Rotation-Based Formulation for Stable Matching

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Abstract. We introduce new CP models for the many-to-many stable matching problem. We use the notion of rotation to give a novel encoding that is linear in the input size of the problem. We give extra filtering rules to maintain arc consistency in quadratic time. Our experimental study on hard instances of sex-equal and balanced stable matching shows the efficiency of one of our propositions as compared with the state-of-the-art constraint programming approach.

1 Introduction

In two-sided stable matching problems the objective is to assign some agents to other agents based on their preferences [14]. The classic exemplar of such problems is the well known *stable marriage* (*SM*) problem, first introduced by Gale and Shapley [6]. In SM the two sets of agents are called men and women. Each man has a preference list over the women and vice versa. The purpose is to find a *matching* where each man (respectively woman) is associated to at most one woman (respectively man) that respects a criterion called *stability*. A matching M in this context is stable if any pair $\langle m, w \rangle$ (where m is a man and w is a woman) that does not belong to M satisfies the property that m prefers his partner in M to w or w prefers her partner in M to m.

This family of problems has gained considerable attention as it has a wide range of applications such as assigning doctors to hospitals, students to college, and in kidney exchange problems. The stable marriage problem itself can be solved in $O(n^2)$ time [6] where n is the maximum number of men/women. This is also true for the general case of many-to-many stable matching; the complexity $O(n^2)$ is given in the proof of Theorem 1 in [1]. However, when facing real world situations the problem often considers additional optimality criteria. In many cases, the problem becomes intractable and specialized algorithms for solving the standard version are usually hard to adapt. The use of a modular approach such as constraint programming is very beneficial to tackle such cases.

Many constraint programming approaches exist in the literature for stable matching problems. Examples of these concern stable marriage [7, 21, 22], hospital residents (HR) [13, 20], many-to-many stable matching [3], and stable roommates [17]. Despite the fact that many-to-many stable matching generalizes HR and SM, it has not gained as much attention as SM and HR in the constraint programming community. In this paper, we follow this line of research by proposing an effective and efficient model for all three variants of stable matching: one-to-one, many-to-one, and many-to-many. Our propositions are based on a powerful structure called rotations. The latter has been used to model the stable roommates problem in [9] (page 194) and [4, 5].

We leverage some known properties related to rotations in order to propose a novel SAT formulation of the general case of many-to-many stable matching. We show that unit propagation on this formula ensures the existence of a particular solution. Next, we use this property to give an algorithm that maintains arc consistency if one considers many-to-many stable matching as a (global) constraint. The overall complexity for arc consistency is $O(L^2)$ time where L is total input size of all preference lists. Our experimental study on hard instances of sex-equal and balanced stable matching show that our approach outperforms the state-of-the-art constraint programming approach [20].

The remainder of this paper is organized as follows. In Section 2 we give a brief overview of constraint programming. We present the stable matching problem in Section 3 as well as various concepts related to rotations. In Section 4 we propose a novel formulation of stable matching based on the notion of rotation. We show in Section 5 some additional pruning rules and show that arc consistency can be maintained in $O(L^2)$ worst case time complexity. Lastly, in Section 6 we present an empirical experimental study on two hard variants of stable matching and show that one of our new models outperforms the state-of-the-art constraint programming approach in the literature.

2 Constraint Programming

We provide a short formal background related to constraint programming. Let $\mathcal{X}$ be a set of integer variables. A *domain* for $\mathcal{X}$, denoted by $\mathcal{D}$, is a mapping from variables to finite sets of integers. For each variable x, we call $\mathcal{D}(x)$ the *domain of the variable* x. A variable is called assigned when $\mathcal{D}(x) = \{v\}$. In this case, we say that v is assigned to x and that x is set to v. A variable is unassigned if it is not assigned. A *constraint* C defined over $[x_1, \ldots, x_k]$ ($k \in \mathbb{N}^*$) is a finite subset of $\mathbb{Z}^k$. The sequence $[x_1, \ldots, x_k]$ is the *scope* of C (denoted by $\mathcal{X}(C)$) and k is called the *arity* of C. A *support* for C in a domain $\mathcal{D}$ is a k-tuple τ such that $\tau \in C$ and $\tau[i] \in \mathcal{D}(x_i)$ for all $i \in [1, \ldots, k]$. Let $x_i \in \mathcal{X}(C)$ and $v \in \mathcal{D}(x_i)$. We say that the assignment of v to x has a support for C in $\mathcal{D}$ iff there exists a support τ for C in $\mathcal{D}$ such that $\tau[i] = v$. The constraint C is *arc consistent* (AC) in $\mathcal{D}$ iff $\forall i \in [1, \ldots, k]$, $\forall v \in \mathcal{D}(x_i)$, the assignment of v to x_i has a support in $\mathcal{D}$. A *filtering algorithm* (or *propagator*) for a constraint C takes as input a domain $\mathcal{D}$ and returns either $\emptyset$ if there is no support for C in $\mathcal{D}$ (i.e., failure) or a domain $\mathcal{D}'$ such that any support for C in $\mathcal{D}$ is a support for C in $\mathcal{D}'$, $\forall x \in \mathcal{X}(C)$, $\mathcal{D}'(x) \subseteq \mathcal{D}(x)$, and $\forall x \notin \mathcal{X}(C)$, $\mathcal{D}'(x) = \mathcal{D}(x)$. A *Boolean* variable has an initial domain equal to $\{0, 1\}$ (0 is considered as *false* and 1 as *true*). A clause is a disjunction of literals where a literal is a Boolean variable or its negation. Clauses are usually filtered with an algorithm called unit propagation [16].

Let $\mathcal{X}$ be a set of variables, $\mathcal{D}$ be a domain, and $\mathcal{C}$ be a set of constraints defined over subsets of $\mathcal{X}$. The *constraint satisfaction problem* (*CSP*) is the question of deciding if an $|\mathcal{X}|-$tuple of integers τ exists such that the projection of τ on the scope of every constraint $C \in \mathcal{C}$ is a support for C in $\mathcal{D}$. We consider in this paper classical backtracking algorithms to solve CSPs by using filtering algorithms at every node of the search tree [19].

3 Stable Matching

We consider the general case of the many-to-many stable matching problem. We follow the standard way of introducing this problem by naming the two sets of agents as *workers* and *firms* [14]. We use a notation similar to that of [3].

Let $n_F, n_W \in \mathbb{N}^*$, $F = \{f_1, f_2, \ldots, f_{n_F}\}$ be a set of firms, $W = \{w_1, w_2, \ldots, w_{n_W}\}$ be a set of workers, and $n = \max\{n_F, n_W\}$. Every firm f_i has a list, P_{f_i}, of workers given in a strict order of preference (i.e., no ties). The preference list of a worker w_i is similarly defined. We denote by $P_W = \{P_{w_i} \mid i \in [1, n_W]\}$ the set of preferences of workers, and by $P_F = \{P_{f_j} \mid j \in [1, n_F]\}$ the set of preferences of firms. We use L to denote the sum of the sizes of the preference lists. Note that the size of the input problem is $O(L)$. Therefore we shall give all our complexity results with respect to L.

