FREE-SURFACE, PURELY AZIMUTHAL EQUATORIAL FLOWS IN SPHERICAL COORDINATES WITH STRATIFICATION

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Abstract. In this paper we derive an exact solution to the governing equations for geophysical fluid dynamics in spherical coordinates which incorporates fluid stratification. This solution represents a steady, purely–azimuthal equatorial flow with an associated free-surface. Following the derivation of the solution we demonstrate that there is a well-defined relationship between the imposed pressure at the free-surface and the resulting distortion of the surface's shape. Finally, the solution for stratified fluid flow is subjected to a short-wavelength stability analysis.

1. Introduction

In this paper we derive an exact solution to the geophysical fluid dynamics (GFD) governing equations for an inviscid, incompressible and stratified fluid. The equations of motion are expressed in terms of a spherical coordinate system fixed at a point on the rotating earth, and are complemented by appropriate boundary conditions on the free-surface and at the rigid bottom. This solution represents a steady, purely–azimuthal equatorial flow with an associated free-surface, and a novel feature of the exact solution we derive is the incorporation of stratification in the fluid by way of modelling the density as a linear function of depth. Exact solutions in fluid mechanics are rare and, accordingly, they perform an extremely useful function which, nonetheless, necessitates some caution. The benefits bestowed by exact solutions are numerous and readily apparent: the existence of an exact solution offers confirmation of the validity of a given model, and furthermore exact solutions provide insight into the mathematical structure of a given problem. From a cautionary perspective, however, it must be remarked that exact solutions often correspond to idealised conditions which do not replicate many of the complexities of observed physical behaviour. Nevertheless, an exact solution represents a framework upon which more complicated and physically realistic solutions can be constructed.

Recently, there has been a flurry of research which has produced exact solutions to the GFD governing equations in various forms. Subsequent to the papers [3–5] a multitude of exact solutions, which are explicit in terms of Lagrangian variables, have been derived which model various oceanic waves [24, 37], and wave–current interactions [12, 19, 20], in the $f-$ and $\beta-$plane approximations, cf. the survey paper [21]. Concurrent, and complimentary, to this work, a variety of exact solutions

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to the GFD governing equations have been derived and generated by way of applying an assortment of classical applied mathematical methodologies and techniques [7–11, 25]. The ethos of these approaches is based upon the assumption that the observed movement of the oceans, and the associated oceanographic processes, are an intrinsic property of a fluid which can be captured by the Euler equations for an inviscid, incompressible and laminar fluid. Even in this regime the GFD equations of motion are nonlinear and highly intractable [13, 18, 41], with Coriolis forces incorporated in the Euler equation; a survey of these approaches, and resulting solutions, can be found in [33].

This paper represents an advancement of this body of work, particularly through the incorporation of stratified effects in the exact solution derived below. The dynamics of the ocean near the equator presents some unique and complex characteristics from a modelling perspective [8, 15, 31, 32, 34, 38], among these being pronounced stratification. Density variation (which is affected by temperature and salinity) is a major component influencing the vertical ocean movement, and substantial vertical fluid stratification is observed in flows that are typical in the equatorial zone of the Pacific (in a band of about 2° from the Equator), cf. [7, 10, 15, 34, 38]. Furthermore, equatorial flow dynamics are dominated by the presence of non-uniform underlying currents, with the most prominent one being the equatorial undercurrent (EUC). This is a depth-dependent current, confined to a depth of no more than 100-200 m, whereby the predominantly westward surface flow (due to the prevailing winds) reverses direction resulting in a (relatively) high-speed jet flowing eastwards whose core is aligned with the thermocline, cf. [7, 8, 31, 32, 39, 40].

The exact solution we derive is presented in Section 3 in terms of the fully nonlinear GFD governing equations in a rotating frame of spherical coordinates: at no stage do we approximate either by way of linearisation, or by passing to cylindrical or $\beta-$plane coordinates. This approach maintains the rich physical structure of the presented solution, while raising a number of significant mathematical complications which we address by way of analytical techniques. The exact solution that we derive allows for an appropriate surface distortion: a Bernoulli-type relation at the free-surface provides an implicit prescription of the relationship between the imposed pressure, and the resulting distortion, at the free-surface. In Section 4 a rigorous and detailed analysis of this intricate relation is performed, without resorting to approximation, through employing the implicit function theorem [14]. In the process, we demonstrate that there is a well-defined relationship between the imposed pressure at the free-surface and the resulting distortion of the surface's shape; furthermore it is established that the relationship exhibits the expected monotonicity properties. Finally, in Section 5, the exact solution for stratified fluid flow is subjected to a short-wavelength stability analysis. Although the mathematical implementation is significantly complicated both by the presence of fluid stratification, and our utilisation of general spherical coordinates, this approach yields physically interesting results.

2. Equations of motion

The governing equations for geophysical fluid dynamics are formulated in a spherical coordinate system which is fixed at a point on the earth's surface as follows. The spherical coordinates are denoted (r, θ, φ), where r is the distance from the centre of the earth, θ (with $0 \le \theta \le \pi$) is the polar angle, and φ (with $0 \le \varphi < 2\pi$) is the azimuthal angle. The location of the North and South poles are at $\theta = 0, \pi$, respectively, while the Equator is situated at $\theta = \frac{\pi}{2}$. The velocity field is denoted with (u, v, w) with respect to the orthonormal system $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi\}$ which corresponds to the (r, θ, φ) variables. Moreover, $\mathbf{e}_\varphi$ points from West to East, and $\mathbf{e}_\theta$ points from North to South. The GFD governing equations, in a spherical coordinate system with its origin at the centre of the sphere, are given by the Euler equation

$$\begin{aligned} u_t + uu_r + \tfrac{v}{r}u_\theta + \tfrac{w}{r\sin\theta}u_\varphi - \tfrac{1}{r}(v^2 + w^2) &= -\tfrac{1}{\rho}p_r + F_r \\ v_t + uv_r + \tfrac{v}{r}v_\theta + \tfrac{w}{r\sin\theta}v_\varphi + \tfrac{1}{r}(uv - w^2\cot\theta) &= -\tfrac{1}{\rho}\tfrac{1}{r}p_\theta + F_\theta \\ w_t + uw_r + \tfrac{v}{r}w_\theta + \tfrac{w}{r\sin\theta}w_\varphi + \tfrac{1}{r}(uw + vw\cot\theta) &= -\tfrac{1}{\rho}\tfrac{1}{r\sin\theta}p_\varphi + F_\varphi, \end{aligned} \tag{2.1}$$

the equation of mass conservation

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho u) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho v\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial(\rho w)}{\partial\varphi} = 0, \tag{2.2a}$$

and the incompressibility condition

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial w}{\partial\varphi} = 0. \tag{2.2b}$$

