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EXISTENCE, POSITIVITY AND CONTRACTIVITY FOR QUANTUM STOCHASTIC FLOWS WITH INFINITE DIMENSIONAL NOISE

J MARTIN LINDSAY AND STEPHEN J WILLS

ABSTRACT. Quantum stochastic differential equations of the form

$$dk_t = k_t \circ \theta^\alpha_\beta \, d\Lambda^\beta_\alpha(t)$$

govern stochastic flows on a C^*-algebra $\mathcal{A}$. We analyse this class of equation in which the matrix of fundamental quantum stochastic integrators Λ is infinite dimensional, and the coefficient matrix θ consists of bounded linear operators on $\mathcal{A}$. Weak and strong forms of solution are distinguished, and a range of regularity conditions on the mapping matrix θ are considered, for investigating existence and uniqueness of solutions. Necessary and sufficient conditions on θ are determined, for any sufficiently regular weak solution k to be completely positive. The further conditions on θ for k to also be a contraction process are found; and when $\mathcal{A}$ is a von Neumann algebra and the components of θ are normal, these in turn imply sufficient regularity for the equation to have a strong solution. Weakly multiplicative and *-homomorphic solutions and their generators are also investigated. We then consider the right and left Hudson-Parthasarathy equations:

$$dX_t = F^\alpha_\beta X_t \, d\Lambda^\beta_\alpha(t), \quad dY_t = Y_t F^\alpha_\beta \, d\Lambda^\beta_\alpha(t),$$

in which F is a matrix of bounded Hilbert space operators. Their solutions are interchanged by a time reversal operation on processes. The analysis of quantum stochastic flows is applied to obtain characterisations of the generators F of contraction, isometry and coisometry processes. In particular weak solutions that are contraction processes are shown to have bounded generators, and to be necessarily strong solutions.

0. INTRODUCTION

The quantum stochastic differential equation (QSDE)

$$dk_t = k_t \circ \theta^\alpha_\beta \, d\Lambda^\beta_\alpha(t), \tag{0.1}$$

in which $\theta = [\theta^\alpha_\beta]$ is a matrix of linear maps on a C^*-algebra and $\Lambda = [\Lambda^\alpha_\beta]$ is the matrix of fundamental noise processes of quantum stochastic calculus ([Par],[Mey]), has been studied from a variety of points of view. Hudson and Evans sought unital *-homomorphic solutions, and called these *quantum diffusions* ([Hud],[Eva],[EvH]); Mohari and Sinha extended this work by allowing the noise and mapping matrix to be infinite dimensional ([MoS],[Moh]); and, extending the original work of Hudson and Parthasarathy ([HuP]), Fagnola and Mohari studied isometric, coisometric and contractive solutions of the related QSDE's

$$dX_t = F^\alpha_\beta X_t \, d\Lambda^\beta_\alpha(t), \tag{0.2}$$

$$dY_t = Y_t F^\alpha_\beta \, d\Lambda^\beta_\alpha(t), \tag{0.3}$$

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in which $F = [F^\alpha_\beta]$ is a matrix of Hilbert space operators, ([Fag],[Moh]). Lindsay and Parthasarathy gave a comprehensive analysis of completely positive (CP) solutions, and completely positive contraction solutions of (0.1) for finite dimensional noise and mapping matrix ([LP1,2]), and applied their results to QSDE's of the form (0.2) whose solutions provide CP flows by conjugation:

$$k_t(a) = X_t^* a X_t. \tag{0.4}$$

One motivation for extending the study of (0.1) from quantum diffusions to CP flows was to develop an approach to Belavkin's quantum filtering theory ([Be1]) in which processes of the form (0.4) arise, by viewing these as "inner" CP flows. As stated in [LP1] the broader framework of completely positive flows is required in order to formulate a quantum theory of measure-valued diffusions — quantum diffusions corresponding to stochastic flows of diffeomorphisms. Another motivation was to see Evans and Hudson's characterisation of the generators of quantum diffusions, and Lindblad's characterisation of quantum dynamical semigroup generators ([Lin],[GKS]), and its subsequent refinements ([ChE]), from a common perspective.

In the present paper we analyse (0.1) in the infinite dimensional case. Whereas each map θ^α_β is assumed to be bounded, the whole matrix θ is not assumed to define a bounded map. Indeed the interplay between boundedness of θ, regularity conditions on θ and the complete positivity and contractivity of solutions is central to our analysis. For us the standing hypothesis is the existence of a weak solution satisfying a mild regularity condition; such solutions are unique and necessarily linear. The set of matrices θ for which the solution is CP (respectively, a CP contraction, unital, weakly multiplicative, or *-homomorphic process) is characterised. When k is a CP contraction process, θ is shown to be completely bounded. A regularity condition on θ ensures the existence of a strong solution, and also that the standing hypothesis is satisfied. It is shown that when θ is completely bounded and $\mathcal{A}$ is a von Neumann algebra, ultraweak continuity of θ is equivalent to a stronger form of regularity devised and exploited by Mohari and Sinha. We thereby obtain an existence theorem for normal CP contraction flows on a von Neumann algebra. The results on the QSDE (0.1) yield a straightforward and illuminating approach to the QSDE's (0.2) and (0.3), and are applied to the questions of existence and uniqueness of solutions, and determining when the solutions are contraction, isometry or coisometry processes, unifying and extending the results of Fagnola and Mohari ([Fag],[Moh]). The interchange of solutions of (0.2) and (0.3) by time reversal is given a very simple explanation. The central tool of our approach is the *semigroup representation* of solutions of (0.1).

The structure of the paper is as follows. Section 1 contains the basic definitions and results on infinite matrices of bounded Hilbert space operators, and of bounded linear maps on a C^*-algebra, and establishes the equivalence of ultraweak continuity and Mohari-Sinha regularity for completely bounded mapping matrices on a von Neumann algebra (Proposition 1.4). Section 2 gives a review of quantum stochastic integration with infinite dimensional noise in a form best suited to our purposes. Weak and strong solutions of (0.1) are considered in Section 3. By choice of domain, the uniqueness of weakly regular weak solutions follows as in the case of finite dimensional noise, as does its representation in terms of the collection of one-parameter semigroups on the algebra whose generators are a linear combination of the θ^α_β. Following Meyer, we give a refinement of the Mohari-Sinha existence theorem, but with simplified proof, in Theorem 3.3. In Section 4 the generators of CP flows are characterised. Unlike the case of finite dimensional noise, here the existence of the flow must be assumed. In Section 5 the generators of CP contraction flows are characterised, again under the assumption of existence. It is shown that the generators of such flows are necessarily completely bounded (Theorem 5.2). Combined with

Proposition 1.4 mentioned above, this gives an existence theorem for CP contraction flows on a von Neumann algebra (Theorem 5.3). In Section 6 we specialise to *-homomorphic and weakly multiplicative flows and their generators. An existence theorem for weakly multiplicative flows is given, and a converse is proved, showing necessity of the weak multiplicativity conditions on the generator. In the final section the above results are applied to the right and left Hudson-Parthasarathy equations, (0.2) and (0.3). Specifically we deduce uniqueness, semigroup representations, and the Mohari-Sinha existence theorem for the two QSDE's, from the corresponding results for the flow equation (Theorem 7.1), yielding a transparent demonstration that solutions of (0.2) and (0.3) are interchanged by time reversal (Theorem 7.2). Conjugation by solutions of the right H-P equation (0.2) is shown to resonate perfectly with the corresponding flow equation (Theorem 7.4). Finally the operator matrices F for which the H-P equations (0.2) and (0.3) have weak solutions which are contraction (respectively isometry or coisometry) processes, are characterised (Theorem 7.5 and Proposition 7.6) — a key point being that the generating operator matrices are necessarily bounded.

The results here build on work of Hudson, Parthasarathy, Evans, Mohari, Sinha, Meyer and Lindsay, and complement the work of Journé, Fagnola and Belavkin. In conjunction with [LP2] the paper is almost self-contained, however the books [Par],[Mey] and [Bia] are recommended company for the nonspecialist reader. The characterisation of CP flow generators was reported in the conference proceedings [LW1], where a version of the existence result Theorem 5.3 was proved under stronger hypotheses (complete positivity and unitality, rather than contractivity). A preliminary account of most of the present results may be found in the PhD thesis [W]. Results characterising the generators of a large class of CP form-valued flows have also been obtained by Belavkin ([Be2,3]). In a very recent preprint Goswami and Sinha have obtained an existence theorem for *-homomorphic flows on a von Neumann algebra, in a coordinate-free quantum stochastic calculus using Hilbert W^*-modules ([GoS]). Their assumptions amount to the coefficient matrix being ultraweakly continuous, and taking the form necessary for a *-homomorphic solution. Free from coordinates they make no separability assumption on the noise dimension space, and indeed exploit this freedom to describe a natural dilation for any quantum dynamical semigroup on a von Neumann algebra having a bounded generator. A converse question to the main concern of the present paper is the algebraic characterisation of the collection of processes on an algebra which satisfy a QSDE of the form (0.1). This is analysed in a sister paper [LW2], where the semigroup representation again plays an important role.

1. Infinite Matrices

In this section we collect together some observations on infinite matrices of Hilbert space operators and C^*-algebra maps. This will provide us with the basic tools for dealing with the infinite dimensional aspects of the analysis of QSDE's of the form (0.1), (0.2) and (0.3). We also introduce our basic conventions and notations. A Hilbert space k, called the *noise dimension space*; another Hilbert space $\mathfrak{h}$, called the *initial space*; and a unital C^*-algebra $\mathcal{A}$, acting on $\mathfrak{h}$, called the *initial algebra* are all fixed throughout the paper. Sometimes $\mathcal{A}$ will be further assumed to be a von Neumann algebra; this will always be indicated. Moreover here the noise dimension space k is infinite dimensional and separable, with fixed orthonormal basis $\eta = (e_i)_{i\geq 1}$, and this basis is extended to a basis $(e_\gamma)_{\gamma\geq 0}$ of $\widehat{\mathsf{k}} := \mathbb{C} \oplus \mathsf{k}$, by putting $e_0 = 1 \in \mathbb{C}$.

General Conventions. Algebraic tensor products are denoted $\odot$, the symbol $\otimes$ being reserved for Hilbert space and von Neumann algebra tensor products. The space of complex sequences whose terms are eventually zero is written $c_{00}(\mathbb{N})$. For Banach spaces X and Y, $\mathcal{B}(X;Y)$ denotes the Banach space of bounded linear operators from X to Y. The notations Dom, Ran, Lin and $\overline{\text{Lin}}$ are used respectively for the domain and range of a linear operator, and the linear span, respectively closed linear span, of a collection of vectors. Hilbert space inner products are linear in their second argument. For a sequence $\{T_\gamma\}$ of nonnegative bounded linear Hilbert space operators, we write $\sum_\gamma T_\gamma < \infty$ to denote weak, and therefore also strong, convergence of the series. For a Hilbert space $\mathcal{K}$, the symmetric Fock space over $\mathcal{K}$ is denoted $\Gamma(\mathcal{K})$. For any linear space V, $\mathrm{M}_\infty(V)$ denotes the collection of infinite arrays $v = [v^\alpha_\beta]^\infty_{\alpha,\beta=0}$ from V, endowed with linear structure via pointwise defined operations. When V carries an involution $\dagger$, this is lifted to an involution on $\mathrm{M}_\infty(V)$ by:

$$v^{\dagger}{}^\alpha_\beta = (v^\beta_\alpha)^\dagger. \tag{1.1}$$

For $v \in \mathrm{M}_\infty(V)$ its *cut-off matrices* $\{v^{[N]} : N \geq 0\}$ are defined by

$$v^{[N]}{}^\alpha_\beta = \begin{cases} v^\alpha_\beta & \text{if } \alpha, \beta \leq N \\ 0 & \text{otherwise} \end{cases}. \tag{1.2}$$

Special Conventions. By means of the fixed bases for k and $\widehat{\mathsf{k}}$ we make the identifications

$$\mathfrak{h} \oplus (\mathfrak{h} \otimes \mathsf{k}) = \mathfrak{h} \otimes \widehat{\mathsf{k}} = \oplus_{\gamma \geq 0} \mathfrak{h} \text{ and } \mathfrak{h} \otimes \mathsf{k} = \mathfrak{h} \otimes (0 \oplus \mathsf{k}) = \oplus_{i \geq 1} \mathfrak{h} \tag{1.3}$$

by continuous linear extension of the isometric map

$$u \otimes (z^0, z^i e_i) = u \otimes z^\gamma e_\gamma \longmapsto (z^\gamma u).$$

Here we meet two conventions: greek indices run from 0 whereas roman indices run from 1, and Einstein's summation convention for repeated indices applies, with sums from 0 or 1 to ∞ — unless otherwise stated. For a vector e in $\widehat{\mathsf{k}}$, define the embedding

$$E_e \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \widehat{\mathsf{k}}) \text{ by } E_e v = v \otimes e, \tag{1.4}$$

and let $E^e = (E_e)^*$. For the basis vectors we abbreviate: $E_\alpha := E_{e_\alpha}$ and $E^\alpha := E^{e_\alpha}$. Thus $E^\alpha \in \mathcal{B}(\oplus_{\gamma \geq 0} \mathfrak{h}; \mathfrak{h})$ is the α^{th} coordinate projection $(u^\gamma) \mapsto u^\alpha$. The following dense subspace of $\oplus_{\gamma \geq 0} \mathfrak{h}$ is useful:

$$\mathfrak{h}_{00} := \{(u^\gamma) \in \oplus_{\gamma \geq 0} \mathfrak{h} : \exists N \text{ such that } u^\gamma = 0 \text{ for all } \gamma \geq N\}. \tag{1.5}$$

Operator Matrices. The elements of $\mathrm{M}_\infty(\mathcal{B}(\mathfrak{h}))$ are called *operator matrices*; the adjoint operation on $\mathcal{B}(\mathfrak{h})$ makes this linear space involutive through (1.1). To an operator matrix F we associate (possibly unbounded) operators $F^\alpha : \oplus_{\gamma \geq 0} \mathfrak{h} \to \mathfrak{h}$, $F_\beta : \mathfrak{h} \to \oplus_{\gamma \geq 0} \mathfrak{h}$ and $\widetilde{F} : \oplus_{\gamma \geq 0} \mathfrak{h} \to \oplus_{\gamma \geq 0} \mathfrak{h}$ by

$$\begin{aligned} F^\alpha(u^\gamma) &= F^\alpha_\beta u^\beta, & \text{Dom}\, F^\alpha &= \mathfrak{h}_{00} \\ F_\beta v &= (F^\alpha_\beta v), & \text{Dom}\, F_\beta &= \{v \in \mathfrak{h} : \textstyle\sum_{\alpha \geq 0} \|F^\alpha_\beta v\|^2 < \infty\} \\ \widetilde{F}(u^\gamma) &= (F^\alpha_\beta u^\beta), & \text{Dom}\, \widetilde{F} &= \{(u^\gamma) \in \mathfrak{h}_{00} : \textstyle\sum_{\alpha \geq 0} \|F^\alpha_\beta u^\beta\|^2 < \infty\}, \end{aligned} \tag{1.6}$$

where the summation convention is in operation; we also associate (extended) non-negative numbers $C^F_n(d,v)$ and $C^F_n(d)$ by

$$C^F_n(d,v)^2 = \sum_{\substack{|\boldsymbol{\alpha}|=n \\ \beta_i \leq d}} \|F^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} v\|^2, \quad C^F_n(d) = \sup\{C^F_n(d,v) : \|v\| = 1\} \tag{1.7}$$

where $F^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} := F^{\alpha_1}_{\beta_1} \cdots F^{\alpha_n}_{\beta_n}$.

