinal variable (in the direction due north), then the governing equations for geophysical ocean waves are given by [13]

$$u_t + uu_x + vu_y + wu_z + 2\Omega w\cos\phi - 2\Omega v\sin\phi = -\frac{1}{\rho}P_x, \tag{1a}$$

$$v_t + uv_x + vv_y + wv_z + 2\Omega u\sin\phi = -\frac{1}{\rho}P_y, \tag{1b}$$

$$w_t + uw_x + vw_y + ww_z - 2\Omega u\cos\phi = -\frac{1}{\rho}P_z - g, \tag{1c}$$

together with the equation for mass conservation

$$\rho_t + u\rho_x + v\rho_y + w\rho_z = 0 \tag{2a}$$

and the equation of incompressibility

$$u_x + v_y + w_z = 0. \tag{2b}$$

Here the variable ϕ represents the latitude, (u, v, w) is the velocity field of the fluid, ρ is the density of the fluid, and P is the pressure of the fluid. The $\beta-$plane approximation of the geophysical governing equations applies when we a working in regions which are within 5° latitude of the equator. There, the latitude ϕ is small and hence the approximations $\sin\phi \approx \phi$, $\cos\phi \approx 1$ are valid, resulting in the $\beta-$plane governing equations [13]

$$u_t + uu_x + vu_y + wu_z + 2\Omega w - \beta yv = -\frac{1}{\rho}P_x, \tag{2ca}$$

$$v_t + uv_x + vv_y + wv_z + \beta yu = -\frac{1}{\rho}P_y, \tag{2cb}$$

$$w_t + uw_x + vw_y + ww_z - 2\Omega u = -\frac{1}{\rho}P_z - g. \tag{2cc}$$

The boundary conditions for the fluid are given by

$$w = \eta_t + u\eta_x + v\eta_y \qquad \text{on } y = \eta(x, y, t), \tag{2d}$$

$$P = P_0 \qquad \text{on } y = \eta(x, y, t). \tag{2e}$$

Here η represents the free-surface and P_0 is the constant atmospheric pressure. The kinematic boundary condition on the surface simply states that all surface particles remain confined to the surface. Since we are interested in waves which are trapped in the equatorial region, we stipulate in the following that the wave surface profile decays in the latitudinal directions away from the equator. Finally, we assume the water to be infinitely deep, with the flow converging to a uniform current rapidly with depth, that is,

$$(u, v) \to (-c_0, 0) \quad \text{as} \quad y \to -\infty. \tag{2f}$$

3 Lagrangian dynamics

In this section we define an exact solution of the $\beta-$ plane governing equations (2). The solution represents steady waves travelling in the longitudinal direction, which have a constant speed of propagation $c > 0$, in the presence of a constant underlying current of strength c_0. We adopt the Lagrangian approach [1], whereby the Eulerian coordinates of fluid particles (x, y, z) are expressed as functions of the Lagrangian labelling variables $(q, r, s) \in (\mathbb{R}, (-\infty, r_0), \mathbb{R})$, and time t, as follows:

$$x = q - c_0 t - \frac{1}{k} e^{k[r-f(s)]} \sin\left[k(q - ct)\right], \tag{4a}$$

$$y = s, \tag{4b}$$

$$z = r + \frac{1}{k} e^{k[r-f(s)]} \cos\left[k(q - ct)\right], \tag{4c}$$

where $r_0 < 0$ and k is the wavenumber. The function $f(s)$ essentially determines the decay of the particle oscillation as it moves in the latitudinal direction away from the equator, and for the present construction we choose

$$f(s) = \frac{c\beta}{2\gamma} s^2, \tag{5}$$

where

$$\gamma = 2\Omega c_0 + g. \tag{6}$$

For notational convenience let us choose

$$\xi = k\left(r - f(s)\right), \ \theta = k(q - ct).$$

Then the Jacobian matrix of the transformation (4) is given by

$$\begin{pmatrix} \frac{\partial x}{\partial q} & \frac{\partial y}{\partial q} & \frac{\partial z}{\partial q} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \end{pmatrix} = \begin{pmatrix} 1 - e^{\xi}\cos\theta & 0 & -e^{\xi}\sin\theta \\ f_s e^{\xi}\sin\theta & 1 & -f_s e^{\xi}\cos\theta \\ -e^{\xi}\sin\theta & 0 & 1 + e^{\xi}\cos\theta \end{pmatrix}. \tag{7}$$