Here $p(r, \theta, \varphi)$ is denotes the pressure in the fluid, $\boldsymbol{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_\varphi\mathbf{e}_\varphi$ is the body-force vector, and ρ denotes the fluid density (which, critically, we do not assume to be constant). The system of equations (2.1) describes the problem in a coordinate system with its origin at the centre of the sphere, whereas ideally we want to incorporate the effects of the earth's rotation into our system. In order to do this we associate $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\varphi\}$ with a point fixed on the sphere which is rotating about its polar axis. Consequently, the left hand side of equations (2.1) must be modified by the additional Coriolis force $2\boldsymbol{\Omega} \times \mathbf{u}$ and centripetal acceleration $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$ terms, with $\mathbf{r} = r\mathbf{e}_r$, $\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_\varphi$, and

$$\boldsymbol{\Omega} = \Omega(\mathbf{e}_r\cos\theta - \mathbf{e}_\theta\sin\theta),$$

where $\Omega \approx 7.29 \times 10^{-5}$ rad s^{-1} is the constant rate of rotation of the earth. The contribution of the Coriolis and centripetal acceleration is therefore given by

$$2\Omega(-w\sin\theta\mathbf{e}_r - w\cos\theta\mathbf{e}_\theta + (u\sin\theta + v\cos\theta)\mathbf{e}_\varphi) - r\Omega^2(\sin^2\theta\mathbf{e}_r + \sin\theta\cos\theta\mathbf{e}_\theta)$$

with the resulting Euler equation taking the form

$$u_t + uu_r + \frac{v}{r}u_\theta + \frac{w}{r\sin\theta}u_\varphi - \frac{1}{r}(v^2+w^2) - 2\Omega w\sin\theta - r\Omega^2\sin^2\theta = -\frac{1}{\rho}p_r + F_r$$

$$v_t + uv_r + \frac{v}{r}v_\theta + \frac{w}{r\sin\theta}v_\varphi + \frac{1}{r}(uv - w^2\cot\theta) - 2\Omega w\cos\theta - r\Omega^2\sin\theta\cos\theta = -\frac{1}{\rho}\frac{1}{r}p_\theta + F_\theta \quad (2.2c)$$

$$w_t + uw_r + \frac{v}{r}w_\theta + \frac{w}{r\sin\theta}w_\varphi + \frac{1}{r}(uw + vw\cot\theta) + 2\Omega u\sin\theta + 2\Omega v\cos\theta = -\frac{1}{\rho}\frac{1}{r\sin\theta}p_\varphi + F_\varphi.$$

It is assumed that the external body-force is due to gravity alone, giving the body-force vector as $-g\mathbf{e}_r$. The governing equations are supplemented by associated boundary conditions as follows. We denote the free surface as $r = R + h(\theta,\varphi)$, where $R \approx 6378$ km is the radius of the earth and $h(\theta,\varphi)$ represents the deviation of the free-surface from a perfect sphere. At the free-surface we require the dynamic condition involving the surface pressure

$$p = P(\theta,\varphi), \quad (2.3a)$$

and the kinematic condition

$$u = \frac{v}{r}\frac{\partial h}{\partial\theta} + \frac{w}{r\sin\theta}\frac{\partial h}{\partial\varphi}, \quad (2.3b)$$

to hold. At the bottom of the ocean, which is an impermeable, solid boundary described by the equation $r = d(\theta,\varphi)$, the associated kinematic condition is

$$u = \frac{v}{r}\frac{\partial d}{\partial\theta} + \frac{w}{r\sin\theta}\frac{\partial d}{\partial\varphi}. \quad (2.3c)$$

We note that this boundary equation is trivially satisfied if, at a sufficient depth, there is no fluid motion.

3. Exact solution

In this section we construct the exact solution to the governing equations (2.2) which represents a steady, purely azimuthal flow in a stratified fluid. The stratification we model takes the form of a linear distribution, whereby the density function is given by $\rho(r) = b - ar$, for $a, b > 0$ constants such that $\rho > 0$. The incorporation of stratification in our fluid model has a number of ramifications, one of which is the particularly intricate form which results for the surface pressure boundary condition (2.3a). This is the Bernoulli condition at the free surface appropriate to our solution, and the relationship between the imposed pressure at the surface of the ocean, and the resulting distortion of that surface, prescribed by this condition will be subjected to a careful analysis in subsequent sections. This section is concerned with the derivation of an exact solution which is compatible with the GFD governing equations for a stratified fluid.

If we assume that the fluid motion is purely in the azimuthal direction, that is $u \equiv v \equiv 0$ everywhere and $w = w(r,\theta)$, equation (2.2c) reduces to

$$\begin{cases} -\frac{w^2}{r} - 2\Omega w \sin\theta - r\Omega^2 \sin^2\theta &= -\frac{1}{\rho} p_r - g, \\ -\frac{w^2}{r}\cot\theta - 2\Omega w\cos\theta - r\Omega^2 \sin\theta\cos\theta &= -\frac{1}{\rho r} p_\theta, \\ 0 &= -\frac{1}{\rho}\frac{1}{r\sin\theta} p_\varphi, \end{cases} \tag{3.1}$$

and furthermore the equation of mass conservation (2.2a) and the incompressibility condition (2.2b) are automatically satisfied. Moreover, from the last equation in (3.1) we see that p is a function that depends only on r and θ. Upon differentiation of the first equation in (3.1) with respect to θ and the second with respect to r we obtain that w satisfies the equation

$$2(r(\cos\theta)w_r - (\sin\theta)w_\theta) = -\frac{r\rho_r\cos\theta}{\rho}(w + \Omega r\sin\theta), \tag{3.2}$$

which, when particularised to the function $\rho(r) = b - ar$, becomes

$$2(r\cos\theta w_r - \sin\theta w_\theta) = \frac{ar\cos\theta}{b - ar}(w + \Omega r\sin\theta). \tag{3.3}$$

Applying the method of characteristics, where we consider the variables r and θ as functions $s \to r(s)$ and $s \to \theta(s)$, and setting $z(s) := w(r(s),\theta(s))$, leads to the equations

$$r'(s) = 2r(s)\cos\theta(s) \tag{3.4a}$$

$$\theta'(s) = -2\sin\theta(s) \tag{3.4b}$$

$$z'(s) = \frac{ar(s)\cos\theta(s)}{b - ar(s)}\big(z(s) + \Omega r(s)\sin\theta(s)\big). \tag{3.4c}$$

Note that from (3.4a) and (3.4b) we obtain

$$\frac{d}{ds}\big(r(s)\sin\theta(s)\big) = 0,$$

which renders equation (3.4c) in the form

$$\frac{z'(s)}{z(s) + \Omega r(s)\sin\theta(s)} = \frac{a}{2}\cdot\frac{r'(s)}{b - ar(s)}.$$

The integration of the latter equation yields that

$$w(r,\theta) = -\Omega r\sin\theta + \frac{F(r\sin\theta)}{\sqrt{b - ar}}, \tag{3.5}$$

for some function F.

Remark 3.1. The impact of stratification on the form of the azimuthal velocity (3.5) is obvious, when compared to the homogeneous case [8], since the second term is essentially divided by $\sqrt{\rho}$. We remark that there is also an appreciable difference between solutions which are derived in terms of either cylindrical or spherical coordinates, since it is shown in [23] that in the simplified setting of cylindrical geometry one must divide instead by a ρ term.