An operator matrix F is *regular* if each F_β is bounded and everywhere defined, and is *bounded* if $\widetilde{F}$ is bounded and $\text{Dom}\, \widetilde{F} = \mathfrak{h}_{00}$. The linear space of regular operator matrices is denoted $\mathrm{M}_\infty(\mathcal{B}(\mathfrak{h}))_{\mathrm{R}}$.

Proposition 1.1. *Let* $F \in \mathrm{M}_\infty(\mathcal{B}(\mathfrak{h}))$.

(a) *For each* γ, $(F^\gamma)^* = (F^\dagger)_\gamma$. *In particular, each* F_β *is a closed operator.*

(b) *The following are equivalent:*

(i) F *is regular.*

(ii) $\mathrm{Dom}\, F_\beta = \mathfrak{h}$ *for all* β.

(iii) $\mathrm{Dom}\, \widetilde{F} = \mathfrak{h}_{00}$.

(iv) $\sum_{\alpha\geq 0}(F^\alpha_\beta)^* T F^\alpha_\gamma$ *converges strongly, for each* β, γ *and all* $T \in \mathcal{B}(\mathfrak{h})$.

(v) $\sum_{\alpha\geq 0}(F^\alpha_\beta)^* F^\alpha_\beta < \infty$, *for each* β.

(vi) $C^F_1(d, v) < \infty$, *for each* d *and* v.

In this case, the limit in (iv) *is* $(F_\beta)^*(T \otimes 1_{\widehat{\mathsf{k}}})F_\gamma$; $(\widetilde{F})^* \supset \widetilde{F^\dagger}$; *and*

$$F_\beta = \widetilde{F}E_\beta, \quad F^\alpha = E^\alpha \widetilde{F}, \quad F^\alpha_\beta = E^\alpha \widetilde{F} E_\beta, \tag{1.8}$$

$$C^F_n(d) \leq (\sqrt{d+1}\max\{\|F_\beta\| : \beta \leq d\})^n. \tag{1.9}$$

(c) *If* F *is bounded, then it is regular.*

Proof. In terms of the cut-off matrices and the orthogonal projections P^N on $\oplus_{\gamma\geq 0}\mathfrak{h}$ given by $(u^\gamma) \mapsto (u^0, \ldots, u^N, 0, \ldots)$, we have the identity

$$\langle (F^{\dagger[N]})_\beta v, (u^\gamma)\rangle = \langle v, F^\beta P^N(u^\gamma)\rangle \quad v \in \mathfrak{h}, (u^\gamma) \in \oplus_{\gamma\geq 0}\mathfrak{h}, N \geq \beta, \tag{1.10}$$

and the characterisation

$$v \in \mathrm{Dom}\, F_\beta \iff ((F^{[N]})_\beta v) \text{ converges in } \oplus_{\gamma\geq 0}\mathfrak{h}. \tag{1.11}$$

(a) If $v \in \mathrm{Dom}\, F^\dagger_\beta$ then $(F^{\dagger[N]})_\beta v = P^N(F^\dagger)_\beta v$ for all $N \geq \beta$, so (1.10) implies that $(F^\dagger)_\beta \subset (F^\beta)^*$. Conversely, if $v \in \mathrm{Dom}(F^\beta)^*$ then $(F^{\dagger[N]})_\beta v = P^N(F^\beta)^* v$, so $v \in \mathrm{Dom}(F^\dagger)_\beta$ by (1.11).

(b) The equivalences follow from (a), the Closed Graph Theorem, polarisation, the inclusion

$$\mathrm{Dom}\, \widetilde{F} \supset \{(u^\gamma) \in \mathfrak{h}_{00} : u^\gamma \in \mathrm{Dom}\, F_\gamma \text{ for each } \gamma\},$$

which follows by an application of the Cauchy-Schwarz inequality, and the fact that if $\mathrm{Dom}\, \widetilde{F} = \mathfrak{h}_{00}$ then in particular it contains all of the vectors having only one nonzero entry.

For regular F, viewing each F_β as an operator $\mathfrak{h} \to \mathfrak{h} \otimes \widehat{\mathsf{k}}$, we have

$$\sum{}_{|\boldsymbol{\alpha}|=n}\|F^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} v\|^2 = \|(F_{\beta_1} \otimes 1_{n-1})\cdots(F_{\beta_{n-1}} \otimes 1_1)F_{\beta_n} v\|^2 \tag{1.12}$$

where 1_j is the identity operator on $\otimes^{(j)}\widehat{\mathsf{k}}$; (1.9) follows.

(c) Note that $F_\beta = \widetilde{F}E_\beta$ for any F and β. So if F is bounded then F_β is bounded and everywhere defined as required. ☐

Dropping the tildes and identifying operator matrices with their corresponding operators, we have

$$\mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}}) \subset \mathrm{M}_\infty(\mathcal{B}(\mathfrak{h}))_{\mathrm{R}}.$$

From (1.8) we see how cut-offs work:

$$F \in \mathrm{M}_\infty(\mathcal{B}(\mathfrak{h}))_{\mathrm{R}} \;\Rightarrow\; F^{[N]}(u^\gamma) \to F(u^\gamma) \quad \forall (u^\gamma) \in \mathfrak{h}_{00}$$
$$F \in \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}}) \;\Rightarrow\; F^{[N]} \to F \text{ strongly.} \tag{1.13}$$

For regular operator matrices F and G,

$$F^\dagger G := [(F_\alpha)^* G_\beta]$$

defines an operator matrix, equal to $F^* G$ when F is bounded.

The following transform on operator matrices, defined in terms of a modified Kronecker matrix, is useful in discussions of QSDE's.

$$\widehat{F}^\alpha_\beta = \begin{cases} 1 + F^\alpha_\beta, & \alpha = \beta \geq 1 \\ F^\alpha_\beta, & \text{otherwise} \end{cases}. \tag{1.14}$$

Thus $\widehat{F} = F + \Delta(1)$, where Δ is defined below in (1.15).

Mapping Matrices. The elements of $\mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))$ are called *mapping matrices*, and the involution on $\mathcal{B}(\mathcal{A})$ given by $\tau^\dagger(a) = \tau(a^*)^*$ makes the linear space $\mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))$ involutive through (1.1). A mapping matrix θ is *real* if $\theta = \theta^\dagger$. Note that a mapping matrix may be viewed as a linear map $\mathcal{A} \to \mathrm{M}_\infty(\mathcal{A}) \subset \mathrm{M}_\infty(\mathcal{B}(\mathfrak{h}))$. Two particular elements $\Delta, \iota \in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))$ are given by

$$\Delta^\alpha_\beta = \begin{cases} \mathrm{id}, & \alpha = \beta \geq 1 \\ 0, & \text{otherwise} \end{cases}; \quad \iota^\alpha_\beta = \begin{cases} \mathrm{id}, & \alpha = \beta \geq 0 \\ 0, & \text{otherwise} \end{cases}. \tag{1.15}$$

In terms of the former we define a transform on $\mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))$ by

$$\widehat{\theta} = \theta + \Delta. \tag{1.16}$$

Following (1.7) we associate (possibly infinite) nonnegative constants to a mapping matrix θ by:

$$\begin{aligned} C^\theta_n(a, v, d)^2 &= \sum\nolimits_{\substack{|\boldsymbol{\alpha}|=n \\ \beta_i \leq d}} \|\theta^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}(a)v\|^2; \\ C^\theta_n(d) &= \sup\{C^\theta_n(a, v, d) : \|a\| = \|v\| = 1\}, \end{aligned} \tag{1.17}$$

where $\theta^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}} = \theta^{\alpha_1}_{\beta_1} \circ \cdots \circ \theta^{\alpha_n}_{\beta_n}$.

A mapping matrix θ is *regular* if for all $\gamma > 0$

$$\sum_{n \geq 0} (n!)^{-1} \gamma^n C^\theta_n(a, u, d)^2 < \infty \quad \forall a, u, d;$$

strongly regular if for all $\gamma > 0$

$$\sum_{n \geq 0} (n!)^{-1} \gamma^n C^\theta_n(d)^2 < \infty \quad \forall d;$$

M-S regular if there is a Hilbert space $\mathcal{K}$ and for each $\beta \geq 0$ an operator $B_\beta \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathcal{K})$ such that

$$\sum_{\alpha \geq 0} \theta^\alpha_\beta(a)^* \theta^\alpha_\beta(a) \leq B^*_\beta(a^* a \otimes 1_\mathcal{K}) B_\beta \quad \forall a; \tag{1.18}$$

completely regular if there is a Hilbert space $\mathcal{K}$ and an operator $B \in \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}}; \mathfrak{h} \otimes \mathcal{K})$ such that

$$\sum_{\alpha \geq 0} \|\theta^\alpha_\beta(a) u^\beta\|^2 \leq \|(a \otimes 1_\mathcal{K}) B(u^\beta)\|^2 \quad \forall a \in \mathcal{A}, (u^\beta) \in \mathfrak{h}_{00}; \tag{1.19}$$

bounded if each operator matrix $\theta(a)$ is bounded; and *completely bounded* if θ is bounded and, as a linear map $\mathcal{A} \to \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}})$, it is completely bounded, that is $\|\theta\|_{\mathrm{cb}} := \sup_n \|\theta^{(n)}\| < \infty$, where $\theta^{(n)} := \theta \otimes \mathrm{id}_{\mathrm{M}_n(\mathbb{C})}$. These are all linear spaces and Δ and ι belong to all of them; in particular for a mapping matrix θ either both θ and $\widehat{\theta}$ belong, or neither belong, to any of these spaces. Note that θ is regular if and only if the sequence $((n!)^{-1} \gamma^n C^\theta_n(a, u, d)^2)_{n \geq 1}$ is bounded for all choices of a, u, d and γ — this is the regularity condition introduced by Meyer in his exposition of the work of Mohari and Sinha.

The strong convergence of cut-offs (1.13) plus two applications of the Uniform Boundedness Principle imply that if θ is bounded then the sequence $(\|\theta^{[N]}\|)$ is bounded. Thus if θ is bounded then $\theta \in \mathcal{B}(\mathcal{A}; \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}}))$. Our definition of M-S regularity is a slight generalisation of the condition given in [MoS] where effectively $\mathcal{K}$ is assumed to be separable. If θ satisfies (1.19) then θ is bounded and satisfies (1.18) with $B_\beta = BE_\beta \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathcal{K})$, and so is M-S regular. Thus a mapping matrix is completely regular if and only if it is bounded and satisfies

$$\theta(a)^*\theta(a) \leq B^*(a^*a \otimes 1_{\mathcal{K}})B \tag{1.19$'$}$$

for some $\mathcal{K}$ and B.

By Proposition 1.1 a mapping matrix θ is M-S regular if and only if each $\theta(a)$ is a regular operator matrix and

$$\theta_\beta(a)^*\theta_\beta(a) \leq (B_\beta)^*(a^*a \otimes 1_{\mathcal{K}})B_\beta \tag{1.18$'$}$$

for some family $\{B_\beta \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathcal{K})\}_{\beta \geq 0}$, where $\theta_\beta(a) := \theta(a)_\beta$.

Let $\mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{R}}$, $\mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{SR}}$, $\mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{MS}}$, $\mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{CR}}$ $\mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{b}}$ and $\mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{cb}}$ denote the collection of regular, strongly regular, M-S regular, completely regular, bounded and completely bounded mapping matrices respectively. When θ is bounded its range lies in $\mathrm{M}_\infty(\mathcal{A})_{\mathrm{b}}$, the subspace of $\mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ consisting of operators whose components with respect to the fixed basis η are elements of $\mathcal{A}$. Since the map $\widehat{\mathsf{k}} \ni e \mapsto E_e \in \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \widehat{\mathsf{k}})$ is a linear isometry, $\mathrm{M}_\infty(\mathcal{A})_{\mathrm{b}}$ is the following norm closed, involutive subspace of $\mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}})$:

$$\{F \in \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}}) : E^e F E_f \in \mathcal{A} \ \forall e, f \in \widehat{\mathsf{k}}\}.$$

In general $\mathrm{M}_\infty(\mathcal{A})_{\mathrm{b}}$ is not closed under multiplication, however

$$\mathcal{A} \otimes_\sigma \mathcal{B}(\widehat{\mathsf{k}}) \subset \mathrm{M}_\infty(\mathcal{A})_{\mathrm{b}} \subset \mathcal{A}'' \otimes \mathcal{B}(\widehat{\mathsf{k}}),$$

where $\otimes_\sigma$ denotes the spatial tensor product of C^*-algebras and $\otimes$ the von Neumann algebra tensor product, the latter inclusion following from (1.13). If $\mathcal{A}$ is finite dimensional then the first inclusion is an equality. The second inclusion is an equality if and only if $\mathcal{A}$ is a von Neumann algebra.