For every firm f_j (respectively, worker w_i), we denote by q_{f_j} (respectively, q_{w_i}) its quota. We denote by $q_W = \{q_{w_i} \mid i \in [1, n_W]\}$ the set of quota for workers, and by $q_F = \{q_{f_j} \mid j \in [1, n_F]\}$ the set of quotas for firms. We use the notation $w_i \succ_{f_k} w_j$ when a firm f_k prefers worker w_i to worker w_j. The operator $\succ_{w_k}$ is defined similarly for any worker w_k.

A pair $\langle w_i, f_j \rangle$ is said to be *acceptable* if $w_i \in P_{f_j}$ and $f_j \in P_{w_i}$. A *matching* M is a set of acceptable pairs. Let $M(w_i) = \{f_j \mid \langle w_i, f_j \rangle \in M\}$, and $M(f_j) = \{w_i \mid \langle w_i, f_j \rangle \in M\}$. A worker w_i (respectively, firm f_j) is said to be *under-assigned* in M if $|M(w_i)| < q_{w_i}$ (respectively, $|M(f_j)| < q_{f_j}$). We define for every worker w_i, $last_M(w_i)$ as the least preferred firm for w_i in $M(w_i)$ if $M(w_i) \neq \emptyset$. For every firm f_j, $last_M(f_j)$ is similarly defined. A pair $\langle w_i, f_j \rangle \notin M$ is said to be *blocking* M if it is acceptable such that the following two conditions are true:

- w_i is under-assigned in M or $\exists f_k \in M(w_i)$ and $f_j \succ_{w_i} f_k$.
- f_j is under-assigned in M or $\exists w_l \in M(f_j)$ and $w_i \succ_{f_j} w_l$.

Definition 1 (Stability). *A matching M is* (pairwise) stable *if* $\forall w_i \in W$, $|M(w_i)| \leq q_{w_i}$, $\forall f_j \in F$, $|M(f_j)| \leq q_{f_j}$, *and there is no blocking pair for M.*

An instance of the many-to-many stable matching problem is defined by the tuple $\langle W, F, P_W, P_F, q_W, q_F \rangle$. The problem is to find a stable matching if one exists.

A pair $\langle w_i, f_j \rangle$ is *stable* if there exists a stable matching M containing $\langle w_i, f_j \rangle$, *unstable* otherwise. A pair $\langle w_i, f_j \rangle$ is fixed if it is included in all stable matchings.

Let M, M' be two stable matchings. A worker w_i prefers M no worse than M' (denoted by $M \succeq_{w_i} M'$) if (1) $M(w_i) = M'(w_i)$ or (2) $|M(w_i)| \geq |M'(w_i)|$

and $last_M(w_i) \succ_{w_i} last_{M'}(w_i)$. It should be noted that every worker (respectively, firm) is assigned to the same number of firms (respectively, workers) in every stable matching [1]. So the condition $|M(w_i)| \geq |M'(w_i)|$ is always true in the case of many-to-many stable matching. Let M, M' be two different stable matchings. We say that M dominates M' (denoted by $M \succeq_W M'$) if $M \succeq_{w_i} M'$ for every worker w_i. This is called the worker-oriented dominance relation. The *firm-oriented dominance relation ($\succeq_F$)* is similarly defined for firms.

The authors of [1] showed that a stable matching always exists and can be found in $O(n^2)$ time. More precisely, the complexity of finding a stable matching is $O(L)$. Moreover, they showed that there always exist worker-optimal and firm-optimal stable matchings (with respect to $\succeq_W$ and $\succeq_F$). We denote these two matchings by M_0 and M_z, respectively.

Example 1 (An instance of many-to-many stable matching (from [3])). Consider the example where $n_W = 5, n_F = 5$, and for all $1 \leq i, j \leq 5$, $q_{w_i} = q_{f_j} = 2$. The preference lists for workers and firms are given in Table 1.

Table 1: Example of preference lists

$P_{w_1} = [f_1, f_2, f_3, f_4, f_5]$	$P_{f_1} = [w_3, w_2, w_4, w_5, w_1]$
$P_{w_2} = [f_2, f_3, f_4, f_5, f_1]$	$P_{f_2} = [w_2, w_3, w_5, w_4, w_1]$
$P_{w_3} = [f_3, f_4, f_5, f_1, f_2]$	$P_{f_3} = [w_4, w_5, w_2, w_1, w_3]$
$P_{w_4} = [f_4, f_5, f_1, f_2, f_3]$	$P_{f_4} = [w_1, w_5, w_3, w_2, w_4]$
$P_{w_5} = [f_5, f_1, f_2, f_3, f_4]$	$P_{f_5} = [w_4, w_1, w_2, w_3, w_5]$

There exist seven stable matchings for this instance:

- $M_0 = \{\langle w_1, f_1\rangle, \langle w_1, f_2\rangle, \langle w_2, f_2\rangle, \langle w_2, f_3\rangle, \langle w_3, f_3\rangle, \langle w_3, f_4\rangle, \langle w_4, f_4\rangle, \langle w_4, f_5\rangle, \langle w_5, f_5\rangle, \langle w_5, f_1\rangle\}$
- $M_1 = \{\langle w_1, f_1\rangle, \langle w_1, f_3\rangle, \langle w_2, f_2\rangle, \langle w_2, f_3\rangle, \langle w_3, f_5\rangle, \langle w_3, f_4\rangle, \langle w_4, f_4\rangle, \langle w_4, f_5\rangle, \langle w_5, f_2\rangle, \langle w_5, f_1\rangle\}$
- $M_2 = \{\langle w_1, f_4\rangle, \langle w_1, f_3\rangle, \langle w_2, f_2\rangle, \langle w_2, f_3\rangle, \langle w_3, f_5\rangle, \langle w_3, f_4\rangle, \langle w_4, f_1\rangle, \langle w_4, f_5\rangle, \langle w_5, f_2\rangle, \langle w_5, f_1\rangle\}$
- $M_3 = \{\langle w_1, f_4\rangle, \langle w_1, f_5\rangle, \langle w_2, f_2\rangle, \langle w_2, f_3\rangle, \langle w_3, f_1\rangle, \langle w_3, f_4\rangle, \langle w_4, f_1\rangle, \langle w_4, f_5\rangle, \langle w_5, f_2\rangle, \langle w_5, f_3\rangle\}$
- $M_4 = \{\langle w_1, f_4\rangle, \langle w_1, f_5\rangle, \langle w_2, f_2\rangle, \langle w_2, f_3\rangle, \langle w_3, f_1\rangle, \langle w_3, f_2\rangle, \langle w_4, f_1\rangle, \langle w_4, f_5\rangle, \langle w_5, f_4\rangle, \langle w_5, f_3\rangle\}$
- $M_5 = \{\langle w_1, f_4\rangle, \langle w_1, f_5\rangle, \langle w_2, f_2\rangle, \langle w_2, f_1\rangle, \langle w_3, f_1\rangle, \langle w_3, f_4\rangle, \langle w_4, f_3\rangle, \langle w_4, f_5\rangle, \langle w_5, f_2\rangle, \langle w_5, f_3\rangle\}$
- $M_z = M_6 = \{\langle w_1, f_4\rangle, \langle w_1, f_5\rangle, \langle w_2, f_2\rangle, \langle w_2, f_1\rangle, \langle w_3, f_1\rangle, \langle w_3, f_2\rangle, \langle w_4, f_3\rangle, \langle w_4, f_5\rangle, \langle w_5, f_4\rangle, \langle w_5, f_3\rangle\}$