Inserting formula (3.5) in (3.1) gives

$$p_r = \frac{F^2(r\sin\theta)}{r} - g(b - ar),$$

and

$$p_\theta = F^2(r\sin\theta)\cot\theta.$$

Thus

$$p = A - gbr + \frac{ga}{2}r^2 + \int_{c\sin\theta}^{r\sin\theta} \frac{F^2(y)}{y}\,dy - \int_\theta^{\pi/2} F^2(c\sin\theta')\cot\theta'\,d\theta',$$

for some constants A and c. Noticing that the kinematic boundary condition (2.3b) on the free surface, written as $r = R + h(\theta)$, is automatically satisfied, as well as the kinematic condition (2.3c) on the bottom $r = d(\theta)$, and taking into account the pressure boundary condition (2.3a), which now reads

$$p = P(\theta) \quad \text{on} \quad r = R + h(\theta),$$

we obtain

$$\begin{aligned} P(\theta) = A - gb[R + h(\theta)] &+ \frac{ga}{2}[R + h(\theta)]^2 \\ &+ \int_{c\sin\theta}^{[R+h(\theta)]\sin\theta} \frac{F^2(y)}{y}dy - \int_\theta^{\pi/2} F^2(c\sin\theta')\cot\theta' d\theta'. \end{aligned} \tag{3.6}$$

This Bernoulli-type expression relates the imposed pressure at the surface of an ocean and the resulting distortion of that surface, and it takes a particularly convoluted form in spherical coordinates (which is complicated even further by the incorporation of stratification into our model).

3.1. **Modelling equatorial flows.** At this stage in the presentation of the solution it is worth commenting on the form of (3.5) derived above, particularly in the context of the equatorial flows for which it may provide an elementary model. Firstly, it is clear from (3.5) that the azimuthal flow velocity is determined by prescribing it at the equator: if the equatorial flow is given by $w(r, \pi/2) = W(r)$, then we simply choose $F(r) = \sqrt{b - ar}\,(W(r) + \Omega r)$, and the flow $w(r, \theta)$ in the neighbouring equatorial region is prescribed by (3.5). Moreover, between the flow at the Equator $\theta = \pi/2$ and the flow at latitude θ there is the relation

$$w(r,\theta) = w(r,\pi/2)\sin\theta + \frac{1}{\sqrt{b - ar}}\left(F(r\sin\theta) - F(r)\sin\theta\right). \tag{3.7}$$

This is a particularly useful property when using the exact solution to model salient features of equatorial flows, an example being the equatorial undercurrent (EUC). The EUC is a current which runs the extent of the Pacific equator whereby the surface flow is predominantly westward (due to the prevailing trade-winds) and a flow reversal occurs beneath the surface leading to an eastward-flowing jet which is confined to depths of $100 - 200$m, cf. [7, 8, 31, 32, 39, 40]. If we suppose that the westward surface velocity has the constant magnitude W_w, and the maximal eastward velocity at the core of the EUC jet has magnitude W_e, then setting

$$F(r) = \begin{cases} \Omega r\sqrt{b-ar} + \sqrt{b-ar}\left[W_e - (W_e+W_w)\left(\frac{r-R_0}{R-R_0}\right)^2\right], & R \geq r \geq \overline{R}, \\ 0, & r < \overline{R}, \end{cases} \tag{3.8}$$

leads to an elementary model for the EUC. Here $r = R$ represents the free-surface at the equator, $r = R_0$ is the depth at which the EUC attains its maximum speed (this typically coincides with the thermocline) and $\overline{R} = R_0 - (R - R_0)\sqrt{W_e/(W_e + W_w)}$ represents a depth beneath which there is essentially no fluid motion.

4. The Bernoulli relation

The aim of this section, having shown that (3.5) represents a solution of the GFD governing equations (2.2) which satisfies the boundary conditions (2.3), is to analyse the relationship between variations of the pressure at the free surface, $P(\theta)$, and variations of the shape of the free surface, $h(\theta)$, as determined by relation (3.6). This relationship is highly convoluted and intractable, with the primary complicating factors being quadratic factors which are due to stratified effects coupled with the integral terms in (3.5). These latter terms are due to the spherical geometry of our system, and can be removed by transforming to a simplified system expressed in terms of cylindrical coordinates, cf. [8, 23]. We present a theoretical analysis of relation (3.6), without resorting to approximation, by way of applying the implicit function theorem [14]. For a given pressure distribution $P(\theta)$, we establish the existence and uniqueness of the implicitly–defined function $h(\theta)$, which represents the distortion of the free surface. Moreover, we conclude by establishing some monotonicity properties of the functions describing the pressure and the distortion of the free surface.

In order to be able to compare physical quantities in a meaningful way we nondimensionalise equation (3.6) as follows. We set first $h \equiv 0$ in (3.6) and we arrive at the simpler scenario whereby the undisturbed surface follows the curvature of the earth away from the Equator. In this setting the pressure required to maintain this shape is given by

$$P_0(\theta) = A - gbR + \frac{gaR^2}{2} + \int_{c\sin\theta}^{R\sin\theta} \frac{F^2(y)}{y} dy - \int_{\theta}^{\pi/2} F^2(c\sin\theta')\cot\theta' d\theta'. \tag{4.1}$$

If the pressure at the Equator (prescribed by setting $\theta = \pi/2$) is the atmospheric pressure P_a then

$$P_a = A - gbR + \frac{gaR^2}{2} + \int_c^R \frac{F^2(y)}{y} dy. \tag{4.2}$$

The identity (3.6) is non-dimensionalised by dividing it by P_a, as defined by (4.2). Introducing the non-dimensional functions

$$\mathfrak{h}(\theta) := \frac{h(\theta)}{R},$$

$$\mathfrak{P}(\theta) := \frac{P(\theta)}{P_a},$$

we obtain the non-dimensionalised equation

$$\alpha - \beta[1+\mathfrak{h}(\theta)] + \gamma[1+\mathfrak{h}(\theta)]^2 + \frac{1}{P_a}\int_{c\sin\theta}^{[1+\mathfrak{h}(\theta)]R\sin\theta} \frac{F^2(y)}{y} dy - \frac{1}{P_a}\int_{\theta}^{\pi/2} F^2(c\sin\theta')\cot\theta' d\theta' - \mathfrak{P}(\theta) = 0, \tag{4.3}$$

where α, β, γ are constants defined by

$$\alpha = \frac{A}{P_a}, \quad \beta = \frac{gbR}{P_a}, \quad \gamma = \frac{gaR^2}{2P_a},$$

Remark 4.1. Equation (4.3) is the appropriate non-dimensional formulation of the Bernoulli relation (3.6) to be subjected to functional-analytic considerations. The polar angle θ is confined to an interval $[\frac{\pi}{2}, \frac{\pi}{2}+\varepsilon]$, for some $\varepsilon > 0$ motivated by geophysical considerations: the choice $\varepsilon = 0.016$ is germane to flows in the equatorial region, corresponding to a strip of 100 km width about the Equator [8].