Complete regularity is a stable property:

Proposition 1.2. *Let $\theta \in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{CR}}$, with associated Hilbert space $\mathcal{K}$ and operator B. For each $n \geq 1$ and $A \in \mathrm{M}_n(\mathcal{A})$,*

$$\theta^{(n)}(A)^*\theta^{(n)}(A) \leq B_n^*(A^*A \otimes 1_{\mathcal{K}})B_n$$

where $B_n = \epsilon_n(B \otimes 1_n)$ and ϵ_n is the flip isomorphism $\mathfrak{h} \otimes \mathcal{K} \otimes \mathbb{C}^n \to \mathfrak{h} \otimes \mathbb{C}^n \otimes \mathcal{K}$, so that $\theta^{(n)} \in \mathrm{M}_\infty(\mathcal{B}(\mathrm{M}_n(\mathcal{A})))_{\mathrm{CR}}$. In particular,

$$\mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{CR}} \subset \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{cb}}.$$

Proof. For $A \in \mathrm{M}_n(\mathcal{A})$ and $\xi \in (\mathfrak{h} \otimes \widehat{\mathsf{k}})^n$,

$$\begin{aligned}\|\theta^{(n)}(A)\xi\|^2 &= \sum_i \|\theta(A^i_j)\xi^j\|^2 \\ &\leq \sum_i \|(A^i_j \otimes 1_{\mathcal{K}})B\xi^j\|^2 = \|(A \otimes 1_{\mathcal{K}})B_n\xi\|^2,\end{aligned}$$

so the result follows. □

Proposition 1.3. *Let $\theta \in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{MS}}$ with associated Hilbert space $\mathcal{K}$ and sequence of operators $\{B_\beta\}$. Then*

$$C_n^\theta(a, v, d)^2 \leq \sum\nolimits_{\substack{|\boldsymbol{\beta}|=n \\ \beta_i \leq d}} \|(a \otimes 1_{\mathcal{K}} \otimes \cdots \otimes 1_{\mathcal{K}})B_{\boldsymbol{\beta}}v\|^2,$$

in the notation $B_{\boldsymbol{\beta}} := (B_{\beta_n} \otimes 1_{\mathcal{K}} \otimes \cdots \otimes 1_{\mathcal{K}}) \cdots (B_{\beta_2} \otimes 1_{\mathcal{K}})B_{\beta_1}$. In particular,

$$\mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{MS}} \subset \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{SR}},$$

with $C_n^\theta(d) \leq (\sqrt{d+1} \max\{\|B_\beta\| : \beta \leq d\})^n$.

Proof. (1.18) in the equivalent form

$$\sum_{\alpha\geq 0} \|(\theta^\alpha_\beta(a)\otimes 1_{\mathcal{L}})\xi\|^2 \leq \|(a\otimes 1_{\mathcal{K}}\otimes 1_{\mathcal{L}})(B_\beta\otimes 1_{\mathcal{L}})\xi\|^2$$

for any Hilbert space $\mathcal{L}$ and $\xi\in\mathfrak{h}\otimes\mathcal{L}$, may be iterated to give

$$\sum_{|\boldsymbol{\alpha}|=n} \|\theta^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}(a)v\|^2 \leq \|(a\otimes 1_{\mathcal{K}}\otimes\cdots\otimes 1_{\mathcal{K}})B_{\boldsymbol{\beta}}v\|^2 \tag{1.20}$$

for each $\boldsymbol{\beta}$ (c.f. (1.12)), from which the result follows. □

The simplest example of a mapping matrix that is real and M-S regular, but not bounded, is given by

$$\theta(a) = \operatorname{diag}[0, a, 2a, \ldots];$$

this satisfies $\sum_{\alpha\geq 0}\theta^\alpha_\beta(a)^*\theta^\alpha_\beta(a) = \beta^2 a^* a$.

When $\mathcal{A}$ is a von Neumann algebra we have

$$\mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{b}} = \mathcal{B}(\mathcal{A};\mathcal{A}\otimes\mathcal{B}(\widehat{\mathsf{k}}))$$

and, combined with Proposition 1.2, the next result shows that

$$\mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{CR}} = \{\theta\in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{cb}} : \theta \text{ normal}\} = \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{cb}}\cap \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{MS}}.$$

Proposition 1.4. *Let $\mathcal{A}$ be a von Neumann algebra, and let $\theta\in\mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))$.*

(a) *If θ is M-S regular then each θ^α_β is ultraweakly continuous.*
(b) *If θ is completely bounded and ultraweakly continuous then θ is completely regular.*
(c) *Let θ be bounded. Then θ is ultraweakly continuous if and only if each θ^α_β is.*
(d) *Let θ be bounded and completely positive. If θ^β_β is ultraweakly continuous then θ^α_β is ultraweakly continuous for each α.*

Proof. The proof of parts (a), (c) and (d) uses the facts that the weak and ultraweak topologies coincide on bounded sets, and that a bounded operator between von Neumann algebras is ultraweakly continuous if it preserves the limit of all bounded increasing nets of positive elements ([Tak], Corollary III.3.11). So let (a_λ) be a bounded increasing net in $\mathcal{A}_+$ with least upper bound a.

(a) In this case $\|\theta^\alpha_\beta(a_\lambda)\|\leq\|\theta^\alpha_\beta\|\|a\|$, so it suffices to show that $\theta^\alpha_\beta(a_\lambda - a)\to 0$ weakly. Let $\{B_\beta\}$ be the operators associated with θ as in (1.18), then $a_\lambda\otimes 1_{\mathcal{K}}\to a\otimes 1_{\mathcal{K}}$ strongly, so

$$\|\theta^\alpha_\beta(a_\lambda - a)v\|^2 \leq \|[(a_\lambda - a)\otimes 1_{\mathcal{K}}]B_\beta v\|^2 \to 0, \quad v\in\mathfrak{h}.$$

Thus $\theta^\alpha_\beta(a_\lambda)\to\theta^\alpha_\beta(a)$ strongly, and so also weakly.

(b) By Haagerup's version of the Wittstock-Paulsen Theorem for normal completely bounded maps ([Haa]), $\theta(a) = V^*(a\otimes 1_{\mathcal{L}})W$ for some Hilbert space $\mathcal{L}$ and operators $V, W\in\mathcal{B}(\mathfrak{h}\otimes\widehat{\mathsf{k}};\mathfrak{h}\otimes\mathcal{L})$. Therefore θ satisfies $(1.19)'$ with $B = \|V\|W$, and so is completely regular.

(c) Let θ be bounded. Suppose first that each θ^α_β is ultraweakly continuous. Since $\|\theta(a_\lambda)\|\leq\|\theta\|\|a\|$, it suffices to show that $\langle (u^\gamma), \theta(a_\lambda - a)(v^\gamma)\rangle\to 0$ for $(u^\gamma), (v^\gamma)$ from the dense subspace $\mathfrak{h}_{00}$ of $\oplus_{\gamma\geq 0}\mathfrak{h}$ defined in (1.5). But this follows immediately from the ultraweak and therefore weak convergence $\theta^\alpha_\beta(a_\lambda)\to\theta^\alpha_\beta(a)$, for each α,β. Conversely, if θ is ultraweakly continuous then, by (1.8), each θ^α_β is a composition of ultraweakly continuous maps, and so is ultraweakly continuous.

(d) This follows from the fact that if θ is bounded and CP then

$$\sum_{\alpha\geq 0}\|\theta^\alpha_\beta(a)u\|^2 = \|S^*\pi(a)SE_\beta u\|^2 \leq \|S\|^2\|\pi(a)SE_\beta u\|^2 = \|S\|^2\langle u,\theta^\beta_\beta(a^*a)u\rangle,$$

where $S^*\pi(\cdot)S$ is a Stinespring decomposition of θ ([Sti],[Tak]). □

Corollary 1.5. *Let $\mathcal{A}$ be finite dimensional, then*

$$\mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{cb}} = \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{b}} = \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{CR}}.$$

Proof. The first equality follows from the fact that bounded linear maps between C^*-algebras are completely bounded when the source space is finite dimensional ([HuT],[Smi]). Since all vector space topologies on a finite dimensional space coincide, $\mathcal{A}$ is a von Neumann algebra and each θ^α_β is ultraweakly continuous. Therefore the second equality follows from parts (b) and (c) of the proposition. □

2. Quantum Stochastic Processes

In this section we describe the approach to quantum stochastic calculus (QSC) with infinite degrees of freedom developed by Mohari and Sinha ([MoS],[Moh],[Par]). Our account allows sufficient flexibility to go beyond the context of unital *-homomorphic flows treated by them. Let $\mathcal{H} = \mathfrak{h}\otimes\mathcal{F}$, the Hilbert space tensor product of the initial space and the symmetric Fock space over $L^2(\mathbb{R}_+;\mathsf{k})$. For each test function $f\in L^2(\mathbb{R}_+;\mathsf{k})$, define the sequence, $\{f^\alpha\}_{\alpha\geq 0}$, of complex-valued functions on $\mathbb{R}_+$ using the orthonormal basis η:

$$f^0(t) = 1;\quad f^i(t) = \langle e_i, f(t)\rangle, i\geq 1.$$

Thus $f^i\in L^2(\mathbb{R}_+)$ for each $i\geq 1$.

We ensure that most of the apparently infinite sums that appear in what follows have only finitely many nonzero terms by working with test functions from the following dense subspace:

$$\mathbb{M} = \mathbb{M}[\eta] := \{f\in(L^2\cap L^\infty_{\mathrm{loc}})(\mathbb{R}_+;\mathsf{k}) : f^i\neq 0 \text{ for only finitely many } i\}.$$

For $f\in\mathbb{M}$, define

$$\dim f := \max\{\alpha\geq 0 : f^\alpha\neq 0\}. \tag{2.1}$$

Processes on a Hilbert Space. A subset $\mathcal{S}$ of $\mathbb{M}$ is *admissible* if

$$f\in\mathcal{S}\ \Rightarrow\ f_{[0,t]}\in\mathcal{S}\quad\forall t\geq 0;$$
$$\mathcal{E}_\mathcal{S} := \mathrm{Lin}\{\varepsilon(f) : f\in\mathcal{S}\} \text{ is dense in } \mathcal{F}.$$

Here $\varepsilon(f) := ((n!)^{-\frac{1}{2}}f^{\otimes n})$ is the exponential vector for the test function f and $f_{[0,t]} := f\mathbf{1}_{[0,t]}$ where $\mathbf{1}_{[0,t]}$ is the indicator function of the time interval $[0,t]$. An admissible subset $\mathcal{S}$ of $\mathbb{M}$ is *fully admissible* if

$$\mathcal{S}^{[N]} := \{(f^1,\dots,f^N) : f\in\mathcal{S}, \dim f\leq N\} \tag{2.2}$$

is an admissible subset of $L^2(\mathbb{R}_+;\mathbb{C}^N)$ for each $N\geq 1$. Full admissibility is a useful concept for reducing proofs to the case of finite dimensional noise. Clearly $\mathbb{M}$ is fully admissible itself; another fully admissible set of particular use for us is

$$\mathbb{S} = \mathbb{S}[\eta] := \mathrm{Lin}\{e_k\mathbf{1}_J : k\in\mathbb{N}, J\subset\mathbb{R}_+ \text{ a bounded interval}\}.$$

Whenever we work with a particular representative of an element of $\mathbb{S}$, for example for the semigroup representation of solutions of QSDE's proved in Theorem 3.1, we will choose the right continuous version.

Operator adaptedness is defined using the increasing time indexed family of subspaces $\mathcal{H}_t := \mathfrak{h}\otimes\mathcal{F}_t$ of $\mathcal{H}$, where $\mathcal{F}_t\subset\mathcal{F}$ is the symmetric Fock space over

$L^2([0,t];\mathsf{k}) \subset L^2(\mathbb{R}_+;\mathsf{k})$. For an admissible set $\mathcal{S}$, an *$\mathcal{E}_\mathcal{S}$-process on $\mathfrak{h}$* is a family $X = (X_t)_{t\geq 0}$ of operators on $\mathcal{H}$ each having domain $\mathfrak{h} \odot \mathcal{E}_\mathcal{S}$, which is *weakly measurable* and *adapted*:

(i) $\forall \zeta \in \mathcal{H}$ the map $t \mapsto \langle \zeta, X_t u\varepsilon(f)\rangle$ is measurable;
(ii) $X_t u\varepsilon(f_{[0,t]}) \in \mathcal{H}_t$, and $X_t u\varepsilon(f) = (X_t u\varepsilon(f_{[0,t]})) \otimes \varepsilon(f_{[t,\infty[})$;

for all $u \in \mathfrak{h}$ and $f \in \mathcal{S}$. Since pointwise limits of measurable functions are measurable, it suffices for (i) to hold for all ζ in some dense subset of $\mathcal{H}$, such as $\mathfrak{h} \odot \mathcal{E}_{\mathcal{S}'}$ for an admissible set $\mathcal{S}'$. An $\mathcal{E}_\mathcal{S}$-process X and $\mathcal{E}_{\mathcal{S}'}$-process Y are called *adjoint processes* if

$$X_t^* \supset Y_t \quad \text{for a.a. } t.$$

Thus an $\mathcal{E}_\mathcal{S}$-process X has an $\mathcal{E}_{\mathcal{S}'}$-adjoint if $\operatorname{Dom} X_t^* \supset \mathfrak{h} \odot \mathcal{E}_{\mathcal{S}'}$ for all t; the $\mathcal{E}_{\mathcal{S}'}$-adjoint process $X_\cdot^*|_{\mathfrak{h}\odot\mathcal{E}_{\mathcal{S}'}}$ is denoted $X^\dagger$. Two $\mathcal{E}_\mathcal{S}$-processes X and X' are *indistinguishable* if

$$\forall \zeta \in \mathcal{H}, \xi \in \mathfrak{h} \odot \mathcal{E}_\mathcal{S} \quad \langle \zeta, (X_t - X'_t)\xi\rangle = 0 \ \text{ for a.a. } t.$$

Clearly $\mathcal{H}$ may be replaced by any dense subset such as $\mathfrak{h} \odot \mathcal{E}_{\mathcal{S}'}$ for an admissible set $\mathcal{S}'$. Indistinguishable processes are identified, and the resulting linear space is denoted $\mathbb{P}(\mathfrak{h}, \mathcal{E}_\mathcal{S})$. The subspace of $\mathcal{E}_\mathcal{S}$-processes having an $\mathcal{E}_{\mathcal{S}'}$-adjoint is denoted $\mathbb{P}^\ddagger(\mathfrak{h}, \mathcal{E}_{(\mathcal{S}',\mathcal{S})})$; the map $\dagger$ is conjugate linear and involutive in the obvious sense:

$$X \in \mathbb{P}^\ddagger(\mathfrak{h}, \mathcal{E}_{(\mathcal{S}',\mathcal{S})}) \Rightarrow X^\dagger \in \mathbb{P}^\ddagger(\mathfrak{h}, \mathcal{E}_{(\mathcal{S},\mathcal{S}')}), \ \text{ and } (X^\dagger)^\dagger = X. \tag{2.3}$$

For discussing products of processes and quantum stochastic integrals, stronger properties are needed. An $\mathcal{E}_\mathcal{S}$-process X is *measurable* (respectively *continuous*) if it satisfies

$$t \mapsto X_t\xi \text{ is strongly measurable (resp. continuous),}$$

By Pettis' Theorem $X \in \mathbb{P}(\mathfrak{h}, \mathcal{E}_\mathcal{S})$ is measurable if and only if each $X_\cdot\xi$ is Lebesgue-essentially separably-valued. For measurable $\mathcal{E}_\mathcal{S}$-processes X and X', indistinguishability is equivalent to

$$\forall \xi \in \mathfrak{h} \odot \mathcal{E}_\mathcal{S} \quad X_t\xi = X'_t\xi \ \text{ for a.a. } t. \tag{2.4}$$

An $\mathcal{E}_\mathcal{S}$-process X on $\mathfrak{h}$ is a *bounded, contraction, isometry, coisometry* or *unitary process* if (for some version of X) each operator X_t has that property. For a bounded process X the weak adaptedness condition amounts to (the continuous extension of) X_t belonging to $\mathcal{B}(\mathcal{H}_t)$.