The determinant of the Jacobian is $1 - e^{2\xi}$, which is time independent, thus it follows that the flow defined by (4) must be volume preserving, ensuring that (2b) holds in the Eulerian setting [1]. We further remark that, in order for the transformation (4) to be well-defined, and to furthermore ensure that our flow has the appropriate decay properties (in both the vertical and the latitudinal directions), we stipulate that

$$r - f(s) \leq r_0 < 0. \tag{8}$$

We note that this relation forces the choice $c > 0$ for our flow. Bearing in mind that we are seeking trapped equatorial waves, we take $v \equiv 0$ throughout the fluid, and we calculate

$$u = \frac{Dx}{Dt} = ce^{\xi}\cos\theta - c_0, \qquad \frac{Du}{Dt} = kc^2 e^{\xi}\sin\theta, \tag{9a}$$
$$v = \frac{Dy}{Dt} = 0, \qquad \frac{Dv}{Dt} = 0, \tag{9b}$$
$$w = \frac{Dz}{Dt} = ce^{\xi}\sin\theta, \qquad \frac{Dw}{Dt} = -kc^2 e^{\xi}\cos\theta, \tag{9c}$$

where D/Dt is the material derivative. We can express (2c) as

$$\begin{aligned}\frac{Du}{Dt} + 2\Omega w &= -\frac{1}{\rho}P_x,\\ \frac{Dv}{Dt} + \beta y u &= -\frac{1}{\rho}P_y,\\ \frac{Dw}{Dt} - 2\Omega u &= -\frac{1}{\rho}P_z - g,\end{aligned}$$

and inserting the terms from (9) in this gives us

$$P_x = -\rho(kc^2 e^{\xi}\sin\theta + 2\Omega c e^{\xi}\sin\theta), \tag{11a}$$
$$P_y = -\rho(\beta s[ce^{\xi}\cos\theta - c_0]), \tag{11b}$$
$$P_z = -\rho(-kc^2 e^{\xi}\cos\theta - 2\Omega c e^{\xi}\cos\theta + \gamma). \tag{11c}$$

Multiplying both sides of (11) by the Jacobian matrix (7) we derive the following expression in terms of the Lagrangian variables

$$\begin{pmatrix} P_q \\ P_s \\ P_r \end{pmatrix} = -\rho \begin{pmatrix} (kc^2 + 2\Omega c - \gamma)e^{\xi}\sin\theta \\ f_s e^{2\xi}(kc^2 + 2\Omega c) + (\beta s c - f_s\gamma)\, e^{\xi}\cos\theta - \beta s c_0 \\ -(kc^2 + 2\Omega c)e^{2\xi} - (kc^2 + 2\Omega c - \gamma)e^{\xi}\cos\theta + \gamma \end{pmatrix}, \tag{12}$$

and so our next task is to determine a suitable expression for the pressure function P for which (12) hold.

3.1 Homogeneous fluid

We now show that, for a fluid which has constant density $\rho = \rho_0$ (and so (2a) holds in a trivial sense), we can construct a suitable pressure function for which (12) is satisfied, thereby proving that the flow which is prescribed by the Lagrangian formulation (4) satisfies (2c). Let us take

$$f(s) = \frac{\beta}{2(kc + 2\Omega)}s^2, \tag{13}$$

and so

$$f_s = \frac{\beta}{(kc+2\Omega)} s.$$

Now, for

$$\tilde{P} = \rho \frac{kc^2 + 2\Omega c}{2k} e^{2\xi} - \rho\gamma r + \frac{\rho\gamma c_0}{c} f(s) + \rho \frac{kc^2 + 2\Omega c - \gamma}{k} e^{\xi} \cos\theta + P_0 \quad (14)$$

we have

$$\begin{aligned}
\tilde{P}_q &= -\rho(kc^2 + 2\Omega c - \gamma) e^{\xi} \sin\theta \\
\tilde{P}_s &= -\rho(kc^2 + 2\Omega c) f_s e^{2\xi} - \rho(kc^2 + 2\Omega c - \gamma) f_s e^{\xi} \cos\theta - \rho\beta s c_0 \\
\tilde{P}_r &= \rho(kc^2 + 2\Omega c) e^{2\xi} - \rho\gamma + \rho(kc^2 + 2\Omega c - \gamma) e^{\xi} \cos\theta,
\end{aligned}$$

which matches the right hand side of (12). Since (2e) we require the pressure to be time independent on the surface, we want there to be no terms containing θ in (14) and therefore it follows that

$$kc^2 + 2\Omega c - \gamma = 0. \quad (15)$$

Now, if $c_0 = c$ then (15) gives us $c = \sqrt{g/k}$, and the geophysical effects have no bearing on the dispersion relation of the wave, which resembles that of the standard Gerstner's wave.