The left-hand side of equation (4.3) may be represented as an operator

$$\mathcal{F}(\mathfrak{h}, \mathfrak{P}) = 0 \tag{4.4}$$

on the function space

$$\mathcal{F} : B \times C\left([\pi/2, \pi/2+\varepsilon]\right) \to C\left([\pi/2, \pi/2+\varepsilon]\right),$$

where B denotes the open ball of radius 10^{-2} from the Banach space

$$C\left(\left[\frac{\pi}{2}, \frac{\pi}{2}+\varepsilon\right]\right),$$

consisting of continuous functions $f : [\pi/2, \pi/2+\varepsilon] \to \mathbb{R}$, equipped with the supremum norm

$$||f|| = \sup_{t\in[\pi/2,\pi/2+\varepsilon]} \{|f(t)|\}.$$

In order to make equation (4.4) amenable to an application of the implicit function theorem, we must first determine elementary solutions of (4.4). Denoting by $\mathfrak{P}_0(\theta)$ the imposed pressure which is required to maintain an undisturbed free-surface of the ocean (corresponding to $\mathfrak{h} \equiv 0$ in (4.3)) we obtain

$$\begin{aligned}\mathfrak{P}_0(\theta) :=& \alpha - \beta + \gamma \\ &+ \frac{1}{P_a}\int_{c\sin\theta}^{R\sin\theta} \frac{F^2(y)}{y}dy - \frac{1}{P_a}\int_{\theta}^{\pi/2} F^2(a\sin\theta')\cot\theta' d\theta',\end{aligned} \tag{4.5}$$

and it follows that

$$\mathcal{F}(0, \mathfrak{P}_0) = 0.$$

The greatest difficulty in determining the derivative

$$D_{\mathfrak{h}}\mathcal{F}(0, \mathfrak{P}_0)(\mathfrak{h}) = \lim_{s\to 0} \frac{\mathcal{F}(s\mathfrak{h}, \mathfrak{P}_0) - \mathcal{F}(0, \mathfrak{P}_0)}{s}$$

involves the differentiation of the fourth term in (4.3); we observe from the mean value theorem for integrals that

$$\lim_{s\to 0} \frac{1}{sP_a}\int_{R\sin\theta}^{[1+s\mathfrak{h}(\theta)]R\sin\theta} \frac{F^2(y)}{y}dy = \lim_{s\to 0}\frac{1}{sP_a}(sR\mathfrak{h}(\theta)\sin\theta)\frac{F^2(x_{\theta,s})}{x_{\theta,s}},$$

where $x_{\theta,s}$ lies between $R\sin\theta$ and $(1+s\mathfrak{h}(\theta))R\sin\theta)$. Thus, the latter limit equals

$$\frac{F^2(R\sin\theta)}{P_a}\mathfrak{h}(\theta),$$

and, as a consequence, we infer that

$$\begin{aligned}D_{\mathfrak{h}}\mathcal{F}(0, \mathfrak{P}_0)(\mathfrak{h}) &= \left(-\beta + 2\gamma + \frac{F^2(R\sin\theta)}{P_a}\right)\mathfrak{h} \\ &= \frac{b-aR}{P_a}[-gR + (w(R\sin\theta) + \Omega R\sin\theta)^2](\mathfrak{h}).\end{aligned} \tag{4.6}$$

It can be easily seen from physical considerations that there is a constant $k < 0$ such that

$$-gR + (w(R\sin\theta) + \Omega R\sin\theta)^2 \le k < 0,$$

hence the derivative

$$D_{\mathfrak{h}}\mathcal{F}(0,\mathfrak{P}_0) : C\left([\pi/2, \pi/2+\varepsilon]\right) \to C\left([\pi/2, \pi/2+\varepsilon]\right)$$

is a linear homeomorphism. Therefore, by the implicit function theorem, for any sufficiently small perturbation $\mathfrak{P}$ of $\mathfrak{P}_0$ there exists a unique $\mathfrak{h} \in C\left([\pi/2, \pi/2+\varepsilon]\right)$ such that (4.3) holds true.

We conclude this section by establishing monotonicity properties of the functions $\mathfrak{P}$ and $\mathfrak{h}$ with respect to each other, noting first that the smoothness properties of $\mathfrak{P}$ can be transferred, in order to confer smoothness upon $\mathfrak{h}$, by way of an iterative bootstrapping procedure, cf [2]. Differentiating (4.3) with respect to θ we get

$$\begin{aligned}\mathfrak{P}' = \mathfrak{h}'\Big[-\beta{+}2\gamma(1+\mathfrak{h}) + \frac{1}{P_a}\cdot\frac{F^2\big((1+\mathfrak{h})R\sin\theta\big)}{1+\mathfrak{h}}\Big]& \\ +\frac{\cot\theta}{P_a}F^2\big((1+\mathfrak{h})R\sin\theta\big)& \\ = \frac{\rho(R+h)}{P_a}\Big[-gR{+}\frac{(w(R+h,\theta)+\Omega(R+h)\sin\theta)^2}{1+\mathfrak{h}}\Big]\mathfrak{h}'& \\ +\frac{\cot\theta}{P_a}F^2\big((1+\mathfrak{h})R\sin\theta\big).& \qquad (4.7)\end{aligned}$$

Given the relation

$$-gR + \frac{(w(R+h,\theta)+\Omega(R+h)\sin\theta)^2}{1+\mathfrak{h}} < 0$$

we conclude from (4.7) that

$$\mathfrak{P}'(\theta) < 0 \quad \text{if} \quad \mathfrak{h}'(\theta) \geq 0 \quad \text{for some} \quad \theta \in (\pi/2, \pi/2+\varepsilon), \tag{4.8}$$

and

$$\mathfrak{h}'(\theta) < 0 \quad \text{if} \quad \mathfrak{P}'(\theta) \geq 0 \quad \text{for some} \quad \theta \in (\pi/2, \pi/2+\varepsilon). \tag{4.9}$$

Therefore, although the Bernoulli relation (3.6) for the exact solution (3.5) is highly convoluted, with the primary complicating factors being related to the use of spherical geometry coordinates and the incorporation of stratification into the fluid model, we have proven that this formula uniquely prescribes a relationship between variations in the imposed surface pressure and the resulting distortion of the ocean's free-surface. Furthermore, this relationship concurs with physical expectations in the sense that the monotonicity properties that we have established in (4.8) and (4.9) hold. These relations have been previously established for exact flow solutions in [8], however this was in the simpler context of a cylindrical coordinate geometry and a homogeneous fluid. The extension to stratified fluid (yet still in the simplified cyclindrical coordinate setting) has recently been achieved in [23].

5. A local stability result

In this section we perform a stability analysis of the exact azimuthal flow solution (3.5) we have derived using the short-wavelength perturbations method for general three-dimensional flows. The short-wavelength method is a particularly elegant analytical approach which was developed by Bayly [1], Friedlander and Vishik [16] and Lifschitz and Hameiri [35]. More precisely, this method investigates the time growth of the amplitude of perturbations to basic flows having a velocity field $\mathbf{u}$ which obeys the Euler equations (2.2c) and the equation of mass conservation (2.2a). The basic flow $\mathbf{u}$ is called stable with respect to the short-wavelength perturbation (5.4)-(5.5) if, for any initial data, the amplitude $\mathbf{A}$ (of the perturbation) is uniformly bounded in time. In the specific setting of GFD, the short-wavelength stability method has proven to be a particularly suitable and powerful tool in the analysis of a variety of recently-derived exact solutions [6,17,22,26,27,29,30]; cf. the survey [28]. The presence of vertical fluid stratification in the solution (3.5) generates some technical complications with regard to the implementation of the short-wavelength perturbation method, the resolution of which is deferred to the Appendix below.