An $\mathcal{E}_\mathcal{S}$-process X on $\mathfrak{h}$ is *$\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-regular* if, for all $f \in \mathcal{S}', g \in \mathcal{S}$ and $T > 0$,

$$\sup\{|\langle u\varepsilon(f), X_t v\varepsilon(g)\rangle| : \|u\| = \|v\| = 1, t \in [0,T]\} < \infty; \tag{2.5}$$

it is *$\mathcal{E}_\mathcal{S}$-regular* if, for all $g \in \mathcal{S}$ and $T > 0$,

$$\sup\{\|X_t v\varepsilon(g)\| : \|v\| = 1, t \in [0,T]\} < \infty. \tag{2.6}$$

An element L of $\mathrm{M}_\infty(\mathbb{P}(\mathfrak{h}, \mathcal{E}_\mathcal{S}))$ is called an *$\mathcal{E}_\mathcal{S}$-process matrix*; when L is in $\mathrm{M}_\infty(\mathbb{P}^\ddagger(\mathfrak{h}, \mathcal{E}_{(\mathcal{S}',\mathcal{S})}))$ its *adjoint* $L^\dagger \in \mathrm{M}_\infty(\mathbb{P}^\ddagger(\mathfrak{h}, \mathcal{E}_{(\mathcal{S},\mathcal{S}')}))$ is defined by

$$(L^\dagger)^\alpha_\beta = (L^\alpha_\beta)^\dagger.$$

Stochastic Integration. The integrator processes of the theory, $[\Lambda^\alpha_\beta]^\infty_{\alpha,\beta=0}$, are defined in terms of the orthonormal basis η by

$$\begin{aligned}
&\textit{Creation:} && \Lambda^0_j(t) := a^*(e_j\mathbf{1}_{[0,t]});\\
&\textit{Conservation:} && \Lambda^i_j(t) := d\Gamma(|e_j\rangle\langle e_i| \otimes \mathrm{M}_{[0,t]});\\
&\textit{Annihilation:} && \Lambda^i_0(t) := a(e_i\mathbf{1}_{[0,t]});\\
&\textit{Time:} && \Lambda^0_0(t) := t1;
\end{aligned}$$

where $a^*, d\Gamma$ and a denote, respectively, Fock space creation, differential second quantisation and annihilation, $\mathrm{M}_{[0,t]}$ is the operator of multiplication by $\mathbf{1}_{[0,t]}$ on

$L^2(\mathbb{R}_+)$, $|e_j\rangle\langle e_i|$ is the Dirac dyad acting on k by $\zeta \mapsto \langle e_i, \zeta\rangle e_j$, and $L^2(\mathbb{R}_+;\mathsf{k})$ is identified with the Hilbert space tensor product $\mathsf{k} \otimes L^2(\mathbb{R}_+)$ (see [HuP],[Mey], [Par]).

An $\mathcal{E}_\mathcal{S}$-process X is *stochastically integrable* if it satisfies

$$t \mapsto X_t\xi \text{ is locally square integrable .}$$

In particular X is measurable. Let $\mathbb{P}_{\mathrm{SI}}(\mathfrak{h}, \mathcal{E}_\mathcal{S})$ denote the collection of stochastically integrable $\mathcal{E}_\mathcal{S}$-processes. For each pair of indices α, β the QS integral $\int_0^t X_s\, d\Lambda^\alpha_\beta(s)$ is defined by Hudson and Parthasarathy, and shown to form a continuous $\mathcal{E}_\mathcal{S}$-process $(\int_0^t X_s\, d\Lambda^\alpha_\beta(s))_{t\geq 0}$, when X is stochastically integrable ([HuP]). An alternative approach to the delicate question of the existence of such QS integrals on the exponential domain is given in [Bia]. Let L be an $\mathcal{E}_\mathcal{S}$-process matrix such that each component process L^α_β is stochastically integrable, thus $L \in \mathrm{M}_\infty(\mathbb{P}_{\mathrm{SI}}(\mathfrak{h}, \mathcal{E}_\mathcal{S}))$. Then, for each $N \geq 0$, we may form the $\mathcal{E}_\mathcal{S}$-process $\Lambda^{[N]}(L)$ given by

$$\Lambda^{[N]}(L)_t = \sum_{\alpha,\beta\leq N} \int_0^t L^\alpha_\beta(s)\, d\Lambda^\beta_\alpha(s).$$

The process matrix L is *stochastically integrable* if

$$\int_0^t \sum_{\alpha\geq 0} \|L^\alpha_\beta(s) u\varepsilon(f)\|^2\, ds < \infty \tag{2.7}$$

for all choices of $u \in \mathfrak{h}, f \in \mathcal{S}, t \geq 0$ and $\beta \leq \dim f$. The collection of stochastically integrable $\mathcal{E}_\mathcal{S}$-process matrices is denoted $\mathrm{M}_\infty(\mathbb{P}(\mathfrak{h}, \mathcal{E}_\mathcal{S}))_{\mathrm{SI}}$.

Theorem 2.1 ([Moh],[MoS],[Par]). *Let $\mathcal{S}$ be an admissible subset of $\mathbb{M}$ and let $L \in \mathrm{M}_\infty(\mathbb{P}(\mathfrak{h}, \mathcal{E}_\mathcal{S}))_{\mathrm{SI}}$. Then there is an $\mathcal{E}_\mathcal{S}$-process denoted $\Lambda(L)$ satisfying*

(i) $\lim_{N\to\infty} \sup_{0\leq t\leq T} \|[\Lambda(L)_t - \Lambda^{[N]}(L)_t] u\varepsilon(f)\| = 0$;

(ii) $\|\Lambda(L)_t u\varepsilon(f)\|^2 \leq 2e^{\nu_f(t)} \sum_{\beta=0}^{\dim f} \int_0^t \sum_{\alpha\geq 0} \|L^\alpha_\beta(s) u\varepsilon(f)\|^2\, d\nu_f(s)$;

for all $u \in \mathfrak{h}, f \in \mathcal{S}$ and $t, T \geq 0$, where $d\nu_f(s) = (1 + \|f(s)\|^2)\, ds$, and $\dim f$ is defined in (2.1).

Notice that the estimate (ii) implies that the process $\Lambda(L)$ is continuous. The *first fundamental formula* of QSC reads

$$\langle u\varepsilon(f), \Lambda(L)_t v\varepsilon(f)\rangle = \int_0^t f_\alpha(s) g^\beta(s) \langle u\varepsilon(f), L^\alpha_\beta(s) v\varepsilon(g)\rangle\, ds, \tag{2.8}$$

with the summation convention operating, and $f_\alpha(s) := \overline{f^\alpha(s)}$, for $f \in \mathbb{M}, g \in \mathcal{S}, u, v \in \mathfrak{h}$ and $t \geq 0$. Here the sum is finite since $f, g \in \mathbb{M}$. If M is a stochastically integrable family of $\mathcal{E}_{\mathcal{S}'}$-processes for another admissible set $\mathcal{S}'$, then

$$\begin{aligned}\langle \Lambda(L)_t u\varepsilon(f), \Lambda(M)_t v\varepsilon(g)\rangle = \int_0^t f_\alpha(s) g^\beta(s) \big\{ &\langle \Lambda(L)_s u\varepsilon(f), M^\alpha_\beta(s) v\varepsilon(g)\rangle \\ &+ \langle L^\beta_\alpha(s) u\varepsilon(f), \Lambda(M)_s v\varepsilon(g)\rangle + \langle L^i_\alpha(s) u\varepsilon(f), M^i_\beta(s) v\varepsilon(g)\rangle \big\}\, ds \end{aligned} \tag{2.9}$$

for $u, v \in \mathfrak{h}, f \in \mathcal{S}, g \in \mathcal{S}'$ and $t \geq 0$, with summation over α, β and i. Here the sum over i is an infinite one — that it converges to an integrable function of s is ensured by the stochastic integrability of L and M . This is the *second fundamental formula* of QSC and is a rigorous formulation of the quantum Itô formula. Thus QS integration may be viewed as a map $\Lambda : \mathrm{M}_\infty(\mathbb{P}(\mathfrak{h}, \mathcal{E}_\mathcal{S}))_{\mathrm{SI}} \to \mathbb{P}(\mathfrak{h}, \mathcal{E}_\mathcal{S})$; its image consists of continuous processes. We mention one important consequence of the first fundamental formula: for $L \in \mathrm{M}_\infty(\mathbb{P}^\ddagger(\mathfrak{h}, \mathcal{E}_{(\mathcal{S}',\mathcal{S})}))$, if L and $L^\dagger$ are both stochastically integrable then $\Lambda(L) \in \mathbb{P}^\ddagger(\mathfrak{h}, \mathcal{E}_{(\mathcal{S}',\mathcal{S})})$ and $\Lambda(L^\dagger) = \Lambda(L)^\dagger$.

Independence of Integrators. The assumption of full admissibility (2.2) reduces the problem of independence of QS integrators to the case of finite dimensional quantum noise.

Proposition 2.2. *Let $\mathcal{S}$ and $\mathcal{S}'$ be fully admissible subsets of $\mathbb{M}$, and suppose that $L \in \mathrm{M}_\infty(\mathbb{P}^\ddagger(\mathfrak{h}, \mathcal{E}_{(\mathcal{S}',\mathcal{S})}))$ with each L^α_β and $L^{\alpha\dagger}_\beta$ stochastically integrable.*

(a) *If, for each $\xi \in \mathfrak{h}\odot\mathcal{E}_\mathcal{S}, \zeta \in \mathfrak{h}\odot\mathcal{E}_{\mathcal{S}'}$ and $t \geq 0$, the sequence $(\langle \zeta, \Lambda^{[N]}(L)_t\xi\rangle)_{N=0}^\infty$ is eventually zero then $L = 0$.*

(b) *If $L \in \mathrm{M}_\infty(\mathbb{P}(\mathfrak{h}, \mathcal{E}_\mathcal{S}))_{\mathrm{SI}}$ and $\Lambda(L) = 0$ then $L = 0$.*

Proof. (a) For each $N \geq 1$ let J_N be the isometry $\mathfrak{h} \otimes \Gamma(L^2(\mathbb{R}_+; \mathbb{C}^N)) \to \mathcal{H}$ defined by continuous linear extension of the prescription

$$J_N : u\varepsilon(\mathbf{f}) \mapsto u\varepsilon(\textstyle\sum_{i=1}^N f^i e_i) \tag{2.10}$$

Fix $N \geq 1$. The hypothesis and (2.8) imply that

$$\langle u\varepsilon(f), \Lambda^{[N]}(L)_t v\varepsilon(g)\rangle = 0$$

for all $t \geq 0, u, v \in \mathfrak{h}$ and $f \in \mathcal{S}', g \in \mathcal{S}$ with $\dim f, \dim g \leq N$. Now the matrix $\{J_N^* L^\alpha_\beta J_N\}_{\alpha,\beta=0}^N$ is a collection of $(N+1)^2$ $\mathcal{E}_{\mathcal{S}^{[N]}}$-processes, with $\mathcal{E}_{\mathcal{S}'^{[N]}}$-adjoints $\{J_N^* L^{\alpha\dagger}_\beta J_N\}$, and by the above they satisfy

$$\int_0^t J_N^* L^\alpha_\beta(s) J_N \, d\Lambda^\beta_\alpha(s) = 0,$$

(with sums only up to N) as a QS integral on $\mathfrak{h} \otimes \Gamma(L^2(\mathbb{R}_+; \mathbb{C}^N))$. The finite dimensional result ([L], Theorem 5.1) implies that $J_N^* L^\alpha_\beta J_N = 0$ for all $\alpha, \beta \leq N$, and the result follows since $\|\xi\| = \lim_{N\to\infty} \|J_N^*\xi\|$ for all $\xi \in \mathcal{H}$.

(b) If $L \in \mathrm{M}_\infty(\mathbb{P}(\mathfrak{h}, \mathcal{E}_\mathcal{S}))_{\mathrm{SI}}$ then $\Lambda(L)$ exists as an $\mathcal{E}_\mathcal{S}$-operator process by Theorem 2.1. If $\Lambda(L) = 0$ then since $\mathcal{S} \subset \mathbb{M}$, each sequence $(\langle \xi, \Lambda^{[N]}(L)_t\xi\rangle)_{N=0}^\infty$ is eventually zero and so (a) applies. □

Remark. The fact that L itself need not be assumed to be stochastically integrable can be useful.