If $c_0 \neq c$ we get

$$c = \frac{\sqrt{\Omega^2 + k\gamma} - \Omega}{k}, \quad (16)$$

since we require $c > 0$ (see the discussion after (8)). This means that

$$c = \frac{\sqrt{\Omega^2 + k(2\Omega c_0 + g)} - \Omega}{k}. \quad (17)$$

We notice that if $c_0 = -\frac{g}{2\Omega}$ then $c = 0$. This scenario is physically quite implausible, and so from now on we assume that $c_0 > -\frac{g}{2\Omega}$, and so $\gamma > 0$. Since (15) also implies that $kc + 2\Omega = \gamma/c$, it follows that (5) matches (13), and so (14) becomes

$$P = \rho\gamma \left(\frac{e^{2\xi}}{2k} - r + \frac{c_0}{c} f(s) \right) + P_0 - \rho g \left(\frac{e^{2kr_0}}{2k} - r_0 \right). \quad (18)$$

Therefore the flow determined by (4) satisfies the governing equations (2c). By considering the boundary conditions (2d) and (2e), we now investigate for which values of the current c_0 this flow is hydrodynamically possible. Let us denote

$$\sigma(r,s) := \frac{e^{2\xi}}{2k} - r + \frac{c_0}{c} f(s) = \frac{e^{2k[r - \frac{c\beta}{2\gamma}s^2]}}{2k} - r + \frac{c_0\beta}{2\gamma}s^2.$$

If, for each fixed s, there exists a unique solution $r(s) \le r_0 < 0$ of the equation

$$\sigma(r(s),s) - \frac{e^{2kr_0}}{2k} + r_0 = 0, \tag{19}$$

then it follows that $r(s)$ determines the free-surface of the fluid (where q is a free-parameter of the surface) and so (2d) holds. Furthermore, it follows from the form of (18) that (2e) is then also satisfied. Proving the existence of such an $r(s)$ will complete our analysis of the homogeneous flow.

Firstly, for $s = 0$, we have $\sigma(r,0) = \frac{e^{2kr}}{2k} - r$, and so $r(0) = r_0$. For $s > 0$, we work as follows. We have $\lim_{r\to-\infty} \sigma(r,s) = \infty$, and also

$$\sigma_r(r,s) = e^{2k[r - \frac{c\beta}{2\gamma}s^2]} - 1 < 0.$$

Hence σ is a monotonically decreasing function of r, and so the existence of a unique solution of (19), for fixed $s > 0$, is equivalent to

$$\lim_{r\uparrow r_0} \sigma(r,s) = \frac{e^{2k[r_0 - \frac{c\beta}{2\gamma}s^2]}}{2k} - r_0 + \frac{c_0\beta}{2\gamma}s^2 < \frac{e^{2kr_0}}{2k} - r_0. \tag{20}$$

If $c_0 \le 0$, it is easy to see that condition (20) holds for all $s > 0$, and so we are done.

For $c_0 > 0$, we can see explicitly that condition (20) will break down for large enough values of s. However, we note that since we are working in the equatorial region where the $\beta-$plane approximation is valid, the variable s will be restricted *de facto* in its range of values. Hence, depending on the size of the current $c_0 > 0$, condition (20) may yet hold for s in this range. To estimate the range of s where condition (20) holds, we remark that for a solution of (19) to exist for s close to zero we must necessarily have

$$\sigma_s(r_0,s) = \frac{\beta s}{\gamma}\left(c_0 - ce^{2k[r_0 - \frac{c\beta}{2\gamma}s^2]}\right) < 0. \tag{21}$$

For a given current $c_0 > 0$, our solutions (4) are dynamically possible in a range of s where (21) holds, and in particular it is necessary that

$$c_0 < ce^{2kr_0}. \tag{22}$$

Finally, when (21) holds, we have

$$\left(r'(s) - \frac{c\beta}{\gamma}s\right) e^{2k[r-\frac{c\beta}{2\gamma}s^2]} - r'(s) + \frac{c_0\beta}{\gamma}s = 0,$$

that is

$$r'(s) = \frac{\beta s}{\gamma}\frac{c_0 - ce^{2k[r-\frac{c\beta}{2\gamma}s^2]}}{1-e^{2k[r-\frac{c\beta}{2\gamma}s^2]}} < 0,$$

and so the even function $s \mapsto r(s)$ is decreasing. Finally, we note that the form of the surface wave, for fixed values of s and t, is an upside down trochoid, cf. [5, 6, 17].