In this instance, $\langle w_1, f_1\rangle$ is a stable pair since $\langle w_1, f_1\rangle \in M_0$ and $\langle w_2, f_4\rangle$ is not stable since it is not included in any stable matching. Regarding the dominance relation, we have $M_1 \succeq_W M_2$, and $M_2 \succeq_W M_3$, Using transitivity, we obtain $M_1 \succeq_W M_3$, Note that M_4 and M_5 are incomparable. □

In the following, we introduce a central notion in this paper called rotation. Consider the matching M_0 from the instance given in Example 1 and the list of pairs $\rho_0 = [\langle w_1, f_2\rangle, \langle w_5, f_5\rangle, \langle w_3, f_3\rangle]$. Notice that every pair in ρ_0 is part of M_0. Consider now the operation of shifting the firms in a cyclic way as follows: f_2 is paired with w_5, f_5 is paired with w_3, and f_3 is paired with w_1. This operation changes M_0 to M_1. In this case, we say ρ_0 is a rotation.

Formally, for any stable matching $M \neq M_z$ and any worker w_i such that $M(w_i) \neq \emptyset$, we define $r_M(w_i)$ to be the most preferred firm f_j for w_i such that $w_i \succ_{f_j} last_M(f_j)$ and $\langle w_i, f_j\rangle \notin M$. In other words, given $\langle w_i, f_j\rangle \notin M$, $r_M(w_i)$ is a firm that is the most preferred firm to w_i such that it prefers w_i to her worst assigned partner in M.

Definition 2 (Rotation [2]). *A rotation ρ is an ordered list of pairs $[\langle w_{i_0}, f_{j_0}\rangle, \langle w_{i_1}, f_{j_1}\rangle, \ldots, \langle w_{i_{t-1}}, f_{j_{t-1}}\rangle]$ such that $t \in [2, \min(n_W, n_F)]$, $i_k \in [1, n_W]$, $j_k \in [1, n_F]$ for all $0 \leq k < t$ and there exists a stable matching M where $\langle w_{i_k}, f_{j_k}\rangle \in M$, $w_{i_k} = last_M(f_{j_k})$, and $f_{j_k} = r_M(w_{i_{k+1 \ mod \ t}})$ for all $0 \leq k < t$. In this case we say that ρ is* exposed *in M.*

Let ρ be a rotation exposed in a stable matching M. The operation of *eliminating* a rotation ρ from M consists of removing each pair $\langle w_{i_k}, f_{j_k}\rangle \in \rho$ from M, then adding $\langle w_{i_{k+1 \ mod \ t}}, f_{j_k}\rangle$. The new set of pairs, denoted by $M_{/\rho}$ constitutes a stable matching that is dominated (w.r.t. workers) by M [3, 8]. We say that ρ produces $\langle w_i, f_j\rangle$ if $\langle w_i, f_j\rangle \in M_{/\rho} \setminus M$.

The following three lemmas are either known in the literature [3] or are a direct consequence of [3].

Lemma 1. *In every stable matching $M \neq M_z$, there exists (at least) a rotation that can be exposed in M.*

Lemma 2. *Every stable matching $M \neq M_0$ can be obtained by iteratively eliminating some rotations, without repetition, starting from M_0.*

Lemma 3. *Any succession of eliminations leading from M_0 to M_z contains all the possible rotations (without repetition).*

We say that a rotation ρ_1 precedes another rotation ρ_2 (denoted by $\rho_1 \prec\prec \rho_2$) if ρ_1 is exposed before ρ_2 in every succession of eliminations leading from M_0 to M_z. Note that this precedence relation is transitive and partial. That is, $\rho_1 \prec\prec \rho_2 \wedge \rho_2 \prec\prec \rho_3$, implies $\rho_1 \prec\prec \rho_3$, and there might exist two rotations ρ_1, and ρ_2 where neither $\rho_1 \prec\prec \rho_2$ nor $\rho_2 \prec\prec \rho_1$.

Example 2 (Rotation precedence). In the previous example we have $\rho_0 \prec\prec \rho_1$, $\rho_1 \prec\prec \rho_2$, $\rho_2 \prec\prec \rho_3$, $\rho_2 \prec\prec \rho_4$. By transitivity we obtain $\rho_0 \prec\prec \rho_4$. Note that in this example neither $\rho_3 \prec\prec \rho_4$ nor $\rho_4 \prec\prec \rho_3$. □

Let R be the set of all rotations. The precedence relation $\prec\prec$ with R forms the *rotation poset Π_R*. Let $G = (V_G, A_G)$ be the directed graph corresponding to the rotation poset. That is, every vertex corresponds to a rotation, and there is an arc $(\rho_j, \rho_i) \in A_G$ iff $\rho_j \prec\prec \rho_i$. The construction of R and G can be performed in $O(L)$ time [3]. For each rotation $\rho_i \in R$, we denote by $N^-(\rho_i)$ the set of rotations having an outgoing edge towards ρ_i, i.e., these rotations dominate ρ_i. We introduce below the notion of closed subset and a very important theorem.

Definition 3 (Closed subset). *A subset of rotations $S \subseteq V_G$ is closed iff $\forall \rho_i \in S$, $\forall \rho_j \in V_G$, if $\rho_j \prec\prec \rho_i$, then $\rho_j \in S$.*

Theorem 1 (From [2]). *There is a one-to-one correspondence between closed subsets and stable matchings.*

The solution corresponding to a closed subset S is obtained by eliminating all the rotations in S starting from M_0 while respecting the order of precedence between the rotations. Recall from Lemma 2 that every stable matching $M \neq M_0$ can be obtained by iteratively eliminating some rotations, without any repetition, starting from M_0. The closed subset corresponding to a stable matching M is indeed the set of rotations in any succession of eliminations of rotations leading to M. Notice that M_0 corresponds to the empty set and that M_z is the set of all rotations.