We consider a perturbation $\mathbf{U} = U\mathbf{e}_r + V\mathbf{e}_\theta + W\mathbf{e}_\varphi$ of the azimuthal flow, from Section 3, with velocity $\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_\varphi$. That is, we seek U, V, W and a pressure function P such that $\mathbf{U} + \mathbf{u}$, $P + p$ satisfy (2.2c), (2.2a) and (2.2b). We then obtain that $\mathbf{U}$ and P satisfy the system

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{U} + 2(\boldsymbol{\Omega} \times \mathbf{U}) = -\frac{\nabla P}{\rho}, \tag{5.1}$$

$$\nabla \cdot (\rho \mathbf{U}) = 0, \tag{5.2}$$

and

$$\nabla \cdot (\mathbf{U}) = 0. \tag{5.3}$$

Moreover, the initial disturbance $\mathbf{U}\big|_{t=0} = \mathbf{U}_0$ is of the form

$$\mathbf{U}_0 := \mathbf{U}(0, r, \theta, \varphi) = \mathbf{A}(0, r, \theta, \varphi) e^{\frac{i}{\epsilon} f(0, r, \theta, \varphi)} =: \mathbf{A}_0(r, \theta, \varphi) e^{\frac{i}{\epsilon} f_0(r, \theta, \varphi)},$$

that is

$$A_0(r, \theta, \varphi) = A(0, r, \theta, \varphi), \quad f_0(r, \theta, \varphi) = f(0, r, \theta, \varphi).$$

Here $\mathbf{A} = A_1\mathbf{e}_r + A_2\mathbf{e}_\theta + A_3\mathbf{e}_\varphi$, f is a scalar function and ϵ plays the role of a small parameter. We further employ the (WKB) ansatz, that is, we seek solutions $\mathbf{U}$ and P of (5.1) and (5.2) having the specific form

$$\mathbf{U}(t, r, \theta, \varphi) \;=\; \mathbf{A}(t, r, \theta, \varphi) e^{\frac{i}{\epsilon} f(t, r, \theta, \varphi)} \;+\; \mathcal{O}(\epsilon) \tag{5.4}$$

$$P(t, r, \theta, \varphi) \;=\; \epsilon B(t, r, \theta, \varphi) e^{\frac{i}{\epsilon} f(t, r, \theta, \varphi)} \;+\; \mathcal{O}(\epsilon^2), \tag{5.5}$$

As a consequence of the equation (5.3) we obtain that $\mathbf{A}$ satisfies the equations

$$\nabla \cdot \mathbf{A} := \frac{1}{r^2}\frac{\partial (r^2 A_1)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial (A_2 \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial A_3}{\partial \varphi} = 0,$$

and

$$\mathbf{A} \cdot \nabla f := A_1 \frac{\partial f}{\partial r} + \frac{A_2}{r}\frac{\partial f}{\partial \theta} + \frac{A_3}{r \sin\theta}\frac{\partial f}{\partial \varphi} = 0. \tag{5.6}$$

Inserting the WKB ansatz in the conservation of momentum equations (2.2c) and, since we are interested in small perturbations $\mathbf{U}$ of $\mathbf{u}$, we obtain (after identification of the coefficients) the following:

$$\begin{aligned} A_1\left(f_t + uf_r + \frac{v}{r}f_\theta + \frac{w}{r\sin\theta}f_\varphi\right) &= 0\\ A_2\left(f_t + uf_r + \frac{v}{r}f_\theta + \frac{w}{r\sin\theta}f_\varphi\right) &= 0\\ A_3\left(f_t + uf_r + \frac{v}{r}f_\theta + \frac{w}{r\sin\theta}f_\varphi\right) &= 0 \end{aligned}$$

and

$$\begin{aligned} A_{1,t} + A_1u_r + \frac{A_2}{r}u_\theta + \frac{A_3}{r\sin\theta}u_\varphi - \frac{A_2v}{r} - \frac{A_3w}{r}\\ +uA_{1,r} + \frac{v}{r}A_{1,\theta} + \frac{w}{r\sin\theta}A_{1,\varphi} - \frac{vA_2}{r} - \frac{wA_3}{r} - 2\Omega A_3\sin\theta = -i\frac{B}{\rho}f_r \end{aligned}$$

$$\begin{aligned} A_{2,t} + A_1v_r + \frac{A_2}{r}v_\theta + \frac{A_3}{r\sin\theta}v_\varphi + \frac{A_2u}{r} - \frac{A_3w}{r}\cot\theta\\ +uA_{2,r} + \frac{v}{r}A_{2,\theta} + \frac{w}{r\sin\theta}A_{2,\varphi} + \frac{vA_1}{r} - \frac{wA_3}{r}\cot\theta - 2\Omega A_3\cos\theta = -i\frac{B}{\rho r}f_\theta \end{aligned}$$

$$\begin{aligned} A_{3,t} + A_1w_r + \frac{A_2}{r}w_\theta + \frac{A_3}{r\sin\theta}w_\varphi + \frac{A_3u}{r} + \frac{A_3v}{r}\cot\theta\\ +uA_{3,r} + \frac{v}{r}A_{3,\theta} + \frac{w}{r\sin\theta}A_{3,\varphi} + \frac{wA_1}{r} + \frac{wA_2}{r}\cot\theta + 2\Omega A_1\sin\theta + 2\Omega A_2\cos\theta\\ = -i\frac{B}{\rho r\sin\theta}f_\varphi \end{aligned}$$

Since the vector A is non-zero we see from above that the phase f satisfies the eikonal equation

$$f_t \;+\; \mathbf{u}\cdot\nabla f \;=\; 0, \tag{5.7}$$

while the amplitude $\mathbf{A}$ obeys

$$\mathbf{A}_t \;+\; (\mathbf{u}\cdot\nabla)\mathbf{A} \;+\; (\mathbf{A}\cdot\nabla)\mathbf{u} \;+\mathfrak{M}(\Omega,\theta)\mathbf{A} = \;-\,i\frac{B}{\rho}\nabla f, \tag{5.8}$$

where

$$\mathfrak{M}(\Omega,\theta) \;:=\; \begin{pmatrix} 0 & 0 & -2\Omega\sin\theta\\ 0 & 0 & -2\Omega\cos\theta\\ 2\Omega\sin\theta & 2\Omega\cos\theta & 0 \end{pmatrix},$$

with the notation

$$\mathfrak{M}(\Omega,\theta)\mathbf{A} := \left(\mathfrak{M}(\Omega,\theta)\begin{pmatrix}A_1\\A_2\\A_3\end{pmatrix}\right)\cdot\begin{pmatrix}\mathbf{e}_r\\\mathbf{e}_\theta\\\mathbf{e}_\varphi\end{pmatrix}$$

and

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_\varphi \end{pmatrix} := a\mathbf{e}_r + b\mathbf{e}_\theta + c\mathbf{e}_\varphi,$$

for any constants $a, b, c \in \mathbb{R}$.