3.2 Heterogeneous fluid

Remarkably, we can accommodate variable density by adapting our flow slightly. Since ρ is no longer constant, we first ensure that (2a) is satisfied. We observe that, for steady waves travelling in the longitudinal direction with relative speed c, we have $\rho(x,y,t) = \rho(x-ct,y)$, and so (2a) becomes

$$\rho_t + u\rho_x + v\rho_y + w\rho_z = -c\rho_x + ce^{\xi}\cos\theta\rho_x + ce^{\xi}\sin\theta\rho_z = 0. \tag{23}$$

Therefore

$$\begin{aligned}\rho_q = \rho_x\frac{\partial x}{\partial q} + \rho_y\frac{\partial y}{\partial q} + \rho_z\frac{\partial z}{\partial q} &= \rho_x(1-e^{\xi}\cos\theta) - \rho_z e^{\xi}\sin\theta \\ &= -\frac{(u-c)\rho_x + w\rho_z}{c} = 0,\end{aligned}$$

and the density ρ is independent of q. Let us prescribe the density function by

$$\rho(r,s) = F\left(\frac{e^{2\xi}}{2k} - r + \frac{c_0}{c}f(s)\right), \tag{24}$$

where $F:(0,\infty)\to(0,\infty)$ is continuously differentiable and non-decreasing. Then we modify (18) slightly by defining

$$P = \gamma\mathcal{F}\left(\frac{e^{2\xi}}{2k} - r + \frac{c_0}{c}f(s)\right) + P_0 - \gamma\mathcal{F}\left(\frac{e^{2kr_0}}{2k} - r_0\right), \tag{25}$$

where $\mathcal{F}' = F$ and $\mathcal{F}(0) = 0$. Now all of the considerations of the preceeding section follow, including the form of the dispersion relation and the function $f(s)$.

3.3 Vorticity

We now calculate the vorticity of the flow prescribed by (4), which turns out to be independent of the density formulation. The inverse of the Jacobian (7) is given by

$$\begin{pmatrix} \frac{\partial q}{\partial x} & \frac{\partial s}{\partial x} & \frac{\partial r}{\partial x} \\ \frac{\partial q}{\partial y} & \frac{\partial s}{\partial y} & \frac{\partial r}{\partial y} \\ \frac{\partial q}{\partial z} & \frac{\partial s}{\partial z} & \frac{\partial r}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{1+e^{\xi}\cos\theta}{1-e^{2\xi}} & 0 & \frac{e^{\xi}\sin\theta}{1-e^{2\xi}} \\ -f_s\frac{e^{\xi}\sin\theta}{1-e^{2\xi}} & 1 & f_s\frac{e^{\xi}\cos\theta-e^{2\xi}}{1-e^{2\xi}} \\ \frac{e^{\xi}\sin\theta}{1-e^{2\xi}} & 0 & \frac{1-e^{\xi}\cos\theta}{1-e^{2\xi}} \end{pmatrix}. \tag{26}$$

Then we have

$$\begin{aligned} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{pmatrix} &= \begin{pmatrix} \frac{\partial q}{\partial x} & \frac{\partial s}{\partial x} & \frac{\partial r}{\partial x} \\ \frac{\partial q}{\partial y} & \frac{\partial s}{\partial y} & \frac{\partial r}{\partial y} \\ \frac{\partial q}{\partial z} & \frac{\partial s}{\partial z} & \frac{\partial r}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial q} & \frac{\partial v}{\partial q} & \frac{\partial v}{\partial q} \\ \frac{\partial u}{\partial s} & \frac{\partial v}{\partial s} & \frac{\partial v}{\partial s} \\ \frac{\partial u}{\partial r} & \frac{\partial v}{\partial r} & \frac{\partial v}{\partial r} \end{pmatrix} \\ &= \frac{cke^{\xi}}{1-e^{2\xi}} \begin{pmatrix} -\sin\theta & 0 & \cos\theta + e^{\xi} \\ f_s(e^{\xi}-\cos\theta) & 0 & -f_s\sin\theta \\ -e^{\xi}+\cos\theta & 0 & \sin\theta \end{pmatrix}, \end{aligned} \tag{27}$$

and so the vorticity is

$$\omega = (w_y - v_z, u_z - w_x, v_x - u_y) \tag{28}$$

$$= \left(-s\frac{kc^2\beta}{g}\frac{e^{\xi}\sin\theta}{1-e^{2\xi}}, -\frac{2kce^{2\xi}}{1-e^{2\xi}}, s\frac{kc^2\beta}{g}\frac{e^{\xi}\cos\theta - e^{2\xi}}{1-e^{2\xi}} \right). \tag{29}$$

We note that since the current is constant, it does not impact on the vorticity of the flow, and the prescription of the vorticity matches that of [6].