We denote by Δ the set of stable pairs. Let $\langle w_i, f_j \rangle$ be a stable pair. There exists a unique rotation containing $\langle w_i, f_j \rangle$ if $\langle w_i, f_j \rangle \notin M_z$ [3]. We denote this rotation by $\rho_{e_{ij}}$. Similarly, $\forall \langle w_i, f_j \rangle \in \Delta \setminus M_0$ there exists a unique rotation ρ such that eliminating ρ produces $\langle w_i, f_j \rangle$. We denote by $\rho_{p_{ij}}$ the rotation that produces the stable pair $\langle w_i, f_j \rangle \in \Delta \setminus M_0$. Notice that it is always the case that $\rho_{p_{ij}} \prec\prec \rho_{e_{ij}}$ for any stable pair that is not part of $M_0 \cup M_z$.

Example 3 (The rotations $\rho_{e_{ij}}$ and $\rho_{p_{ij}}$). For the previous example, we have $\rho_{e_{23}} = \rho_4$, and $\rho_{p_{31}} = \rho_2$ since ρ_2 produces the pair $\langle w_3, f_1 \rangle$. □

Lastly, we denote by FP the set of fixed pairs, SP is the set of stable pairs that are not fixed, and NSP is the set of non stable pairs. Note that $\langle w_i, f_j \rangle \in FP$ iff $\langle w_i, f_j \rangle \in M_0 \cap M_z$. These three sets can be constructed in $O(L)$ [3].

4 A Rotation-based Formulation

We first show that the problem of finding a stable matching can be formulated as a SAT formula using properties from rotations. Next, we show that for any input domain $\mathcal{D}$, if unit propagation is performed without failure, then there exists necessarily a solution in $\mathcal{D}$. Recall that there exists an algorithm (called the Extended Gale-Shapley algorithm) to find a solution to the many-to-many stable matching that runs in $O(L)$ time [1,3]. However, using a CP formulation such as the one that we propose in this section is very beneficial when dealing with NP-Hard variants of the problem.

In out model, a preprocessing step is performed to compute M_0, M_z, SP, FP, NSP, the graph posed, $\rho_{e_{ij}}$ for all $\langle w_i, f_j \rangle \in SP \setminus M_z$, and $\rho_{p_{ij}}$ for all $\langle w_i, f_j \rangle \in SP \setminus M_0$. This preprocessing is done in $O(L)$ time [3].

4.1 A SAT Encoding

We introduce for each pair $\langle w_i, f_j \rangle$ a Boolean variable $x_{i,j}$. The latter is set to true iff $\langle w_i, f_j \rangle$ is part of the stable matching. Moreover, we use for each rotation ρ_k a Boolean variable r_k (called *rotation variable*) to indicate whether the rotation ρ_k is in the closed subset that corresponds to the solution.

Observe first that for all $\langle w_i, f_j \rangle \in FP$, $x_{i,j}$ has to be true, and for all $\langle w_i, f_j \rangle \in NSP$, $x_{i,j}$ has to be false.

We present three lemmas that are mandatory for the soundness and completeness of the SAT formula. Let M be a stable matching and S its closed subset (Theorem 1).

Lemma 4. *$\forall \langle w_i, f_j \rangle \in SP \cap M_0 : \langle w_i, f_j \rangle \in M$ iff $\rho_{e_{ij}} \notin S$.*

Proof. $\Rightarrow$ Suppose that $\rho_{e_{ij}} \in S$. Let *Sequence* be an ordered list of the rotations in S such that exposing the rotations of S starting from M_0 leads to M. For all $a \in [1, |S|]$, we define M'_a to be the stable matching corresponding the closed subset $S'_a = \{Sequence[k] \mid k \in [1, a]\}$. We also use M'_0 to denote the particular case of M_0 and $S'_0 = \emptyset$. Notice that $M'_{|S|} = M$ and $S'_{|S|} = S$. Let $a \in [1, |S|]$ such that $Sequence[a] = \rho_{e_{ij}}$. We know that exposing the rotation $\rho_{e_{ij}}$ from S'_{a-1} moves worker w_i to a partner that is worse than f_i. For any matching M'_b where $b \in [a, |S|]$, w_i either has the same partners in M'_{b-1} or is assigned a new partner that is worse than f_i. Hence $\langle w_i, f_j \rangle$ cannot be part of $M'_{|S|} = M$.

$\Leftarrow$ $\langle w_i, f_j \rangle$ must be part of the solution since it is part of M_0 and $\rho_{e_{ij}} \notin S$. $\square$

Lemma 5. *$\forall \langle w_i, f_j \rangle \in SP \cap M_z : \langle w_i, f_j \rangle \in M$ iff $\rho_{p_{ij}} \in S$.*

Proof. $\Rightarrow$ Suppose that $\rho_{p_{ij}} \notin S$. The pair $\langle w_i, f_j \rangle$ cannot be produced when eliminating rotations in S since $\rho_{p_{ij}}$ is unique. Therefore $\rho_{p_{ij}} \in S$.

$\Leftarrow$ Suppose that $\rho_{p_{ij}} \in S$. The pair $\langle w_i, f_j \rangle$ must be part of the solution since $\rho_{p_{ij}} \in S$ and it can never be eliminated by any rotation since $\langle w_i, f_j \rangle \in M_z$. $\square$

Lemma 6. *$\forall \langle w_i, f_j \rangle \in SP \setminus (M_0 \cup M_z) : \langle w_i, f_j \rangle \in M$ iff $\rho_{p_{ij}} \in S \land \rho_{e_{ij}} \notin S$.*

Proof. $\Rightarrow$ Suppose that $\langle w_i, f_j \rangle$ is part of M.

- If $\rho_{p_{ij}} \notin S$, then $\langle w_i, f_j \rangle$ can never be produced when eliminating rotations in S. Therefore $\rho_{p_{ij}} \in S$.
- If $\rho_{e_{ij}} \in S$, similarly to the proof of Lemma 4, we can show that the pair $\langle w_i, f_j \rangle$ cannot be part of the solution.