Processes on a C^*-algebra. For an admissible set $\mathcal{S}$, an *$\mathcal{E}_\mathcal{S}$-process on $\mathcal{A}$*, the initial algebra fixed at the beginning of Section 1, is a map $k : \mathcal{A} \to \mathbb{P}(\mathfrak{h}, \mathcal{E}_\mathcal{S})$ such that each $k_t(a)$ is affiliated to the von Neumann algebra $\mathcal{A}'' \otimes \mathcal{B}(\mathcal{F})$, that is

$$k_t(a)(a' \otimes 1_\mathcal{F})\xi = (a' \otimes 1_\mathcal{F})k_t(a)\xi \quad \forall a' \in \mathcal{A}', \xi \in \mathfrak{h} \odot \mathcal{E}_\mathcal{S}. \tag{2.11}$$

The linear space of $\mathcal{E}_\mathcal{S}$-processes on $\mathcal{A}$ is denoted $\mathbb{P}(\mathcal{A}, \mathcal{E}_\mathcal{S})$. An $\mathcal{E}_\mathcal{S}$-process k on $\mathcal{A}$ is *continuous* if each $\mathcal{E}_\mathcal{S}$-process $k.(a)$ on $\mathfrak{h}$ is (see (2.4)); it is *linear* if each k_t is; it is *unital* if each $k_t(1)$ equals $1|_{\mathfrak{h}\odot\mathcal{E}_\mathcal{S}}$; it is a *bounded process* if each operator $k_t(a)$ is bounded and each k_t, viewed as a map $\mathcal{A} \to \mathcal{B}(\mathcal{H})$, is a bounded linear operator; and is a *contraction process* if furthermore each k_t is a contraction. Thus bounded $\mathcal{E}_\mathcal{S}$-processes on $\mathcal{A}$ satisfy

$$k_t(\mathcal{A}) \subset \mathcal{A}'' \otimes \mathcal{B}(\mathcal{F}).$$

An $\mathcal{E}_\mathcal{S}$-process k on $\mathcal{A}$ is *$\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-regular* if

$$\sup\left\{ \frac{|\langle u\varepsilon(f), [k_t(a_1) - k_t(a_2)]v\varepsilon(g)\rangle|}{\|u\|\|a_1 - a_2\|\|v\|} : t \in [0,T], u, v, a_1 - a_2 \neq 0 \right\} < \infty, \tag{2.12}$$

and *$\mathcal{E}_\mathcal{S}$-regular* if

$$\sup\left\{ \frac{\|[k_t(a_1) - k_t(a_2)]v\varepsilon(g)\|}{\|a_1 - a_2\|\|v\|} : t \in [0,T], v, a_1 - a_2 \neq 0 \right\} < \infty, \tag{2.13}$$

for all $T > 0, f \in \mathcal{S}'$ and $g \in \mathcal{S}$. Thus contraction $\mathcal{E}_{\mathcal{S}}$-processes are $\mathcal{E}_{\mathcal{S}}$-regular. Regularity properties of processes are key to the uniqueness of QSDE solutions; they are closely related to the regularity of QSDE coefficients; and they play an important role in the algebraic characterisation of QSDE solutions ([LW2]).

A linear $\mathcal{E}_{\mathcal{S}}$-process on $\mathcal{A}$ is *completely positive* if

$$\sum_{p,q} \langle \xi^p, k_t(a_p^* a_q)\xi^q \rangle \geq 0$$

for all $t \geq 0$ and each finite collection $\{(a_p, \xi^p)\}$ from $\mathcal{A} \times (\mathfrak{h} \odot \mathcal{E}_{\mathcal{S}})$, and it is **-homomorphic* if it satisfies

$$\langle \xi, k_t(a^*a)\xi \rangle = \|k_t(a)\xi\|^2,$$

in other words $k_t(a^*a) = k_t(a)^* k_t(a)$.

If $k \in \mathbb{P}(\mathcal{A}, \mathcal{E}_{\mathcal{S}})$ and each $k_\cdot(a) \in \mathbb{P}^\ddagger(\mathfrak{h}, \mathcal{E}_{(\mathcal{S}',\mathcal{S})})$ then, following (2.3),

$$k_t^\dagger(a) := k_t(a^*)^\dagger$$

defines an adjoint $\mathcal{E}_{\mathcal{S}'}$-process $k^\dagger$ to k; the collection of such processes is denoted $\mathbb{P}^\ddagger(\mathcal{A}, \mathcal{E}_{(\mathcal{S}',\mathcal{S})})$. Processes in $\mathbb{P}^\ddagger(\mathcal{A}, \mathcal{E}_{(\mathcal{S},\mathcal{S})})$ satisfying $k^\dagger = k$ are called *real*. Note the obvious symmetry in the above definitions: for $k \in \mathbb{P}^\ddagger(\mathcal{A}, \mathcal{E}_{(\mathcal{S}',\mathcal{S})})$, k is $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-regular if and only if $k^\dagger$ is $\mathcal{E}_{(\mathcal{S},\mathcal{S}')}$-regular. A process $k \in \mathbb{P}^\ddagger(\mathcal{A}, \mathcal{E}_{(\mathcal{S}',\mathcal{S})})$ is *weakly multiplicative* if it satisfies

$$\langle k_t^\dagger(a^*)\xi, k_t(b)\zeta \rangle = \langle \xi, k_t(ab)\zeta \rangle,$$

in other words $k_t(ab) = k_t^\dagger(a^*)^* k_t(b)$. Thus $k \in \mathbb{P}(\mathcal{A}, \mathcal{E}_{\mathcal{S}})$ is *-homomorphic if and only if it is linear, real and weakly multiplicative.

The following relationships mirror familiar C^*-algebraic facts ([LP2], Proposition 2.1).

Lemma 2.3. *Let k be an $\mathcal{E}_{\mathcal{S}}$-process on $\mathcal{A}$. If k is *-homomorphic then k is a CP contraction process with $k_\cdot(1)$ orthogonal projection valued. If k is CP then k is real, and is a contraction process if and only if $k_\cdot(1)$ is a contraction process on $\mathfrak{h}$.*

When $\mathcal{A}$ is a von Neumann algebra, a bounded process k on $\mathcal{A}$ is *normal* if each map k_t is normal.

Terminology refinement. In [LP2] what we here call an $\mathcal{E}_{\mathcal{S}}$-process on $\mathcal{A}$ was called an $\mathcal{E}_{\mathcal{S}}$-flow, and $\mathbb{P}(\mathcal{A}, \mathcal{E}_{\mathcal{S}})$ there denoted the collection of linear maps $\mathcal{A} \to \mathbb{P}(\mathfrak{h}, \mathcal{E}_{\mathcal{S}})$, without the restriction of affiliation imposed here (2.11). Here we reserve the term flow for $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-regular $\mathcal{E}_{\mathcal{S}}$-processes on $\mathcal{A}$ weakly satisfying a QSDE of the form (0.1). Theorem 3.1 below shows that flows (in the present sense) are uniquely determined by the coefficients of the QSDE, and that they are automatically linear and satisfy the affiliation condition.

3. Quantum Stochastic Differential Equations

In this section we analyse the linear quantum stochastic differential equation

$$dk_t = k_t \circ \theta^\alpha_\beta \, d\Lambda^\beta_\alpha(t), \qquad k_0(a) = a \otimes 1, \tag{3.1}$$

governed by a mapping matrix θ, for $\mathcal{E}_{\mathcal{S}}$-flows on the C^*-algebra $\mathcal{A}$. In the work of Mohari and Sinha various relations were imposed on the mapping matrix θ in order to ensure that the solution to the QSDE enjoyed certain algebraic properties. However at this stage we impose only minimal conditions on θ in order to study the equation in a more general setting, before deriving the necessary and sufficient structure equations for complete positivity of the flow in the next section. In particular we discuss the notion of weak and strong solutions, giving conditions that ensure the existence of solutions, and we prove that weak solutions, subject to the very mild condition of weak regularity, are unique and satisfy a semigroup representation property, as in the case of finite dimensions of quantum noise.

Weak and Strong Solutions. Let $\mathcal{S}$ and $\mathcal{S}'$ be admissible sets. An $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$*-weak solution* of (3.1) is an $\mathcal{E}_{\mathcal{S}}$-process k on $\mathcal{A}$ that satisfies

$$s \mapsto f_\alpha(s) g^\beta(s) \langle u\varepsilon(f), k_s(\theta^\alpha_\beta(a)) v\varepsilon(g)\rangle \text{ is locally integrable,} \tag{3.2a}$$

and

$$\begin{aligned} \langle u\varepsilon(f), k_t(a) v\varepsilon(g)\rangle =& \langle u\varepsilon(f), av\varepsilon(g)\rangle \\ &+ \int_0^t f_\alpha(s) g^\beta(s) \langle u\varepsilon(f), k_s(\theta^\alpha_\beta(a)) v\varepsilon(g)\rangle\, ds \end{aligned} \tag{3.2b}$$

for all $u, v \in \mathfrak{h}, f \in \mathcal{S}', g \in \mathcal{S}, a \in \mathcal{A}$ and $t \geq 0$. An $\mathcal{E}_{\mathcal{S}}$*-strong solution* of (3.1) is an $\mathcal{E}_{\mathcal{S}}$-process k such that

$$[k_\cdot(\theta^\alpha_\beta(a))] \in \mathrm{M}_\infty(\mathbb{P}(\mathfrak{h}, \mathcal{E}_{\mathcal{S}}))_{\mathrm{SI}} \quad \forall a \in \mathcal{A}, \tag{3.3a}$$

(see (2.7)) and the corresponding integral equation holds:

$$k_t(a) = a \otimes 1 + \int_0^t k_s(\theta^\alpha_\beta(a))\, d\Lambda^\beta_\alpha(s). \tag{3.3b}$$

By the first fundamental formula, $k \in \mathbb{P}(\mathcal{A}, \mathcal{E}_{\mathcal{S}})$ is an $\mathcal{E}_{\mathcal{S}}$-strong solution if and only if k satisfies (3.3a) and is an $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-weak solution for any/every admissible set $\mathcal{S}'$; by the second fundamental formula $\mathcal{E}_{\mathcal{S}}$-strong solutions are continuous.

We shall make use of the following notation: given $\phi \in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))$ and $w, z \in c_{00}(\mathbb{N})$, define a bounded linear map $\phi_{w,z}$ on $\mathcal{A}$ by

$$\phi_{w,z}(a) = w_\alpha \phi^\alpha_\beta(a) z^\beta, \tag{3.4}$$

where $w_\alpha := \overline{w^\alpha}$ and we set $w^0 = z^0 = 1$. Thus if $k \in \mathbb{P}(\mathcal{A}, \mathcal{E}_{\mathcal{S}})$ is a *linear* process satisfying (3.2a) then k is an $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-weak solution of (3.1) if and only if it satisfies

$$\langle u\varepsilon(f), k_t(a) v\varepsilon(g)\rangle = \langle u\varepsilon(f), av\varepsilon(g)\rangle + \int_0^t \langle u\varepsilon(f), k_s(\theta_{f(s),g(s)}(a)) v\varepsilon(g)\rangle\, ds,$$

where $f(s) = \{f^i(s)\} \in c_{00}(\mathbb{N})$ and similarly for $g(s)$, for representatives from the measure equivalence classes of f and g.

Uniqueness of Solutions and the Semigroup Representation. A consequence of working with test functions from the set $\mathbb{M}$ is that results in the finite dimensional noise setting, whose proofs rely only on manipulations of matrix elements of the form $\langle u\varepsilon(f), k_t(a) v\varepsilon(g)\rangle$, often carry over to the infinite dimensional setting, since only a finite (but arbitrarily large) number of components of the test functions f and g are nonzero. In particular the uniqueness result and semigroup representation of weak solutions given in Theorem 3.1 of [LP2] both remain valid with a very similar proof.

Theorem 3.1. *Let $\mathcal{S}$ and $\mathcal{S}'$ be admissible sets.*

- (ai) *For $\theta \in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))$, the* QSDE (3.1) *has at most one $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-regular $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-weak solution k. If such a k exists then it is linear.*
- (aii) *For $k \in \mathbb{P}(\mathcal{A}, \mathcal{E}_{\mathcal{S}})$, if $\mathcal{S} \cap \mathcal{S}' \supset \mathbb{S}$ then k is an $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-weak solution of* (3.1) *for at most one mapping matrix θ.*
- (b) *For $\theta \in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))$ and $k \in \mathbb{P}(\mathcal{A}, \mathcal{E}_{\mathcal{S}})$, if $\mathcal{S} \cup \mathcal{S}' \subset \mathbb{S}$ then the following are equivalent:*
  - (i) *k is an $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-regular $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-weak solution of* (3.1).

(ii) *For $f \in \mathcal{S}'$ and $g \in \mathcal{S}$, with discontinuities contained in the set* $\{0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = \infty\}$,

$$\langle u\varepsilon(f), k_t(a)v\varepsilon(g)\rangle = \langle u, \mathcal{P}^{[0]}_{t_1-t_0} \circ \cdots \circ \mathcal{P}^{[i]}_{t-t_i}(a)v\rangle\langle\varepsilon(f), \varepsilon(g)\rangle \tag{3.5}$$

for $t \in [t_i, t_{i+1}[$ *where, in the notation* (3.4), $\mathcal{P}^{[k]}$ *is the one-parameter semigroup on* $\mathcal{A}$ *with generator* $\theta_{w,z}$ *where* $w = f(t_k)$ *and* $z = g(t_k)$.

Proof. (ai) Let ${}^1k$ and ${}^2k$ be $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-regular $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-weak solutions of (3.1). Application of (3.2b) to $\langle u\varepsilon(f), [\,{}^1k_t(a) - {}^2k_t(a)]v\varepsilon(g)\rangle$ may be iterated; $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-regularity applied after n iterations yields a bound of the form $(n!)^{-1}C_1C_2^n$ as in [LP2] — the only difference being that here the constants depend on the dimensions of the test functions. The necessary linearity then follows from the same argument by applying (3.2b) to $\langle u\varepsilon(f), [k_t(a + \lambda b) - k_t(a) - \lambda k_t(b)]v\varepsilon(g)\rangle$ and iterating.

(aii) Suppose that k is an $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-weak solution of (3.1) for mapping matrices ${}^1\theta$ and ${}^2\theta$. Since k satisfies (3.2b) for both matrices

$$0 = \int_0^t f_\alpha(s)g^\beta(s)\langle u\varepsilon(f), [k_s({}^1\theta^\alpha_\beta(a)) - k_s({}^2\theta^\alpha_\beta(a))]v\varepsilon(g)\rangle\, ds.$$

Putting $f = w\mathbf{1}_{[0,1]}$ and $g = z\mathbf{1}_{[0,1]}$, where $w, z \in c_{00}(\mathbb{N})$, continuity of the integrand on $[0, 1]$ means that it vanishes identically on that interval. Considering the value at 0, we obtain the identity

$$w_\alpha[{}^1\theta^\alpha_\beta(a) - {}^2\theta^\alpha_\beta(a)]z^\beta = 0;$$

varying w and z gives ${}^1\theta(a) = {}^2\theta(a)$ as required.

(bi $\Rightarrow$ bii) This is proved exactly as in the finite dimensional case ([LP2]).

(bii $\Rightarrow$ bi) This follows easily from the Fundamental Theorem of Calculus. □

Remarks. (i) It is clear from the semigroup representation (3.5) that any $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-regular $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-weak solution k to the QSDE (3.1) satisfies the affiliation condition (2.11).