We will determine in the sequel more information about the phase f. To this end we note that the velocity of the basic flow satisfies

$$\begin{aligned} \mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_\varphi = \frac{d}{dt}(r(t)\mathbf{e}_r) = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r \\ = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + r\dot{\varphi}(\sin\theta)\mathbf{e}_\varphi, \end{aligned} \tag{5.9}$$

where

$$t \to r(t), \quad t \to \theta(t), \quad t \to \varphi(t)$$

represents a streamline of the basic flow. Thus, the equations of a streamline for the azimuthal flow (with $u = v = 0, w = w(r, \theta)$) found in Section 3 and passing through $(r_0, \theta_0, \varphi_0)$ are

$$\frac{dr}{dt} = 0, \quad \frac{d\theta}{dt} = 0, \quad \frac{d\varphi}{dt} = \frac{w}{r\sin\theta}, \tag{5.10}$$

$$r(0) = r_0, \quad \theta(0) = \theta_0, \quad \varphi(0) = \varphi_0, \tag{5.11}$$

with

$$w(r, \theta) = -\Omega r\sin\theta + \frac{F(r\sin\theta)}{\sqrt{b - ar}}.$$

Clearly, the system (5.10)-(5.11) has the general solution

$$r(t) \equiv r_0, \quad \theta(t) \equiv \theta_0, \quad \varphi(t) = \int_0^t \frac{w(r(s), \theta(s))}{r(s)\sin\theta(s)}\, ds + \varphi_0. \tag{5.12}$$

The equation (5.7) for f becomes

$$f_t + \frac{w(r, \theta)}{r\sin\theta} f_\varphi = 0,$$

whose general solution is

$$f = \mathcal{F}\left(\varphi - \int_0^t \frac{w(r(s), \theta(s))}{r(s)\sin\theta(s)}\, ds\right),$$

for some function $\mathcal{F}$. Therefore, along the streamlines (5.12), we have that f is constant. Consequently, $\nabla f \equiv 0$. Thus, we obtain from (5.8), that the evolution of $\mathbf{A}$ is subject to

$$\mathbf{A}_t + (\mathbf{u} \cdot \nabla)\mathbf{A} = -(\mathbf{A} \cdot \nabla)\mathbf{u} - \mathfrak{M}(\theta, \Omega)\mathbf{A},$$

which, along the streamlines of the flow $u = v = 0, w = w(r,\theta)$, assumes the shape

$$\begin{aligned}
\left[A_{1,t} + \tfrac{wA_{1,\varphi}}{r\sin\theta} - \tfrac{wA_3}{r}\right]\Big|_{r=r_0,\theta=\theta_0} &= \left[\tfrac{A_3 w}{r} + 2\Omega(\sin\theta)A_3\right]\Big|_{r=r_0,\theta=\theta_0} \\
\left[A_{2,t} + \tfrac{wA_{2,\varphi}}{r\sin\theta} - \tfrac{wA_3}{r}\cot\theta\right]\Big|_{r=r_0,\theta=\theta_0} &= \left[\tfrac{A_3 w}{r}\cot\theta + 2\Omega(\cos\theta)A_3\right]\Big|_{r=r_0,\theta=\theta_0} \\
\left[A_{3,t} + \tfrac{wA_{3,\varphi}}{r\sin\theta} + \tfrac{wA_1}{r} + \tfrac{wA_2}{r}\cot\theta\right]\Big|_{\substack{r=r_0\\ \theta=\theta_0}} &= \left[-A_1 w_r - \tfrac{A_2}{r}w_\theta - 2\Omega(\sin\theta)A_1 - 2\Omega(\cos\theta)A_2\right]\Big|_{\substack{r=r_0\\ \theta=\theta_0}}
\end{aligned} \tag{5.13}$$

We seek to write the previous system as one containing the derivatives of A_1, A_2 and A_3 along the trajectories (5.12) of the basic flow $\mathbf{u}$. To this end we note that

$$\begin{aligned}
\frac{d}{dt}(A_1(t, r(t), \theta(t), \varphi(t)) &= A_{1,t} + A_{1,r}\cdot r'(t) + A_{1,\theta}\cdot\theta'(t) + A_{1,\varphi}\cdot\varphi'(t) \\
&= A_{1,t} + \frac{wA_{1,\varphi}}{r\sin\theta},
\end{aligned} \tag{5.14}$$

by means of (5.9) and (5.10). Similarly,

$$\frac{d}{dt}(A_2(t, r(t), \theta(t), \varphi(t)) = A_{2,t} + \frac{wA_{2,\varphi}}{r\sin\theta},$$

and

$$\frac{d}{dt}(A_3(t, r(t), \theta(t), \varphi(t)) = A_{3,t} + \frac{wA_{3,\varphi}}{r\sin\theta}.$$

Therefore, the system (5.13) can be rewritten as

$$\begin{cases}
\frac{d}{dt}\left(A_1(t, r(t), \theta(t), \varphi(t))\right) = \left(2\frac{w(r_0,\theta_0)}{r_0} + 2\Omega\sin\theta_0\right)A_3 \\
\frac{d}{dt}\left(A_2(t, r(t), \theta(t), \varphi(t))\right) = \left(2\frac{w(r_0,\theta_0)}{r_0} + 2\Omega\sin\theta_0\right)(\cot\theta_0)A_3 \\
\frac{d}{dt}\left(A_3(t, r(t), \theta(t), \varphi(t))\right) = \alpha_1 A_1 + \beta_1 A_2
\end{cases} \tag{5.15}$$

where

$$\alpha_1 = -\left(w_r(r_0,\theta_0) + \frac{w(r_0,\theta_0)}{r_0} + 2\Omega\sin\theta_0\right),$$

and

$$\beta_1 = -\left(\frac{w_\theta(r_0,\theta_0)}{r_0} + \frac{w(r_0,\theta_0)\cot\theta_0}{r_0} + 2\Omega\cos\theta_0\right).$$

Since

$$\frac{d}{dt}\left(A_1(t, r(t), \theta(t), \varphi(t))\right) = (\tan\theta_0)\frac{d}{dt}\left(A_2(t, r(t), \theta(t), \varphi(t))\right),$$

the study of (5.15) reduces to the study of the system

$$\begin{pmatrix} \frac{d}{dt}\left(A_2(t,r(t),\theta(t),\varphi(t))\right) \\ \\ \frac{d}{dt}\left(A_3(t,r(t),\theta(t),\varphi(t))\right) \end{pmatrix} = \mathfrak{A} \begin{pmatrix} A_2(t,r(t),\theta(t),\varphi(t)) \\ \\ A_3(t,r(t),\theta(t),\varphi(t)) \end{pmatrix} + \begin{pmatrix} 0 \\ \\ \gamma \end{pmatrix},$$

where

$$\mathfrak{A} = \left(\begin{array}{c|c} 0 & \cot\theta_0 \left(2\frac{w(r_0,\theta_0)}{r_0} + 2\Omega\sin\theta_0 \right) \\ \hline \alpha_1\tan\theta_0 + \beta_1 & 0 \end{array} \right), \tag{5.16}$$

and γ is a constant depending on the initial data. The time growth of the amplitude $\mathbf{A}$ is contingent on the signs of the eigenvalues of matrix $\mathfrak{A}$ which, as the following result shows, can be explicitly ascertained for certain choices of the azimuthal velocity w.

Theorem 5.1. *The flow with $u = v = 0$ and w given by (3.5) with $F(r\sin\theta) = cr\sin\theta$ for some constant $c \in \mathbb{R}$, $c \neq 0$ is stable under short-wavelength perturbations.*

Proof. Taking into account formula (3.5) for w we see that the eigenvalues of matrix $\mathfrak{A}$ from (5.16) are solutions to the equation

$$\lambda^2 + \frac{2c^2\sin^2\theta_0}{b - ar_0}\left(2 + \frac{b}{b - ar_0}\right) + \frac{4c^2\cos^2\theta_0}{b - ar_0} = 0. \tag{5.17}$$

Since equation (5.17) does not have real solutions the amplitude $\mathbf{A}(t)$ remains bounded in time. $\square$

6. Appendix

In this Appendix we address some technical issues concerning the local stability conditions which apply for a stratified flow of non-constant (continuous) density and exhibiting Coriolis effects. The main achievement of this section is a derivation of an energy estimate for the remainder term of a perturbation in the corresponding WKB expansion of the solution of the linearised problem. More precisely, we will show that the remainder terms in the perturbation remain uniformly bounded (in the sense of the L^2 norm) with respect to the parameter ϵ introduced in (6.5) below and on any fixed time interval.