$\Leftarrow$ Suppose that $\rho_{p_{ij}} \in S$ and $\rho_{e_{ij}} \notin S$. The pair $\langle w_i, f_j \rangle$ must be part of the solution since it is produced by $\rho_{p_{ij}}$ and not eliminated since $\rho_{e_{ij}} \notin S$. $\square$

Using Lemmas 4, 5, 6, we can formulate the problem of finding a stable matching as follows.

$$\forall \rho_i \in R, \forall \rho_j \in N^-(\rho_i) : \ \neg r_i \lor r_j \tag{1}$$

$$\forall \langle w_i, f_j\rangle \in SP \cap M_0 : \neg x_{i,j} \vee \neg r_{e_{ij}} \ ; \ x_{i,j} \vee r_{e_{ij}} \tag{2}$$

$$\forall \langle w_i, f_j\rangle \in SP \cap M_z : \neg x_{i,j} \vee r_{p_{ij}} \ ; \ x_{i,j} \vee \neg r_{p_{ij}} \tag{3}$$

$$\forall \langle w_i, f_j\rangle \in SP \setminus (M_0 \cup M_z) : \neg x_{i,j} \vee r_{p_{ij}} \ ; \ \neg x_{i,j} \vee \neg r_{e_{ij}} \ ; \ x_{i,j} \vee \neg r_{p_{ij}} \vee r_{e_{ij}} \tag{4}$$

$$\forall \langle w_i, f_j\rangle \in FP : x_{i,j} \tag{5}$$

$$\forall \langle w_i, f_j\rangle \in NSP : \neg x_{i,j} \tag{6}$$

We denote this formula by Γ. Clauses 1 make sure that the set of rotation variables that are set to true corresponds to a closed subset. Clauses 2, 3, and 4 correspond (respectively) to Lemmas 4, 5,and 6. Lastly, Clauses 5 and 6 handle the particular cases of fixed and non stable pairs (respectively). Observe that each clause is of size at most 3. Moreover, since the the number of edges in the graph poset is bounded by $O(L)$ [3], then the size of this formula is $O(L)$.

The only CP formulation for the case of many-to-many stable matching was proposed in [3]. It is a straightforward generalization of the CSP model proposed for the hospital/residents problem in [13]. The authors use q_{w_i} variables per worker, and q_{f_j} variables per firm. The variables related to a worker w_i represent the rank of the firm assigned at each position (out of the q_{w_i} available positions). A similar set of variables is used for firms. The model contains $|W| \times (\sum_i q_{w_i} + |F| \times (1 + \sum_j q_{f_j} \times (2 + \sum_i (q_{w_i} - 1))))$ constraints related to workers. Likewise, $|F| \times (\sum_j q_{f_j} + |W| \times (1 + \sum_i q_{w_i} \times (2 + \sum_j (q_{f_j} - 1))))$ constraints are used for firms.

4.2 Properties Related to Unit Propagation

In the following, we show that once unit propagation is performed without failure then there exists necessarily a solution.

Suppose that $\mathcal{D}$ is a domain where unit propagation has been performed without failure. Let S_1 be the set of rotation variables that are set to 1.

Lemma 7. *S_1 is a closed subset.*

Proof. Let ρ_i be a rotation in S_1 and let ρ_j be rotation such that $\rho_j \prec\prec \rho_i$. Unit propagation on Clauses 1 enforces r_j to be true. Therefore $\rho_j \in S_1$. Hence S_1 is a closed subset. □

Let M_1 be the stable matching corresponding to S_1 (Theorem 1). We show that M_1 is part of the solution space in $\mathcal{D}$.

Lemma 8. *For any $x_{i,j}$ that is set to 1, $\langle w_i, f_j\rangle \in M_1$.*

Proof. The case where $\langle w_i, f_j\rangle$ is a fixed pair or non stable is trivial. Take a non-fixed stable pair $\langle w_i, f_j\rangle$ and suppose that $\mathcal{D}(x_{i,j}) = \{1\}$. There are three cases to distinguish.

1. $\langle w_i, f_j\rangle \in SP \cap M_0$: Unit propagation on clauses 2 enforces $r_{e_{ij}}$ to be false. Therefore, $\rho_{e_{ij}} \notin S_1$. Hence by Lemma 4 we obtain: $\langle w_i, f_j\rangle \in M_1$.

2. $\langle w_i, f_j \rangle \in SP \cap M_z$: Unit propagation on clauses 3 enforces $r_{p_{ij}}$ to be true. Therefore, $\rho_{p_{ij}} \in S_1$. Hence by Lemma 5 we obtain: $\langle w_i, f_j \rangle \in M_1$.
3. $\langle w_i, f_j \rangle \in SP \setminus (M_0 \cup M_z)$: Unit propagation on clauses 4 enforces $r_{p_{ij}}$ to be true and $r_{e_{ij}}$ to be false. Therefore, $\rho_{p_{ij}} \in S_1$, $\rho_{e_{ij}} \notin S_1$. Hence by Lemma 6 we obtain: $\langle w_i, f_j \rangle \in M_1$. □

Lemma 9. *For any $x_{i,j}$ that is set to 0, $\langle w_i, f_j \rangle \notin M_1$.*

Proof. The case where $\langle w_i, f_j \rangle$ is a fixed pair or non-stable is trivial. Take a non-fixed stable pair $\langle w_i, f_j \rangle$ and suppose that $\mathcal{D}(x_{i,j}) = \{0\}$. There are three cases to distinguish.

1. $\langle w_i, f_j \rangle \in SP \cap M_0$: Unit propagation on Clauses 2 enforces $r_{e_{ij}}$ to be true. Therefore, $\rho_{e_{ij}} \in S_1$. Hence by Lemma 4 we obtain: $\langle w_i, f_j \rangle \notin M_1$.
2. $\langle w_i, f_j \rangle \in SP \cap M_z$: Unit propagation on Clauses 3 enforces $r_{p_{ij}}$ to be false. Therefore, $\rho_{p_{ij}} \notin S_1$. Hence by Lemma 5 we obtain: $\langle w_i, f_j \rangle \notin M_1$.
3. $\langle w_i, f_j \rangle \in SP \setminus (M_0 \cup M_z)$: We distinguish two cases:
   (a) $\mathcal{D}(\rho_{p_{ij}}) \neq \{1\}$: In this case $\rho_{p_{ij}} \notin S_1$ hence by Lemma 6 we obtain: $\langle w_i, f_j \rangle \notin M_1$
   (b) $\mathcal{D}(\rho_{p_{ij}}) = \{1\}$: In this case, unit propagation on Clauses 4 enforces $r_{e_{ij}}$ to be true. Therefore, $\rho_{e_{ij}} \in S_1$. Hence by Lemma 6 we obtain: $\langle w_i, f_j \rangle \notin M_1$. □

Recall that Γ denotes the SAT formula defined in Section 4.1.

Theorem 2. *Let $\mathcal{D}$ be a domain such that unit propagation is performed without failure on Γ. There exists at least a solution in $\mathcal{D}$ that satisfies Γ.*

Proof. We show that M_1 corresponds to a solution under $\mathcal{D}$. To do so, one needs to set every unassigned variable to a particular value. We propose the following assignment. Let $x_{i,j}$ be an unassigned variable. Note that $\langle w_i, f_i \rangle$ has to be part of SP.