References

[1] A. Bennett, *Lagrangian fluid dynamics*, Cambridge University Press, Cambridge, 2006.

[2] A. Constantin, On the deep water wave motion, *J. Phys. A* **34** (2001), 1405–1417.

[3] A. Constantin, Edge waves along a sloping beach, *J. Phys. A* **34** (2001), 9723–9731.

[4] A. Constantin, The trajectories of particles in Stokes waves, *Invent. Math.* **166** (2006), 523–535.

[5] A. Constantin, *Nonlinear Water Waves with Applications to Wave-Current Interactions and Tsunamis*, CBMS-NSF Conference Series in Applied Mathematics, Vol. 81, SIAM, Philadelphia, 2011.

[6] A. Constantin, An exact solution for equatorially trapped waves, *J. Geophys. Res.* **117** (2012), C05029.

[7] A. Constantin, On the modelling of Equatorial waves, *Geophys. Res. Lett.*, **39** L05602 (2012).

[8] A. Constantin and J. Escher, Symmetry of steady deep-water waves with vorticity, *Eur. J. Appl. Math.* **15** (2004), 755–768.

[9] A. Constantin, M. Ehrnström and G. Villari, Particle trajectories in linear deep-water waves *Nonl. Anal. B* **9** (2008), 1336-1344.

[10] A. Constantin and G. Villari, Particle trajectories in linear water waves, *J. Math. Fluid Mech.* **10** (2008), 1–18.

[11] B. Cushman-Roisin and J.-M. Beckers, *Introduction to Geophysical Fluid Dynamics: Physical and Numerical Aspects*, Academic, Waltham, Mass., 2011.

[12] A. V. Fedorov and J. N. Brown, Equatorial waves, in *Encyclopedia of Ocean Sciences*, edited by J. Steele, pp. 3679–3695, Academic, San Diego, Calif., 2009.

[13] I. Gallagher and L. Saint-Raymond, On the influence of the Earths rotation on geophysical flows, in *Handbook of Mathematical Fluid Mechanics*, vol. 4, edited by S. Friedlander and D. Serre, pp. 201–329, North-Holland, Amsterdam, 2007.

[14] F. Gerstner, Theorie der Wellen samt einer daraus abgeleiteten Theorie der Deichprofile, *Ann. Phys.* **2** (1809), 412–445.

[15] D. Henry, Particle trajectories in linear periodic capillary and capillary-gravity deep-water waves, *J. Nonlinear Math. Phys.* **14** (2007), 1–7.

[16] D. Henry, Particle trajectories in linear periodic capillary and capillary-gravity water waves, *Philos. Trans. R. Soc. Lond. Ser. A*, **365** (2007), 2241–2251.

[17] D. Henry, On Gerstner's water wave, *J. Nonl. Math. Phys.* **15** (2008), 87–95.

[18] D. Henry, On the deep-water Stokes wave flow, *Int. Math. Res. Not.* (2008), Art. ID rnn071, 7 pages.

[19] D. Ionescu-Kruse, Particle trajectories in linearized irrotational shallow water flows, *J. Nonlinear Math. Phys.* **15** (2008), 13–27.

[20] T. Izumo, The equatorial current, meridional overturning circulation, and their roles in mass and heat exchanges during the El Niño events in the tropical Pacific Ocean, *Ocean Dyn.*, **55** (2005), 110–123.

[21] A.-V. Matioc, On particle trajectories in linear water waves, *Nonl. Anal. B* **11** (2010), 4275–4284.

[22] A.-V. Matioc, On particle trajectories in linear deep-water waves, *Commun. Pure Appl. Anal.* **11** (2012), 1537–1547.

[23] A. V. Matioc, An exact solution for geophysical equatorial edge waves over a sloping beach, *J. Phys. A* **45** 365501 (2012).

[24] E. Mollo-Christensen, Gravitational and Geostrophic Billows: Some Exact Solutions, *J. Atmos. Sci.* **35** (1978), 1395–1398.

[25] R. Stuhlmeier, On edge waves in stratified water along a sloping beach, *J. Nonlinear Math. Phys.* **18** (2011), 127–137.