Let us first remark that we can extend the flow to the entire space and consider perturbations defined throughout the entire space: in keeping with the approach employed in [16, 35], which concerns low-regularity flow, a C^1 Whitney–type extension suffices. Let $\mathbf{u}$, p denote the velocity field and the pressure, respectively, associated with the basic flow. Likewise, we denote with $\mathbf{U}$, P the perturbations (along the streamlines of the basic flow) of the velocity field and of the pressure, respectively. Then, the equations for the basic flow are

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho}\nabla p + 2\boldsymbol{\Omega} \times \mathbf{u} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = 0, \tag{6.1}$$

$$\operatorname{div}(\rho\mathbf{u}) = 0, \tag{6.2}$$

$$\operatorname{div}(\mathbf{u}) = 0 \tag{6.3}$$

where ρ is the (non constant) density, $\boldsymbol{\Omega}$ is the rotation vector and $\boldsymbol{r}$ represents the position vector of some point in the flow.

Asking $\mathbf{u}+\mathbf{U}, p+P$ to satisfy (6.1) and (6.2) and retaining only linear terms, we see that the equations satisfied by the perturbation $\mathbf{U}$, P are

$$\frac{D\mathbf{U}}{Dt} + \frac{\partial \mathbf{u}}{\partial x}\mathbf{U} + \frac{1}{\rho}\nabla P + 2\boldsymbol{\Omega} \times \mathbf{U} = 0, \tag{6.4a}$$

(with $\frac{\partial \mathbf{u}}{\partial x}$ being the Jacobian of $\mathbf{u}$),

$$\operatorname{div}(\rho\mathbf{U}) = 0, \tag{6.4b}$$

$$\operatorname{div}(\mathbf{U}) = 0, \tag{6.4c}$$

with initial conditions

$$\mathbf{U}(0, x) = \mathbf{U}_0(x). \tag{6.4d}$$

We use the WKB Ansatz

$$\mathbf{U} = [\mathbf{U}^0 + \epsilon\mathbf{U}^1] \exp\left(\frac{if}{\epsilon}\right) + \epsilon\mathbf{W}(\epsilon), \quad P = [P^0 + \epsilon P^1] \exp\left(\frac{if}{\epsilon}\right) + \epsilon Q(\epsilon) \tag{6.5}$$

aiming at showing that the remainders $\mathbf{W}$ and Q are uniformly bounded in ϵ (with respect to the L^2 norm) on each fixed time interval $[0, T]$.

We insert the WKB Ansatz in the equations (6.4) satisfied by the perturbation.

The coefficient of ϵ^{-1} in $\operatorname{div}(\mathbf{U})$ is $i(\mathbf{U}^0 \cdot \nabla f) \exp\left(\frac{if}{\epsilon}\right)$, thus the equation

$$\operatorname{div}(\mathbf{U}) = 0$$

gives that

$$\mathbf{U}^0 \cdot \nabla f = 0. \tag{6.6}$$

Identifying the coefficient of ϵ^{-1} from equation (6.4a) with zero we obtain first

$$\mathbf{U}^0(f_t + \nabla f \cdot \mathbf{u}) + \frac{P^0 \nabla f}{\rho} = 0. \tag{6.7a}$$

Multiplying the above equation with $\mathbf{U}^0 \neq 0$ and taking into account (6.6) we obtain that $P^0 = 0$, and, as a consequence, we see that the phase f satisfies the eikonal equation

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f = 0, \tag{6.7b}$$

with $f(0,x) = f_0(x)$. We also note that the eikonal equation (6.7b) implies

$$\frac{D(\nabla f)}{Dt} + \left(\frac{\partial \mathbf{u}}{\partial x}\right)^T \nabla f = 0. \tag{6.7c}$$

Identifying now the coefficients of ϵ^0 and ϵ^1 from (6.4a) with zero and using (6.7b) we obtain

$$\begin{aligned} i\frac{1}{\rho}P^1\nabla f &= -\left(\frac{D\mathbf{U}^0}{Dt} + \frac{\partial \mathbf{u}}{\partial x}\mathbf{U}^0 + \frac{1}{\rho}\nabla P^0 + 2\boldsymbol{\Omega}\times\mathbf{U}^0\right), \\ &= -\left(\frac{D\mathbf{U}^0}{Dt} + \frac{\partial \mathbf{u}}{\partial x}\mathbf{U}^0 + 2\boldsymbol{\Omega}\times\mathbf{U}^0\right), \end{aligned} \tag{6.7d}$$

$$\begin{aligned} &\frac{D\mathbf{W}}{Dt} + \frac{\partial \mathbf{u}}{\partial x}\mathbf{W} + \frac{\nabla Q}{\rho} + 2\boldsymbol{\Omega}\times w = \\ &\qquad -\exp\left(\frac{if}{\epsilon}\right)\left(\frac{D\mathbf{U}^1}{Dt} + \frac{\partial \mathbf{u}}{\partial x}\mathbf{U}^1 + \frac{\nabla P^1}{\rho} + 2\boldsymbol{\Omega}\times\mathbf{U}^1\right), \end{aligned} \tag{6.7e}$$

while from (6.4b) the following equations emerge

$$\nabla f\cdot(\rho\mathbf{U}^0) = 0 \tag{6.8a}$$

$$i\nabla f\cdot(\rho\mathbf{U}^1) = -\mathrm{div}\,(\rho\mathbf{U}^0) \tag{6.8b}$$

$$\mathrm{div}\,(\rho\mathbf{W}) = -\exp\left(\frac{if}{\epsilon}\right)\mathrm{div}\,(\rho\mathbf{U}^1) \tag{6.8c}$$

We will denote for simplicity $a := \rho\mathbf{U}^0$. Subsequently we now show that $\mathbf{U}^1$ and P^1 can be obtained in terms of a. To this end, we note first that (6.8b) gives,

$$\rho\mathbf{U}^1 = i\frac{\mathrm{div}\,a}{|\nabla f|^2}\nabla f. \tag{6.9}$$

Considering the material derivative in (6.8a) and using (6.7c) deliver

$$\frac{Da}{Dt}\cdot\nabla f = \frac{\partial \mathbf{u}}{\partial x}a\cdot\nabla f. \tag{6.10}$$

Using also that

$$\frac{D(\rho\mathbf{U}^0)}{Dt} = \rho\frac{D\mathbf{U}^0}{Dt} + (\mathbf{u}\cdot\nabla\rho)\mathbf{U}^0 \tag{6.11}$$

we see from (6.7d) that

$$iP^1\nabla f = -\frac{D(\rho\mathbf{U}^0)}{Dt} + (\mathbf{u}\cdot\nabla\rho)\mathbf{U}^0 - \frac{\partial \mathbf{u}}{\partial x}(\rho\mathbf{U}^0) - 2\boldsymbol{\Omega}\times(\rho\mathbf{U}^0),$$

which implies, via (6.10), that

$$P^1 = 2i\frac{\partial \mathbf{u}}{\partial x}a\cdot\frac{\nabla f}{|\nabla f|^2} - i\left(\frac{\mathbf{u}}{\rho}\cdot\nabla\rho\right)a\cdot\frac{\nabla f}{|\nabla f|^2} + 2i(\boldsymbol{\Omega}\times a)\cdot\frac{\nabla f}{|\nabla f|^2}. \tag{6.12}$$