1. If $\langle w_i, f_j \rangle \in SP \cap M_0$: $x_{i,j}$ is set to 1 if $\rho_{e_{ij}} \notin S_1$; and 0 otherwise.
2. If $\langle w_i, f_j \rangle \in SP \cap M_z$: $x_{i,j}$ is set to 1 if $\rho_{p_{ij}} \in S_1$; and 0 otherwise.
3. If $\langle w_i, f_j \rangle \in SP \setminus (M_0 \cup M_z)$: $x_{i,j}$ is set to 1 if $\rho_{p_{ij}} \in S_1 \wedge \rho_{e_{ij}} \notin S_1$; and 0 otherwise.

This assignment corresponds to a solution as a consequence of Lemmas 4, 5, 6, 8, and 9. Therefore, once unit propagation is established without failure, we know that there exists at least one solution. □

5 Arc Consistency

We propose in this section a procedure to filter more of the search space. We assume in the rest of this section that I is a stable matching instance defined by $\langle W, F, P_W, P_F, q_W, q_F \rangle$ using the same notations introduced in Section 3.

Let $\mathcal{X}(M2M) = \{x_{1,1}, \ldots x_{n_W,n_F}, r_1, \ldots r_{|R|}\}$ be the set of Boolean variables defined in Section 4.1. We define the many-to-many stable matching constraint as $M2M(I, \mathcal{X}(M2M))$. Given a complete assignment of the variables in $\mathcal{X}(M2M)$, this constraint is satisfied iff the set M of pairs corresponding to Boolean variables $x_{i,j}$ that are set to 1 is a solution to I and the set of rotations corresponding to Boolean variables r_k that are set to 1 is the closed subset corresponding to M.

Example 4 shows an instance with a particular domain where unit propagation on Γ is not enough to establish arc consistency on the $M2M$ constraint.

Example 4 (Missing Support). Consider the example where $n_W = 4, n_F = 4$, and for all $1 \leq i, j \leq 4$, $q_{w_i} = q_{f_j} = 1$. The preference lists for workers and firms are given in Table 2.

Table 2: Preference lists

$P_{w_1} = [f_3, f_2, f_4, f_1]$	$P_{f_1} = [w_1, w_2, w_4, w_3]$
$P_{w_2} = [f_2, f_4, f_1, f_3]$	$P_{f_2} = [w_3, w_1, w_2, w_4]$
$P_{w_3} = [f_4, f_1, f_3, f_2]$	$P_{f_3} = [w_2, w_3, w_4, w_1]$
$P_{w_4} = [f_1, f_2, f_3, f_4]$	$P_{f_4} = [w_4, w_1, w_2, w_3]$

Consider the domain such that all the variables are unassigned except for $x_{1,4}$, $x_{3,1}$, $x_{3,3}$, $x_{4,2}$, and $x_{4,3}$ where the value 0 is assigned to each of these variables. Unit propagation on the encoding Γ of this instance does not trigger a failure. It also does not change the domain of $x_{2,1}$ (i.e., $\{0, 1\}$). However, the assignment of 1 to $x_{2,1}$ does not have a support in $\mathcal{D}$ for $M2M$. $\square$

In the following, we assume that unit propagation is established on an input domain $\mathcal{D}$ and that it propagated the clauses without finding a failure. In the rest of this section, we use the term 'support' to say 'support for $M2M(I, \mathcal{X}(M2M))$'. We shall use unit propagation to find a support for any assignment using the property we showed in Theorem 2.

In order to construct supports, we need to introduce the following two lemmas.

Lemma 10. *For any rotation ρ_i where $\mathcal{D}(r_i) = \{0, 1\}$, assigning 1 to r_i has a support.*

Proof. Consider the set of rotations $S = S_1 \cup \{r_j \mid r_j \prec\prec r_i\}$. Clearly S is a closed subset (Lemma 7). Let M be the corresponding stable matching of S. We show that M corresponds to a valid support.

By construction, we have any variable $x_{i,j}$ set to 1 is part of M and any variable set to 0 is not. Consider now the rotation variables. Recall that S_1 is the set of rotation variables that are set to 1. Observe that $\{r_j | r_j \prec\prec r_i\}$ can only contain rotations that are unassigned because otherwise, unit propagation would assign 0 to r_i. In our support, every rotation variable whose rotation is in $\{r_j | r_j \prec\prec r_i\}$ is set to 1. Consider $x_{i,j}$ an unassigned variable. We set $x_{i,j}$ as follows

1. If $\langle w_i, f_j \rangle \in SP \cap M_0 : x_{i,j}$ is set to 1 if $\rho_{e_{ij}} \notin S$; and 0 otherwise.
2. If $\langle w_i, f_j \rangle \in SP \cap M_z : x_{i,j}$ is set to 1 if $\rho_{p_{ij}} \in S$; and 0 otherwise.
3. If $\langle w_i, f_j \rangle \in SP \setminus (M_0 \cup M_z) : x_{i,j}$ is set to 1 if $\rho_{p_{ij}} \in S \wedge \rho_{e_{ij}} \notin S$; and 0 otherwise.

This assignment corresponds by construction to M as a consequence of Lemmas 4, 5, 6, 8, and 9. □

Lemma 11. *For any rotation ρ_i where $\mathcal{D}(r_i) = \{0, 1\}$, assigning 0 to r_i has a support.*

Proof. Recall that S_1 is the the set of rotation variables that are set to 1 and that M_1 is its corresponding stable matching. By construction, we can show that M_1 corresponds to a support. □

Consider now an unassigned variable $x_{i,j}$. Notice that $\langle w_i, f_i \rangle \in SP$. Lemma 12 show that there is always a support for 0.

Lemma 12. *For any unassigned variable $x_{i,j}$, assigning 0 to $x_{i,j}$ has a support.*

Proof. We distinguish three cases:

1. $\langle w_i, f_j \rangle \in SP \cap M_0$: Observe that $\rho_{e_{ij}}$ is unassigned. We know by Lemma 10 that assigning 1 to $r_{e_{ij}}$ has a support. In this support 0 is assigned to $x_{i,j}$.
2. $\langle w_i, f_j \rangle \in SP \cap M_z$: In this case $\rho_{p_{ij}}$ is unassigned. We know by Lemma 11 that assigning 0 to $r_{p_{ij}}$ has a support. In this support, 0 is assigned to $x_{i,j}$.
3. $\langle w_i, f_j \rangle \in SP \setminus (M_0 \cup M_z)$: Note that 0 cannot be assigned to $\rho_{p_{ij}}$ because otherwise $x_{i,j}$ would be set to 0. We distinguish two cases:
   (a) $\rho_{p_{ij}}$ is set to 1: In this case $\rho_{e_{ij}}$ is unassigned (otherwise $x_{i,j}$ would be assigned). We know by Lemma 10 that assigning 1 to $r_{e_{ij}}$ has a support. In this support 0 is assigned to $x_{i,j}$.
   (b) $\rho_{p_{ij}}$ is unassigned: We know by Lemma 11 that assigning 0 to $r_{p_{ij}}$ has a support. In this support 0 is assigned to $x_{i,j}$. □

In the case of finding supports when assigning 1 to $x_{i,j}$, there are three cases. These cases are detailed in Lemmas 13, 14, and 15.