(ii) If the coefficient matrix θ is a function of time, with sufficiently regular dependence, then the semigroups in the representation (3.5) should be replaced by evolution operators governed by equations of the form

$$d\mathcal{P}_t = \mathcal{P}_t \circ \theta_{w,z}(t, \cdot)\, dt, \quad \mathcal{P}_0 = \mathrm{id}_{\mathcal{A}}.$$

Flows and Generators. The previous theorem allows us to establish the following terminology: when (3.1) has an $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-regular $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-weak solution k, we call θ *the QS generator* of k, and k *the* $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$*-flow weakly generated by* θ.

Our next result shows that reality passes from generator to flow and vice versa.

Proposition 3.2. *Suppose that* $\theta \in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))$ *weakly generates an* $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$*-flow* k*, for some admissible sets* $\mathcal{S}$ *and* $\mathcal{S}'$*. Then the following hold.*

(a) *The following are equivalent:*
  (i) $k \in \mathbb{P}^\ddagger(\mathcal{A}, \mathcal{E}_{(\mathcal{S}',\mathcal{S})})$.
  (ii) $\theta^\dagger$ *weakly generates an* $\mathcal{E}_{(\mathcal{S},\mathcal{S}')}$*-flow* j.
  In this case $j = k^\dagger$.
(b) *If* $\mathcal{S} = \mathcal{S}' \supset \mathbb{S}$ *then the following are equivalent:*
  (i) k *is real.*
  (ii) θ *is real.*
(c) *If* $\mathcal{A}$ *is a von Neumann algebra and* k *is bounded, then the following are equivalent:*
  (i) k *is normal.*
  (ii) *each* θ^α_β *is ultraweakly continuous.*

Proof. (ai $\Rightarrow$ aii) This is an immediate consequence of (3.2b) applied to

$$\langle k_t(a^*)u\varepsilon(f), v\varepsilon(g)\rangle - \langle u\varepsilon(f), av\varepsilon(g)\rangle.$$

(aii $\Rightarrow$ ai) This yields to the argument used in the proof of Theorem 3.1(ai), since application of (3.2b) to the difference

$$\langle j_t(a^*)u\varepsilon(f), v\varepsilon(g)\rangle - \langle u\varepsilon(f), k_t(a)v\varepsilon(g)\rangle,$$

may be iterated.

(b) This follows from (a) and Theorem 3.1(aii).

(c) Since the set of ultraweakly continuous maps in $\mathcal{B}(\mathcal{A})$ is norm closed, the equivalence follows easily from the semigroup representation (3.5) and the fact that the ultraweak and weak topologies coincide on bounded sets. □

Remark. The implications (ai $\Rightarrow$ aii) and (bi $\Rightarrow$ bii) are valid without weak regularity — the existence of a weak solution suffices.

Existence of Solutions. If $\theta \in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))$ has only finitely many nonzero components then it is a straightforward matter to generate the unique, $\mathcal{E}_{\mathbb{M}}$-regular, $\mathcal{E}_{\mathbb{M}}$-strong solution to the equation (3.1) using a Picard iteration method (see, for example, [Eva] or [HLP]). However the estimates involved in the construction of these solutions depend on the number of nonzero components, and so more sophisticated methods are required in the general case. The first existence theorem for (3.1) with an infinite number of nonzero coefficients was obtained by Mohari and Sinha ([MoS]). We give a refinement, with a simplified proof, as described in [Mey].

Recall the weak and strong regularity conditions (2.12) and (2.13) for processes on the algebra $\mathcal{A}$.

Theorem 3.3. *Let $\theta \in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{R}}$. Then* (3.1) *has an $\mathcal{E}_{\mathbb{M}}$-strong solution k which is $\mathcal{E}_{(\mathbb{M},\mathbb{M})}$-regular. Moreover, if $\theta \in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{SR}}$ then the process k is $\mathcal{E}_{\mathbb{M}}$-regular.*

Proof. Consider the Picard iteration scheme

$$k_t^{\{0\}}(a) = a \otimes 1; \quad k_t^{\{n+1\}}(a) = a \otimes 1 + \int_0^t k_s^{\{n\}}(\theta^\alpha_\beta(a))\, d\Lambda^\beta_\alpha(s).$$

Using the relationship

$$\sum_{\alpha \geq 0} \sum_{\beta \leq d} C_n^\theta(\theta^\alpha_\beta(a), u, d)^2 = C_{n+1}^\theta(a, u, d)^2, \tag{3.6}$$

an inductive argument shows that $[k_\cdot^{\{n\}}(\theta^\alpha_\beta(a))]$ is stochastically integrable, so that the scheme is well-defined, and for $f \in \mathbb{M}$ and $t \in [0, T]$,

$$\|[k_t^{\{n\}}(a) - k_t^{\{n-1\}}(a)]u\varepsilon(f)\|^2 \leq (n!)^{-1} C(f,t)^n C_n^\theta(a,u,d)^2 \|\varepsilon(f)\|^2$$

where $C(f,t) = 2\nu_f(t) \exp \nu_f(t)$ and $d = \dim f$. This implies the estimate

$$\|k_t^{\{n\}}(a)u\varepsilon(f)\| \leq \sum_{p=0}^n (p!)^{-1/2} C(f,T)^{p/2} C_p^\theta(a,u,d) \|\varepsilon(f)\|. \tag{3.7}$$

It follows that $k_t(a)u\varepsilon(f) = \lim_n k_t^{\{n\}}(a)u\varepsilon(f)$ exists and defines an adapted measurable process k satisfying

$$\|k_t(a)u\varepsilon(f)\| \leq \sum_{n \geq 0} (n!)^{-1/2} C(f,T)^{n/2} C_n^\theta(a,u,d) \|\varepsilon(f)\|. \tag{3.8}$$

Introducing a factor of $2^{-n} \times 2^n$ into the summand above, the Cauchy-Schwarz inequality and (3.6) then gives

$$\sum_{\alpha \geq 0} \sum_{\beta \leq d} \|k_t(\theta^\alpha_\beta(a))u\varepsilon(f)\|^2 \leq \sum_{n \geq 0} (n!)^{-1} C(f,T)^n 2^{n+1} C_{n+1}^\theta(a,u,d)^2 \|\varepsilon(f)\|^2, \tag{3.9}$$

and the right hand side converges since θ is regular. Thus $[k_{\cdot}(\theta^{\alpha}_{\beta}(a))]$ is stochastically integrable, and the estimate

$$\|[k_t(a) - k_t^{\{n\}}(a)]u\varepsilon(f)\| \leq \sum_{p\geq n+1} (p!)^{-1/2} C(f,T)^{p/2} C^{\theta}_p(a,u,d)\|\varepsilon(f)\|$$

now allows us to deduce that k satisfies the integral equation (3.3b).

Examination of the iteration scheme that generates k reveals the identity:

$$\langle u\varepsilon(f), k_t(a)v\varepsilon(g)\rangle = \langle\varepsilon(f),\varepsilon(g)\rangle \sum_{n\geq 0} \sum_{\substack{|\boldsymbol{\alpha}|=|\boldsymbol{\beta}|=n \\ 0\leq\alpha_i,\beta_i\leq d'}} \langle u, \theta^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}}(a)v\rangle \int_0^t dt_n \cdots \int_0^{t_2} dt_1\, h^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}(\mathbf{t}) \quad (3.10)$$

where $d' = \max\{\dim f, \dim g\}$, $h^{\boldsymbol{\beta}}_{\boldsymbol{\alpha}}(\mathbf{t}) = \prod_{i=1}^n h^{\beta_i}_{\alpha_i}(t_i)$ and $h^{\beta}_{\alpha} = f_{\alpha} g^{\beta}$. It follows that

$$|\langle u\varepsilon(f), k_t(a)v\varepsilon(g)\rangle| \leq |\langle\varepsilon(f),\varepsilon(g)\rangle| \|u\|\|a\|\|v\| \sum_{n\geq 0} (n!)^{-1} \left(\int_0^t p(s)ds\right)^n \quad (3.11)$$

where $p(s) = \sum_{\alpha,\beta} \|h^{\beta}_{\alpha}(s)\theta^{\alpha}_{\beta}\|$, thus k is $\mathcal{E}_{(\mathbb{M},\mathbb{M})}$-weakly regular. If $\theta \in \mathrm{M}_{\infty}(\mathcal{B}(\mathcal{A}))_{\mathrm{SR}}$ then $\mathcal{E}_{\mathbb{M}}$-strong regularity follows from (3.8). □

The solution constructed in Theorem 3.3 will be denoted k^{θ}. Propositions 1.3 and 1.4, and Corollary 1.5 thus give useful sufficient conditions for the existence of a strong solution of (3.1).

The following property is used in the discussion of *-homomorphic processes in Section 6. An $\mathcal{E}_{\mathcal{S}}$-process k on $\mathcal{A}$ is *sequentially strongly continuous* if it satisfies

$$a, a_1, a_2, \ldots \in \mathcal{A}, a_n \to a \text{ strongly} \implies k_t(a_n)\xi \to k_t(a)\xi \quad \forall \xi, t.$$

Sequential weak continuity is defined analogously.

Proposition 3.4. *Let $\theta \in \mathrm{M}_{\infty}(\mathcal{B}(\mathcal{A}))_{\mathrm{MS}}$. Then k^{θ} and each of its Picard approximants $k^{\{n\}}$ are sequentially strongly continuous.*

Proof. Let $f \in \mathbb{M}, a, a_1, a_2, \ldots \in \mathcal{A}$ with $a_i \to a$ strongly, and put $M = \sup_i \|a_i\|$ and $d = \dim f$. In the notation (1.17), the iterated form of M-S regularity (1.20) implies that $C^{\theta}_n(a_i - a, u, d) \to 0$ for each N, and strong regularity implies that $C^{\theta}_n(a_i - a, u, d) \leq 2M\|u\|C^{\theta}_n(d)$. Thus the sequential strong continuity of each $k^{\{n\}}$ follows from (3.7). The estimate (3.8) now gives $\|k^{\theta}_t(a_i - a)u\varepsilon(f)\| \to 0$ by the Dominated Convergence Theorem for series. □

Continuous Dependence on Test Functions. We conclude this section with a continuity result for the solution to (3.1) when θ is a regular mapping matrix. From Theorem 3.3 we see that the map $(u, a, v) \mapsto \langle u\varepsilon(f), k_t(a)v\varepsilon(g)\rangle$ is continuous for fixed f and g if θ is regular. Now we show that, if u, a and v are fixed, there is a continuous dependence on the test functions within certain restrictions on the number of nonzero components.

Proposition 3.5. *Let $\theta \in \mathrm{M}_{\infty}(\mathcal{B}(\mathcal{A}))_{\mathrm{R}}$. For $f, g \in \mathbb{M}$ and $T > 0$, if $\{f_n\}$ and $\{g_n\}$ are sequences in $\mathbb{M}$ satisfying*

(i) $\lim_{n\to\infty} \|f_n - f\|_{L^2} = \lim_{n\to\infty} \|g_n - g\|_{L^2} = 0$,
(ii) $\sup_n\{\max(\dim f_n, \dim g_n)\} < \infty$,

then

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |\langle u\varepsilon(f_n), k^{\theta}_t(a)v\varepsilon(g_n)\rangle - \langle u\varepsilon(f), k^{\theta}_t(a)v\varepsilon(g)\rangle| = 0$$

for all $u, v \in \mathfrak{h}$ and $a \in \mathcal{A}$.

Proof. Letting d' equal the maximum dimension of all of the test functions involved, the result follows from the representation (3.10) as in Proposition 3.5 of [LP2]. □

4. Completely Positive Flows

In this section *we assume that θ weakly generates an $\mathcal{E}_{(\mathcal{S},\mathcal{S})}$-flow*, for some sufficiently large admissible set $\mathcal{S}$. Under this assumption we establish necessary and sufficient conditions on θ for the flow to be completely positive. This is the infinite dimensional extension of Theorem 4.1 in [LP2], where the boundedness of each of the finite number of maps θ^α_β ensures that θ generates a flow. Here, with this existence assumption, we may exploit the results of [LP2].

Recall the definition of the transformed mapping matrices (1.16).

Theorem 4.1. *Let $\theta \in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))$ and suppose that θ weakly generates an $\mathcal{E}_{(\mathcal{S},\mathcal{S})}$-flow k, where $\mathcal{S}$ is a fully admissible set such that each $\mathcal{S}^{[N]}$ is dense in $L^2(\mathbb{R}_+;\mathbb{C}^N)$.*

(a) *The following are equivalent:*

  (i) *k is completely positive.*

  (ii) *θ^0_0 is real and there is a quadruple $\mathcal{Q} = (\pi, \mathcal{K}, \delta, \{W_i\})$ consisting of a representation $(\pi,\mathcal{K})$ of $\mathcal{A}$, a π-derivation $\delta : \mathcal{A} \to \mathcal{B}(\mathfrak{h};\mathcal{K})$ and a sequence $\{W_i\}$ in $\mathcal{B}(\mathfrak{h};\mathcal{K})$ such that*

$$\left.\begin{aligned} \partial\theta^0_0(a_1,a_2) &= \delta(a_1)^*\pi(1)\delta(a_2)\\ \theta^i_0(a) - W_i^*\pi(1)\delta(a) &= \theta^i_0(1)a\\ \theta^0_j(a) - \delta^\dagger(a)\pi(1)W_j &= a\theta^0_j(1)\\ \theta^i_j(a) - \delta^i_j a &= W_i^*\pi(a)W_j. \end{aligned}\right\} \tag{4.1}$$

  (iii) *There is a quadruple $\mathcal{P} = (\pi,\mathcal{K},\{K_\gamma\},\{W_\gamma\})$ consisting of a representation $(\pi,\mathcal{K})$ of $\mathcal{A}$ and sequences of operators $\{K_\gamma\}$ in $\mathcal{B}(\mathfrak{h})$ and $\{W_\gamma\}$ in $\mathcal{B}(\mathfrak{h};\mathcal{K})$ such that $\widehat{\theta} = \psi + \phi$, where*

$$\psi^\alpha_\beta(a) = W_\alpha^*\pi(a)W_\beta \text{ and } \phi^\alpha_\beta(a) = \delta^\alpha_0 aK_\beta + \delta^0_\beta K_\alpha^* a. \tag{4.1$'$}$$