We devote the remaining part of this section to the analysis of the remainder terms $\mathbf{W}$ and Q. More precisely, we will show that $\mathbf{W}$ and Q are bounded uniformly in ϵ (in the sense of the L^2 norm) on any fixed time interval $[0,T]$. Taking into account the formula (6.11) for $\mathbf{W}$ and $\mathbf{U}^1$ and using (6.9), (6.12) we can rewrite equation (6.7e) as

$$\frac{D(\rho\mathbf{W})}{Dt} + \left[\frac{\partial\mathbf{u}}{\partial x} - \operatorname{diag}\left(\mathbf{u}\cdot\frac{\nabla\rho}{\rho}\right)\right](\rho\mathbf{W}) + \nabla Q + 2\mathbf{\Omega}\times(\rho\mathbf{W}) = F, \tag{6.13}$$

where

$$\operatorname{diag}\left(\mathbf{u}\cdot\frac{\nabla\rho}{\rho}\right) = \begin{bmatrix} \mathbf{u}\cdot\frac{\nabla\rho}{\rho} & 0 & 0 \\ 0 & \mathbf{u}\cdot\frac{\nabla\rho}{\rho} & 0 \\ 0 & 0 & \mathbf{u}\cdot\frac{\nabla\rho}{\rho} \end{bmatrix},$$

$$F = \exp\left(\frac{if}{\epsilon}\right)\widetilde{F},$$

for

$$\begin{aligned}\widetilde{F} = &-i\left\{\frac{D}{Dt}\left(\frac{\operatorname{div}a}{|\nabla f|^2}\nabla f\right) + \left[\frac{\partial\mathbf{u}}{\partial x} - \operatorname{diag}\left(\mathbf{u}\cdot\frac{\nabla\rho}{\rho}\right)\right]\left(\frac{\operatorname{div}a}{|\nabla f|^2}\nabla f\right)\right\} \\ &-i\nabla\left\{2\frac{\partial\mathbf{u}}{\partial x}a\cdot\frac{\nabla f}{|\nabla f|^2} - \left(\frac{\mathbf{u}}{\rho}\cdot\nabla\rho\right)a\cdot\frac{\nabla f}{|\nabla f|^2} + 2(\mathbf{\Omega}\times a)\cdot\frac{\nabla f}{|\nabla f|^2}\right\}. \qquad (6.14) \\ &+ i\cdot 2\mathbf{\Omega}\times\frac{\operatorname{div}a}{|\nabla f|^2}\nabla f.\end{aligned}$$

Moreover, we have

$$\operatorname{div}(\rho\mathbf{W}) = G := \exp\left(\frac{if}{\epsilon}\right)\widetilde{G}, \tag{6.15}$$

for

$$\widetilde{G} = -i\operatorname{div}\left(\frac{\operatorname{div}a}{|\nabla f|^2}\nabla f\right).$$

Passing to the divergence of equation (6.13) and employing (6.15) we obtain

$$-\Delta Q = -\operatorname{div}F + \frac{DG}{Dt} + \operatorname{div}\big(\widetilde{\mathbf{u}}(\rho\mathbf{W})\big) + 2\operatorname{div}\big(\mathbf{\Omega}\times(\rho\mathbf{W})\big)$$

where

$$\widetilde{\mathbf{u}} := \frac{\partial\mathbf{u}}{\partial x} - \operatorname{diag}\left(\mathbf{u}\cdot\frac{\nabla\rho}{\rho}\right).$$

We now obtain from above that

$$q = \frac{1}{4\pi|x|} * \left(-\operatorname{div}F + \frac{DG}{Dt} + \operatorname{div}\big(\widetilde{\mathbf{u}}(\rho\mathbf{W})\big) + 2\operatorname{div}\big(\mathbf{\Omega}\times(\rho\mathbf{W})\big)\right). \tag{6.16}$$

Remark 6.1. Using the eikonal equation satisfied by the phase f and the Leibniz rule we infer that

$$\frac{DG}{Dt} = \exp\left(\frac{if}{\epsilon}\right)\frac{D\widetilde{G}}{Dt}.$$

Employing Parseval's theorem, Remark 6.1 and using properties of the Fourier transform pertaining to convolution, multiplication and differentiation we infer from (6.16) that there are constants $c_1, c_2, c_3 \in \mathbb{R}$ that depend only on T such that

$$(Q, Q) \leq c_1(\rho\mathbf{W}, \rho\mathbf{W}) + c_2(\widetilde{F}, \widetilde{F}) + c_3\left(\frac{D\widetilde{G}}{Dt}, \frac{D\widetilde{G}}{Dt}\right), \tag{6.17}$$

where by $(\cdot, \cdot)$ we mean the inner product in the spaces of square integrable functions. Following direct computations, equation (6.13) now implies that

$$\begin{aligned}\frac{d}{dt}(\rho\mathbf{W}, \rho\mathbf{W}) = &- \big((\widetilde{\mathbf{u}} + \widetilde{\mathbf{u}}^T)\rho\mathbf{W}, \rho\mathbf{W}\big) + (\rho\mathbf{W}, F) + (F, \rho\mathbf{W}) + (G, Q) + (Q, G)\\ &+ \Big(2\mathbf{\Omega} \times (\rho\mathbf{W}), \rho\mathbf{W}\Big) + \Big(\rho\mathbf{W}, 2\mathbf{\Omega} \times (\rho\mathbf{W})\Big) \qquad (6.18)\\ = &- \big((\widetilde{\mathbf{u}} + \widetilde{\mathbf{u}}^T)\rho\mathbf{W}, \rho\mathbf{W}\big) + (\rho\mathbf{W}, F) + (F, \rho\mathbf{W}) + (G, Q) + (Q, G).\end{aligned}$$

Equations (6.18) and (6.17) give

$$\frac{d}{dt}(\rho\mathbf{W}, \rho\mathbf{W}) \leq \widetilde{c_1}(\rho\mathbf{W}, \rho\mathbf{W}) + \widetilde{c_2}(\widetilde{F}, \widetilde{F}) + \widetilde{c_3}(\widetilde{G}, \widetilde{G}) + \widetilde{c_4}\left(\frac{D\widetilde{G}}{Dt}, \frac{D\widetilde{G}}{Dt}\right).$$

Multiplying the previous inequality by $\exp(\widetilde{c_1}(t - s))$, integrating the result from 0 to t and using that $\mathbf{W}(0) = 0$ yields the inequality

$$(\rho\mathbf{W}, \rho\mathbf{W}) \leq \int_0^t \exp(\widetilde{c_1}(t - s))\left[\widetilde{c_2}(\widetilde{F}(s), \widetilde{F}(s)) + \widetilde{c_3}(\widetilde{G}(s), \widetilde{G}(s)) + \widetilde{c_4}\left(\frac{D\widetilde{G}(s)}{Dt}, \frac{D\widetilde{G}(s)}{Dt}\right)\right] ds,$$

which shows that $(\rho\mathbf{W}, \rho\mathbf{W})$ is uniformly bounded in ϵ on the fixed time interval $[0, T]$. We can now easily conclude that $(\mathbf{W}, \mathbf{W})$ and (Q, Q) are uniformly bounded in ϵ on the fixed time interval $[0, T]$.

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