Lemma 13. *If $\langle w_i, f_j \rangle \in SP \cap M_0$, then assigning 1 to $x_{i,j}$ has a support.*

Proof. In this case $\rho_{e_{ij}}$ is unassigned. We know by Lemma 11 that assigning 0 to $r_{e_{ij}}$ has a support. In this support 1 is assigned to $x_{i,j}$. □

Lemma 14. *If $\langle w_i, f_j \rangle \in SP \cap M_z$, then assigning 1 to $x_{i,j}$ has a support.*

Proof. In this case $\rho_{p_{ij}}$ is unassigned. We know by Lemma 10 that assigning 1 to $r_{p_{ij}}$ has a support. In this support $x_{i,j}$ is set to 1. □

Algorithm 1: Arc Consistency for $M2M(I, \mathcal{X}(M2M))$

1 $\mathcal{D} \leftarrow \mathrm{UP}(\mathcal{D})$;
  if $\mathcal{D} \neq \emptyset$ **then**
2   **foreach** $\langle w_i, f_j \rangle \in SP \wedge \langle w_i, f_j \rangle \notin M_0 \cup M_z \wedge \mathcal{D}(r_{p_{ij}}) = \{0, 1\}$ **do**
3     $\mathcal{D}' \leftarrow \mathrm{UP}(\mathcal{D}^1_{x_{i,j}})$;
      if $\mathcal{D}' = \emptyset$ **then**
4       $\mathcal{D}(x_{i,j}) = \{0\}$;
  return $\mathcal{D}$

Let $\mathcal{D}^1_{x_{i,j}}$ be the domain identical to $\mathcal{D}$ except for $\mathcal{D}(x_{i,j}) = \{1\}$.

Lemma 15. *If $\langle w_i, f_j \rangle \in SP \setminus M_0 \cup M_z$, then*

- If $\mathcal{D}(r_{p_{ij}}) = \{1\}$*, then assigning* 1 *to* $x_{i,j}$ *has a support.*
- *Otherwise, we have* $\mathcal{D}(r_{p_{ij}}) = \{0, 1\}$ *and assigning* 1 *to* $x_{i,j}$ *has a support iff unit propagation on* $\mathcal{D}^1_{x_{i,j}}$ *does not fail.*

Proof. For the first case, we can argue that $\rho_{e_{ij}}$ is unassigned (otherwise $x_{i,j}$ would be assigned). By Lemma 11, we have a support if we set $r_{e_{ij}}$ to 0. In this support $x_{i,j}$ is set to 1.

For the second case, we have necessarily $\mathcal{D}(r_{p_{ij}}) = \{0, 1\}$ (otherwise $x_{i,j}$ would be assigned) and it is easy to see that there exists a support iff unit propagation does not fail on $\mathcal{D}^1_{x_{i,j}}$ by Theorem 2. □

We summarize all the properties of the previous lemmas in Algorithm 1. This algorithm shows a pseudo-code to maintain arc consistency on $M2M(I, \mathcal{X}(M2M))$. In this algorithm, $\mathrm{UP}(\mathcal{D})$ is the output domain after performing unit propagation on a domain $\mathcal{D}$. The output of $\mathrm{UP}(\mathcal{D})$ is $\emptyset$ iff a failure is found.

Suppose that $\mathcal{D}$ is a domain where unit propagation is established without failure. First, for any variable that is set to a value v, the assignment of v to this variable has a support in $\mathcal{D}$ since there exists necessarily a solution (Theorem 2). Second, we know that any assignment of any rotation variable has a support in $\mathcal{D}$ by Lemmas 10 and 11. Also, the assignment of 0 to any unassigned variable $x_{i,j}$ has a support (Lemma 12). Lastly, by Lemmas 13, 14, and 15, we know that we need to check supports only for the assignment of 1 to some particular unassigned variables $x_{i,j}$. These variables correspond to the pairs of the set $\Psi = \{\langle w_i, f_j \rangle | \langle w_i, f_j \rangle \in SP \wedge \langle w_i, f_j \rangle \notin M_0 \cup M_z \wedge \mathcal{D}(r_{p_{ij}}) = \{0, 1\}\}$ (Lemma 15).

Algorithm 1 first performs unit propagation on the input domain $\mathcal{D}$ in Line 1. If a failure is not found, we loop over the pairs in Ψ in Line 2 and call unit propagation on the new domain $\mathcal{D}^1_{x_{i,j}}$ in Line 3 for each $\langle w_i, f_j \rangle \in \Psi$. If this call results in failure then $x_{i,j}$ does not have a support for the value 1. In this case, such a variable is set to 0 in Line 4.

We discuss now the complexity of Algorithm 1. Observe first that since the SAT formula contains only clauses of size at most 3, and since the number of clauses is $O(L)$, then unit propagation takes $O(L)$ time. Notice that by using the two-watched literal procedure [16], there is no data structure to update between the different calls. Lastly, observe that the number of calls to unit propagation in Line 3 is bounded by the number of unassigned variables. Therefore the worst-case time complexity to maintain arc consistency is $O(U_x \times L)$ where U_x is the number of unassigned $x_{i,j}$ variables. Therefore the overall complexity is $O(L^2)$.

6 Experimental Results

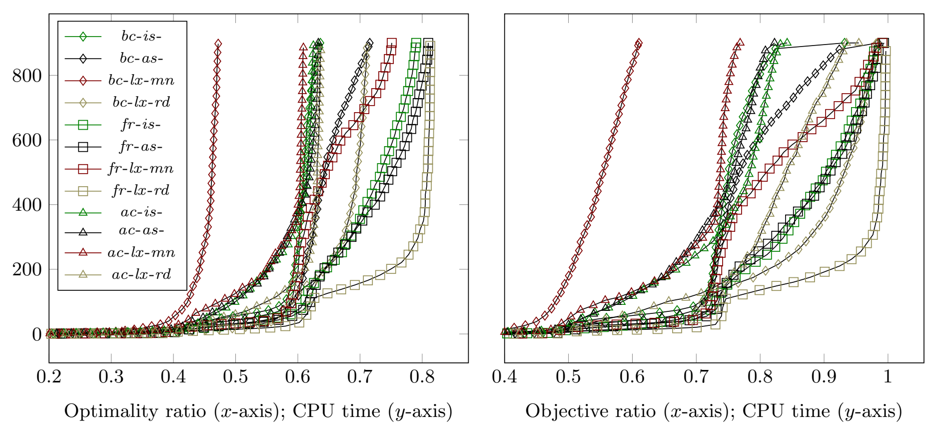
In the absence of known hard problems for many to many stable matching, we propose to evaluate our approach on two NP-hard variants of stable marriage called sex-equal stable matching and balanced stable matching [14]. Let M be a stable marriage. Let C^m_M (respectively C^w_M) be the sum of the ranks of each man's partner (respectively woman's partner). In balanced stable matching, the problem is to find a stable matching M with the minimum value of $max\{C^m_M, C^w_M\}$. In sex-equal stable matching, the problem is to find a stable matching M with the minimum value of $|C^m_M - C^w_M|$ [14]. Modeling these problems in constraint programming is straightforward by using an integer variable X_i for each man m_i whose domain represents the rank of the partner of m_i.