(bi) *$\mathcal{Q}$ may be chosen so that*

$$(\pi,\mathcal{K}) \text{ is unital,} \tag{4.2}$$

and the following minimality condition is satisfied:

$$\mathcal{K} = \overline{\mathcal{K}_0} \text{ where } \mathcal{K}_0 = \mathrm{Lin}\left\{\delta(a)u^0 + \pi(a)W_iu^i : a \in \mathcal{A}, (u^\gamma) \in \mathfrak{h}_{00}\right\}. \tag{4.3}$$

(bii) *$\mathcal{P}$ may be chosen so that $(\pi,\mathcal{K})$ is unital, $K_\gamma \in \mathcal{A}''$ and*

$$\mathcal{K} = \overline{\mathcal{K}_0} \text{ where } \mathcal{K}_0 = \mathrm{Lin}\left\{\pi(a)W_\alpha u^\alpha - W_0au^0 : a \in \mathcal{A}, (u^\gamma) \in \mathfrak{h}_{00}\right\}. \tag{4.3$'$}$$

(c) *Quadruples $\mathcal{Q}$ satisfying* (4.1), (4.2) *and* (4.3) *are unique: if $\mathcal{Q}'$ satisfies* (4.1) *then there is a unique isometry $V : \mathcal{K} \to \mathcal{K}'$ such that*

$$VW_i = \pi'(1)W_i'; \quad V\delta(a) = \pi'(1)\delta'(a); \quad V\pi(a) = \pi'(a)V. \tag{4.4}$$

Moreover, V is an isomorphism if and only if $\mathcal{Q}'$ also satisfies (4.2) *and* (4.3)

(d) *Suppose that $\mathcal{A}$ is a von Neumann algebra and k is completely positive. If θ^α_α is ultraweakly continuous, then so are θ^α_β and θ^β_α for each β. If every θ^α_β is ultraweakly continuous and $\mathcal{Q}$ is a quadruple satisfying* (4.1) *and the minimality condition* (4.3)*, then π is normal.*

Terminology. (i) The coboundary-like operator ∂ acts on linear maps ψ on the algebra $\mathcal{A}$ to give sesquilinear maps $\partial\psi : \mathcal{A}\times\mathcal{A} \to \mathcal{A}$ in the following manner:

$$\partial\psi(a_1,a_2) = \psi(a_1^*a_2) - a_1^*\psi(a_2) - \psi(a_1^*)a_2 + a_1^*\psi(1)a_2.$$

The map $\partial\psi$ determines ψ up to the sum of a derivation and an anticommutator on $\mathcal{A}$.

(ii) If $(\pi, \mathcal{K})$ is a representation of the algebra $\mathcal{A}$ then a linear map $\delta : \mathcal{A} \to \mathcal{B}(\mathfrak{h}; \mathcal{K})$ is a π-*derivation* if it satisfies the π-Leibnitz identity:

$$\delta(a_1 a_2) = \delta(a_1)a_2 + \pi(a_1)\delta(a_2).$$

It is well known that derivations $\mathcal{A} \subset \mathcal{B}(\mathfrak{h}) \to \mathcal{B}(\mathfrak{h})$ are bounded ([Rin]). An elementary argument with block matrices shows that π-derivations are automatically bounded too.

Proof. First note that if (4.1) holds then, replacing $\mathcal{K}$ by $\pi(1)\mathcal{K}$, δ by $\pi(1)\delta$ and W_j by $\pi(1)W_j$, and restricting π to the Hilbert space $\pi(1)\mathcal{K}$, it may be seen that (4.1) holds for a unital π. Similarly for (4.1)$'$.

Let $\theta^{[d]}$ be the cut-off matrix (see (1.2)) viewed as an element of $\mathrm{M}_{d+1}(\mathcal{B}(\mathcal{A}))$, and let ${}^dk = k^\phi$, the strong solution to the QSDE (3.1) on $\mathfrak{h} \otimes \Gamma(L^2(\mathbb{R}_+; \mathbb{C}^d))$ for $\phi = \theta^{[d]}$, so that ${}^dk$ is an $\mathcal{E}_{\mathbb{M}^{[d]}}$-process ([LP2]). If J_d denotes the linear isometry $\mathfrak{h} \otimes \Gamma(L^2(\mathbb{R}_+; \mathbb{C}^d)) \to \mathcal{H}$ defined through (2.10), then by uniqueness of weakly regular weak solutions ([LP2], Theorem 3.1(b)),

$$J_d^* k_t(a) J_d = {}^dk_t(a)|_{\mathfrak{h} \odot \mathcal{E}_{\mathcal{S}^{[d]}}}, \tag{4.5a}$$

and, for any $\xi \in \mathfrak{h} \odot \mathcal{E}_\mathcal{S}$,

$$\langle \xi, k_t(a)\xi \rangle = \langle J_d^*\xi, {}^dk_t(a) J_d^*\xi \rangle \tag{4.5b}$$

for all sufficiently large d. By continuity in test functions ([LP2], Proposition 3.5), it follows from (4.5) that k is completely positive if and only if each ${}^dk$ is completely positive. Thus the implications (aii $\Rightarrow$ ai, aiii $\Rightarrow$ ai) follow from their finite dimensional counterparts ([LP2], Theorem 4.1).

(ai $\Rightarrow$ aii, aiii), (b): Suppose that k is completely positive. Then, for each d, ${}^dk$ is completely positive and so Theorem 4.1 of [LP2] guarantees the existence of a sequence of quadruples $({}^d\mathcal{K}, {}^d\pi, {}^d\delta, {}^dW)_{d \geq 1}$ whose terms consist of a unital representation $({}^d\pi, {}^d\mathcal{K})$ of $\mathcal{A}$, a ${}^d\pi$-derivation ${}^d\delta : \mathcal{A} \to \mathcal{B}(\mathfrak{h}; {}^d\mathcal{K})$ and an operator ${}^dW \in \mathcal{B}(\mathfrak{h}^d; {}^d\mathcal{K})$, where $\mathfrak{h}^d = \oplus_{i=1}^d \mathfrak{h}$, such that

$$\widehat{\theta}^{[d]}(a) = \begin{bmatrix} \theta_0^0(a) & a\,{}^dC^* + {}^d\delta^\dagger(a)\,{}^dW \\ {}^dCa + {}^dW^*\,{}^d\delta(a) & {}^dW^*\,{}^d\pi(a)\,{}^dW \end{bmatrix} \tag{4.6a}$$

where ${}^dC = [\theta_0^1(1) \cdots \theta_0^d(1)]^\top$, θ_0^0 is real,

$$\partial\theta_0^0(a,b) = {}^d\delta(a)^*\,{}^d\delta(b) \tag{4.6b}$$

and ${}^d\mathcal{K} = \overline{{}^d\mathcal{K}_0}$ where

$${}^d\mathcal{K}_0 = \mathrm{Lin}\{{}^d\delta(a)u^0 + {}^d\pi(b)\mathbf{u} : a, b \in \mathcal{A}, u^0 \in \mathfrak{h}, \mathbf{u} \in \mathfrak{h}^d\}.$$

For all $e \geq d$ let $I^{ed} \in \mathcal{B}(\mathfrak{h}^d; \mathfrak{h}^e)$ be the isometry $\mathbf{u} \mapsto (u^1, \ldots, u^d, 0, \ldots, 0)$. It follows that $({}^e\pi, {}^e\mathcal{K}, {}^e\delta, {}^eWI^{ed})$ is a quadruple that satisfies (4.6), and so by the uniqueness part of Theorem 4.1 of [LP2] there is a (unique) isometry ${}^{ed}U : {}^d\mathcal{K} \to {}^e\mathcal{K}$ such that

$${}^{ed}U\,{}^dW = {}^eWI^{ed}; \quad {}^{ed}U\,{}^d\delta(a) = {}^e\delta(a); \quad {}^{ed}U\,{}^d\pi(a) = {}^e\pi(a)\,{}^{ed}U.$$

The identity $I^{fe}I^{ed} = I^{fd}$ for all $f \geq e \geq d$ leads to the consistency relation ${}^{fe}U{}^{ed}U = {}^{fd}U$, as can be seen by checking the actions on the dense set ${}^d\mathcal{K}_0$, and thus $(\{{}^d\mathcal{K}\}, \{{}^{ed}U\})$ is a directed system of Hilbert spaces ([KaR], p.886). Let $(\mathcal{K}, \{{}^dU\})$ be its inductive limit. For $1 \leq i \leq d$ let $I_i^d \in \mathcal{B}(\mathfrak{h}; \mathfrak{h}^d)$ denote the embedding of $\mathfrak{h}$ into the ith coordinate, that is $I_i^d u = (0, \ldots, 0, u, 0, \ldots, 0)$. Since ${}^eU\,{}^{ed}U = {}^dU$ for all $e \geq d$ the vector

$${}^dU\,{}^d\pi(a)[{}^d\delta(b)u + {}^d\pi(c)\,{}^dWI_i^d v], \tag{4.7}$$

is independent of the choice of d $(\geq i)$. It follows that

$$\delta(a) := {}^d U^d \delta(a) \text{ and } W_i := {}^d U^d W I_i^d \ (d \geq i),$$

define operators $\delta(a)$ and W_i in $\mathcal{B}(\mathfrak{h};\mathcal{K})$ and since by construction $\mathcal{K}_0 = \bigcup_{d\geq 1} {}^d U^d \mathcal{K}_0$ is dense in $\mathcal{K}$, and each ${}^d U$ is isometric, it also follows that each $a \in \mathcal{A}$ uniquely determines a bounded operator $\pi(a)$ on $\mathcal{K}$ through the identities

$$\pi(a)\,{}^d U = {}^d U\,{}^d\pi(a).$$

The map $a \mapsto \pi(a)$ defines a unital representation $(\pi, \mathcal{K})$ of $\mathcal{A}$, the map $a \mapsto \delta(a)$ defines a π-derivation, and the vector (4.7) equals

$$\pi(a)[\delta(b)u + \pi(c)W_i v].$$

The finite-dimensional structure (4.6) can now be rewritten in terms of the quadruple $\mathcal{Q} = (\pi, \mathcal{K}, \delta, \{W_i\})$, which evidently satisfies (4.1), (4.2) and (4.3). Thus (aii) holds and (bi) is proved.

Finally, start with a quadruple $\mathcal{Q}$ satisfying (4.1–4). Since $\partial\theta_0^0(\mathcal{A}\times\mathcal{A}) \subset \mathcal{A}$ it follows from Theorem 2.1 of [ChE] that δ is implemented by an element W_0 in the ultraweak closure of $\mathrm{Lin}\{\delta(a)b : a, b \in \mathcal{A}\}$, that is $\delta(a) = \pi(a)W_0 - W_0 a$. Manipulating (4.1) gives

$$(\delta(b_1)c_1)^*\pi(a)\delta(b_2)c_2, W_i^*\delta(b)c \in \mathcal{A} \quad \forall a, b_k, c_k \in \mathcal{A}$$

and so $W_0^*\pi(a)W_0, W_i^*W_0 \in \mathcal{A}''$. Define $\chi : \mathcal{A} \to \mathcal{A}''$ by $\chi(a) = \theta_0^0(a) - W_0^*\pi(a)W_0$. Then $\partial\chi = 0$, and so $\chi(a) = \alpha(a) + (ac + ca)$ for some real derivation $\alpha : \mathcal{A} \to \mathcal{A}''$ and $c = c^* \in \mathcal{A}''$ ([LP2], Proposition 1.2). Applying Theorem 2.1 of [ChE] once more produces an $h = h^* \in \mathcal{A}''$ such that $\alpha(a) = i(ha - ah)$. Putting $K_0 = c - ih$ and $K_j = \theta_j^0(1) - W_0^* W_j$ gives a quadruple $\mathcal{P} = (\pi, \mathcal{K}, \{K_\gamma\}, \{W_\gamma\})$ that satisfies $(4.1)'$ and $(4.3)'$, with $K_\gamma \in \mathcal{A}''$. Thus (aiii) holds too, and (b) is proved.

(c): Let $\mathcal{Q} = (\pi, \mathcal{K}, \delta, \{W_i\})$ be any quadruple satisfying (4.1–4), and let $\mathcal{Q}'$ be a similar quadruple satisfying (4.1). The identity

$$\begin{aligned}\langle \pi'(1)\delta'(a)u^0 + \pi'(b)W_i'u^i, \pi'(1)\delta'(c)v^0 + \pi'(d)W_j'v^j\rangle \\ = \langle u^0, \partial\theta_0^0(a,b)v^0\rangle + \langle u^i, [\theta_0^i(b^*c) - \theta_0^i(b^*)c]v^0\rangle \\ + \langle [\theta_0^j(d^*a) - \theta_0^j(d^*)a]u^0, v^j\rangle + \langle u^i, \theta_j^i(b^*d)v^j\rangle,\end{aligned}$$

for $(u^\gamma), (v^\gamma) \in \mathfrak{h}_{00}$, together with its unprimed version, and the totality condition (4.3), imply the existence of a unique linear isometry $V : \mathcal{K} \to \mathcal{K}'$ satisfying

$$V\{\delta(a)u^0 + \pi(b)W_j u^j\} = \pi'(1)\delta'(a)u^0 + \pi'(b)W_j'u^j.$$

The identities (4.4) follow easily, moreover any other isometry $V' \in \mathcal{B}(\mathcal{K};\mathcal{K}')$ satisfying (4.4) agrees with V on the dense subspace $\mathcal{K}_0$, and so $V = V'$. It is clear from the definition of V that it is unitary if $\mathcal{Q}'$ also satisfies (4.3–4), and for the converse note that the range of V lies in $\pi'(1)\mathcal{K}'$.