We implemented our two propositions in the Mistral-2.0 [10] solver (denoted by *fr* for the first formulation and *ac* for the arc consistency algorithm) and we compare them against the bound ($\mathcal{D}$) consistency algorithm of [20] implemented in the same solver (denoted by *bc*). We restrict the search strategy to branch on the sequence $[X_1, \ldots, X_n]$ since it is sufficient to decide the problem. We used four different heuristics: a lexicographic branching (*lx*) with random value selection (*rd*); *lx* with random min/max value selection (*mn*); activity based search (*as*) [15]; and impact-based search (*is*) [18]. We use geometric restarts and we run 5 randomization seeds. There is a time cutoff of 15 minutes for each model on each instance.

We first run all the configurations on purely random instances with complete preference lists of size up to 500×500 and observed that these instances are extremely easy to solve for all configurations without valuable outcome. We therefore propose to use a new benchmark of hard instances.

Irving and Leather [12] described a family of stable marriage instances, where the number of solutions for stable matching grows exponentially. In this family, the number of stable matchings $g(n)$ for an instance of size $n \times n$ respects the recursive formula $g(n) \geq 2 \times g(n/2)^2$, and $g(1) = 1$, where n, the number of men, is of the form 2^k. To give an idea of the exponential explosion, when $n = 16$, the number of solutions is 195472, and when $n = 32$, the number of solutions is 104310534400. We generate instances of sizes $n \in \{32, 64, 128, 256\}$ as follows. For each size, we generate the instance as in [12], then swap $\alpha\%$ of n random pairs from the preference lists of men. We apply the same swapping procedure for woman. We generated 50 instances for each size with $\alpha = 10$, $\alpha = 20$, and $\alpha = 30$. This gives us a total of 600 instances available in http://siala.github.io/sm/sm.zip.

In the following figures we represent every configuration by “A-B” where A$\in$ $\{fr, ac, bc\}$ is the constraint model for stability and B $\in$ $\{$*lx-rd*, *lx-mn*, *as*, *is*$\}$ is the search strategy. In Figures 1a and 2a we give the cactus plots of proving optimality for these instances on the two problems. That is, after a given CPU time in seconds (y-axis), we give the percentage of instances proved to optimality for each configuration on the x-axis. In Figures 1b and 2b we study the quality of solutions by plotting the normalized objective value of the best solution found by the configuration h (x-axis) after a given time in seconds (y-axis) [11]. Let $h(I)$ be the objective value of the best solution found using model h on instance I and $lb(I)$ (resp. $ub(I)$) the lowest (resp. highest) objective value found by any model on I. We use a normalized score in the interval $[0, 1]$: $score(h, I) = \frac{ub(I)-h(I)+1}{ub(I)-lb(I)+1}$. The value of $score(h, I)$ is equal to 1 if h has found the best solution for this instance among all models, decreases as $h(I)$ gets further from the optimal objective value, and is equal to 0 if and only if h did not find any solution for I. Note that for *fr* and *ac* the CPU time in all there figures includes the $O(L)$ preprocessing step that we mentioned at the beginning of Section 4.

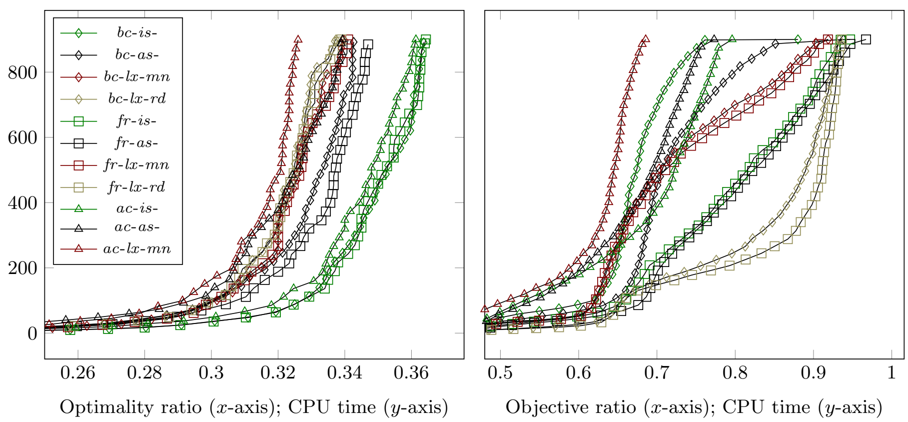


(a) Number of proofs

(b) Objective value

Fig. 1: Performance Cactus, Sex Equal Stable Matching

These figures show that the arc consistency model (*ac*) does not pay off as it considerably slows down the speed of exploration. It should be noted that between *bc* and *ac* there is no clear winner. The SAT formulation (*fr*), on the other hand, outperforms both *bc* and *ac* using any search strategy. This is true for both finding proofs of optimality and finding the best objective values. In fact, *fr* clearly finds better solutions faster than any other approach.



(a) Number of proofs (b) Objective value

Fig. 2: Performance Cactus, Balanced Stable Matching

Lastly, we note that the best search strategy for sex-equal stable matching is, surprisingly, the one branching lexicographically using a random value selection (Figures 1a and 1b)). For the case of balanced stable matching, clearly impact-based search is the best choice for finding proofs (Figure 2a) whereas activity based search finds better solutions (Figure 2b).

7 Conclusion

We addressed the general case of many-to-many stable matching in a constraint programming context. Using fundamental properties related to the notion of rotation in stable matching we presented a novel SAT formulation of the problem then showed that arc consistency can be maintained in quadratic time. Our experimental study on two hard variants of stable matching called sex-equal and balanced stable matching showed that our SAT formulation outperforms the best CP approach in the literature. In the future, it would be interesting to experimentally evaluate our propositions on hard variants in the many-to-many setting.

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