(d): Suppose that $\mathcal{A}$ is a von Neumann algebra and k is completely positive. If θ_α^α is ultraweakly continuous then, letting $\mathcal{P}$ be a quadruple as in (aiii), the estimate

$$|\langle u, W_\beta^*\pi(a)W_\alpha v\rangle|^2 \leq \|W_\beta u\|^2\langle v, W_\alpha^*\pi(a^*a)W_\alpha v\rangle$$

shows that θ_α^β is ultraweakly continuous moreover, by the reality of θ, θ_β^α is ultraweakly continuous too. Now suppose that each θ_β^α is ultraweakly continuous and let $\mathcal{Q}$ be a quadruple satisfying (4.1) and (4.3). To prove the normality of π it suffices to check that $\langle\xi, \pi(d_\lambda)\eta\rangle \to \langle\xi, \pi(d)\eta\rangle$ for any bounded increasing net (d_λ) in $\mathcal{A}_+$ with least upper bound d, and for all vectors $\xi, \eta \in \mathcal{K}_0$. By sesquilinearity we can write the inner product on the left hand side as a finite sum of terms of the following three types:

$$\langle \delta(a)u^0, \pi(d_\lambda)\delta(c)v^0\rangle = \langle u^0, [\theta^0_0(a^*d_\lambda c) - a^*\theta^0_0(d_\lambda c) - \theta^0_0(a^*d_\lambda)c + a^*\theta^0_0(d_\lambda)c]v^0\rangle,$$
$$\langle \pi(a)W_iu^i, \pi(d_\lambda)\delta(c)v^0\rangle = \langle u^i, [\theta^i_0(a^*d_\lambda c) - \theta^i_0(a^*d_\lambda)c]v^0\rangle,$$
$$\langle \pi(a)W_iu^i, \pi(d_\lambda)\pi(c)W_jv^j\rangle = \langle u^i, \widehat{\theta}^i_j(a^*d_\lambda c)v^j\rangle,$$

where the equalities here arise from the identities (4.1). Thus each θ^α_β being ultraweakly continuous entails the required convergence. □

Remark. The density assumption on $\mathcal{S}^{[d]}$ is used in the above proof to deduce the complete positivity of each $\mathcal{E}_{\mathbb{M}^{[d]}}$-process ${}^dk$. If however k is assumed to be a bounded process then the complete positivity of each ${}^dk$ may be proved by alternative means (as in the proof of Proposition 5.1 below). Thus the conclusion of the above result remains true under the alternative assumption that the flow k is a bounded process.

We shall refer to (4.1) (equivalently (4.1)′) as the *CP structure relations.*

Proposition 4.2. *Let* θ *and* $\mathcal{Q} = (\pi, \mathcal{K}, \delta, \{W_i\})$ *be as in* Theorem 4.1(aii) *and* (bi)*, and suppose that the Hilbert space* $\mathfrak{h}$ *is separable. If either*

(i) $\mathcal{A}$ *is a separable* C^**-algebra, or*
(ii) $\mathcal{A}$ *is a von Neumann algebra and each* θ^α_α *is ultraweakly continuous,*

then $\mathcal{K}$ *is separable.*

Proof. By [ChE], Theorem 2.1, $\delta(a) = \pi(a)W_0 - W_0a$ for some $W_0 \in \mathcal{B}(\mathfrak{h};\mathcal{K})$. For any subset $S \subset \mathcal{A}$ and $T \subset \mathfrak{h}$ put

$$\mathcal{K}_{S,T} = \{W_0u + \textstyle\sum_{\gamma=0}^n \pi(a_\gamma)W_\gamma v^\gamma : n \in \mathbb{N}, u, v^\gamma \in T, a_\gamma \in S\},$$

then since the quadruple $\mathcal{Q}$ satisfies the minimality condition (4.3) we see that $\mathcal{K}_{\mathcal{A},\mathfrak{h}}$ is total for $\mathcal{K}$. Now let T be a countable dense subset of $\mathfrak{h}$ and let S be any countable subset of $\mathcal{A}$ (so that $\mathcal{K}_{S,T}$ is countable). If $\pi(S)$ is strongly dense in $\pi(\mathcal{A})$ then it follows that $\overline{\mathrm{Lin}}\mathcal{K}_{S,T} = \mathcal{K}$, in particular $\mathcal{K}$ is separable. Case (i) is immediate.

Case (ii): The closed unit ball of $\mathcal{A}$ is metrisable ([Tak], Proposition II.2.7) and compact in the ultraweak topology, and so has a countable ultraweakly dense subset S_0. Let S be the algebra of polynomials over $S_0 \cup S_0^*$ with complex-rational coefficients, then S is countable and ultraweakly dense in $\mathcal{A}$. Since π is normal (Theorem 4.1(d)), $\pi(S)$ is ultraweakly dense in the algebra $\pi(\mathcal{A})$, and so also strongly dense ([Tak]). □

5. CP Contraction Flows

In this section we characterise the generators of completely positive contraction flows. From this characterisation, and the general results of Sections 1 and 3, we obtain existence theorems when $\mathcal{A}$ is a von Neumann algebra.

Consider the following condition on a mapping matrix θ:

$$\theta^{[d]}(1) \le 0 \quad \forall d. \tag{5.1}$$

Anticipating the next result we shall refer to these as the *CP contractivity relations.* When θ is bounded they are equivalent to

$$\theta(1) \le 0. \tag{5.1$'$}$$

Proposition 5.1. *Let* $\theta \in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))$ *and suppose that* θ *weakly generates an* $\mathcal{E}_{(\mathcal{S},\mathcal{S})}$*-flow* k*, where* $\mathcal{S}$ *is fully admissible.*

(a) k *is unital if and only if* $\theta(1) = 0$.
(b) k *is a completely positive contraction process if and only if* θ *satisfies* (aii) (*equivalently* (aiii)) *of* Theorem 4.1 *and* (5.1).

Proof. We adopt the notations $\theta^{[d]}$, J_d and ${}^d\!k$ from the proof of Theorem 4.1, and first show that k is a unital, contraction or CP contraction process if and only if each ${}^d\!k$ is. By (4.5b), if each ${}^d\!k$ is unital, or a contraction, then k_t has that property. Conversely, by (4.5a) and [LP2], Theorem 3.1(c),

$$J_d^* k_t(a) J_d \subset {}^d\!k_t(a) \subset {}^d\!j_t(a^*)^* \tag{5.2}$$

where ${}^d\!j = k^\psi$ for $\psi = \theta^{[d]\dagger} = \theta^{\dagger[d]}$. Thus if k is unital, then

$$1|_{\mathfrak{h}\odot\mathcal{E}_{\mathcal{S}^{[d]}}} \subset {}^d\!k_t(1) \subset {}^d\!j_t(1)^*.$$

Since ${}^d\!j_t(1)^*$ is closed, densely defined and agrees with the identity on a dense subspace of its domain, it equals the identity. Thus ${}^d\!k$ is unital. If on the other hand k is a CP contraction process, then (5.2) implies that ${}^d\!j_t(a^*)^*$ is bounded with norm at most $\|a\|$, and so ${}^d\!k$ is a contraction process; if furthermore k is completely positive, then (5.2) implies that ${}^d\!k$ is also completely positive, by continuity.

Since $\theta(1) = 0$ if and only if $\theta^{[d]}(1) = 0$ for each d, (a) now follows from its finite dimensional counterpart ([LP2], Theorem 5.1). If k is a CP contraction process, then each ${}^d\!k$ is a CP contraction process and the proof of (ai $\Rightarrow$ aii) in Theorem 4.1 is valid (see the remark after the proof), moreover θ satisfies (5.1) by [LP2], Theorem 5.1(e). Conversely, if θ satisfies (aii) or (aiii) of Theorem 4.1 and (5.1), then $\theta^{[d]}$ satisfies the conditions for ${}^d\!k$ to be a CP contraction process (by [LP2]), and so k is a CP contraction process too. □

Remark. In view of the identity

$$\langle \xi, k_t(a)\xi\rangle = \langle \xi, (a\otimes 1)\xi\rangle + \int_0^t \langle x(s), k_s^{(d+1)}(\theta^{[d]}(a))x(s)\rangle\, ds \tag{5.3}$$

in which $\xi = \sum_{p=1}^n u_p \varepsilon(f_{(p)}) \in \mathfrak{h}\odot\mathcal{E}_{\mathcal{S}}$ and $x^\alpha(s) = \sum_{p=1}^n f^\alpha_{(p)}(s) u_p \varepsilon(f_{(p)}) \in \mathcal{H}$, some of the implications in the above result require neither full admissibility of $\mathcal{S}$ nor weak regularity of the solution k.

Recall the identifications (1.3) and embeddings (1.4).

Theorem 5.2. *Let $\theta \in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))$ and suppose that θ satisfies the CP contractivity relations* (5.1)*. The following are equivalent:*

(i) *θ satisfies the CP structure relations* (4.1)$'$.
(ii) *θ is bounded, and there is a completely positive map $\psi : \mathcal{A} \to \mathcal{B}(\mathfrak{h}\otimes\widehat{\mathsf{k}})$ and an operator $K \in \mathcal{B}(\mathfrak{h}\otimes\widehat{\mathsf{k}};\mathfrak{h})$ such that*

$$(\widehat{\theta} - \psi)(a) = K^* a E^0 + E_0 a K.$$

The pair (ψ, K) may be chosen so that $KE_e \in \mathcal{A}''$ for all $e \in \widehat{\mathsf{k}}$.

Proof. (ii $\Rightarrow$ i) This is an immediate consequence of the Stinespring decomposition of completely positive maps, and Theorem 4.1.

(i $\Rightarrow$ ii) Let $\mathcal{P}$ be as in Theorem 4.1(aiii) with π unital and $K_\gamma \in \mathcal{A}''$, let $(u^\gamma) \in \mathfrak{h}_{00}$, and pick d such that $u^\gamma = 0$ for all $\gamma > d$. Let $\mathbf{u}$ denote the vector $(u^1, \dots, u^d) \in \mathfrak{h}^d$, and T the bounded operator on $\mathfrak{h}^d$ obtained by deleting the first row and column of the cut-off matrix $\widehat{\theta}^{[d]}(1)$. By (5.1) we have $T \leq 1$, and thus

$$\begin{aligned}\|W_\gamma u^\gamma\|^2 \leq 2\|W_0 u^0\|^2 + 2\|W_i u^i\|^2 &= 2\|W_0 u^0\|^2 + 2\langle \mathbf{u}, T\mathbf{u}\rangle \\ &\leq 2\|W_0\|^2\|u^0\|^2 + 2\|\mathbf{u}\|^2,\end{aligned}$$

so there is a bounded operator $S : \oplus_{\gamma\geq 0}\mathfrak{h} \to \mathcal{K}$ satisfying $S(u^\gamma) = W_\gamma u^\gamma$. Now

$$\begin{aligned}\|\mathbf{u}\|^2 = \langle (u^\alpha), \Delta(1)(u^\beta)\rangle \geq & \langle (u^\alpha), [\widehat{\theta}^\alpha_\beta(1)](u^\beta)\rangle \\ = & \langle u^\alpha, W^*_\alpha W_\beta u^\beta\rangle + \langle u^0, K_\beta u^\beta\rangle + \langle K_\alpha u^\alpha, u^0\rangle \\ = & \|S(u^\gamma)\|^2 + 2\mathrm{Re}\langle u^0, K_\beta u^\beta\rangle,\end{aligned}$$

and so it follows that

$$2|\langle u^0, K_i u^i\rangle| \leq 2\|K_0\|\|u^0\|^2 + \|\mathbf{u}\|^2.$$

Thus there is a bounded operator $K : \oplus_{\gamma\geq 0}\mathfrak{h} \to \mathfrak{h}$ such that $K(u^\gamma) = K_\gamma u^\gamma$, in other words $K_\gamma = KE_\gamma$. Therefore in (4.1)′ ψ and ϕ are bounded with $\psi(a) = S^*\pi(a)S$ and $\phi(a) = E_0 aK + K^* aE^0$. □

Specialising to the case where $\mathcal{A}$ is a von Neumann algebra we have the following characterisation and existence result.

Theorem 5.3. *Let $\theta \in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))$. If $\mathcal{A}$ is a von Neumann algebra then the following are equivalent:*

(i) *θ satisfies* (4.1) *and* (5.1)*, and each θ^α_α is ultraweakly continuous.*

(ii) *There is a Hilbert space $\mathcal{L}$ and operators $S \in \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}}; \mathfrak{h} \otimes \mathcal{L})$ and $K \in \mathcal{B}(\mathfrak{h} \otimes \widehat{\mathsf{k}}; \mathfrak{h})$ such that*

$$\widehat{\theta}(a) = S^*(a \otimes 1_{\mathcal{L}})S + K^* aE^0 + E_0 aK.$$

where $S^*S + K^*E^0 + E_0K \leq \Delta(1)$.

(iii) *θ weakly generates a normal CP contraction $\mathcal{E}_{(\mathcal{S},\mathcal{S})}$-flow k, for some fully admissible set $\mathcal{S}$.*

In this case, θ is M-S regular and k^θ is a normal CP contraction process extending k.

Proof. If (i) holds then, by Theorem 5.2 and Proposition 1.4(d), (c) and (b), θ is bounded, ultraweakly continuous and M-S regular, and so k^θ is a normal CP contraction process by Propositions 3.2(c) and 5.1 — in particular (iii) holds. If (iii) holds then, in view of Proposition 5.1, Theorem 5.2 and Proposition 3.2(c), (ii) follows from the Stinespring representation of normal CP maps, and the standard form for normal representations of a von Neumann algebra ([Tak], Theorem IV.5.5). Finally (i) easily follows from (ii). □

Applying Corollary 1.5 we have

Corollary 5.4. *Let $\theta \in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))$. If $\mathcal{A}$ is finite dimensional then the following are equivalent:*

(i) *θ satisfies* (4.1) *and* (5.1).

(ii) *θ weakly generates a CP contraction $\mathcal{E}_{(\mathcal{S},\mathcal{S})}$-flow k, for some fully admissible set $\mathcal{S}$.*

In this case θ is completely regular, and k^θ is a normal CP contraction process extending k.

6. *-homomorphic Flows

In this section we give a proof of the Mohari-Sinha existence theorem for *-homomorphic flows, and also give a converse result which confirms the global form of the structure relations of *-homomorphic flows. By Lemma 2.3 *-homomorphic processes are necessarily contraction processes as well as being completely positive. Proposition 5.1 and Theorem 5.2 therefore combine to give the following result which was not readily apparent from the analysis of Mohari and Sinha.

Proposition 6.1. *Suppose that $\theta \in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))$ weakly generates an $\mathcal{E}_{(\mathcal{S},\mathcal{S})}$-flow k for some fully admissible set $\mathcal{S}$. If k is a (not necessarily unital) *-homomorphic process then θ is bounded.*

The *structure relations for weak multiplicativity* are

$$\theta^\alpha_i(a)\theta^i_\beta(b) = \theta^\alpha_\beta(ab) - \theta^\alpha_\beta(a)b - a\theta^\alpha_\beta(b), \tag{6.1}$$

with weak convergence in the sum on the left hand side. When each of the operator matrices $\theta^\dagger(a)$ and $\theta(b)$ is regular, this may be written in the form

$$\theta^\alpha_\beta(ab) = \theta^\alpha_\beta(a)b + a\theta^\alpha_\beta(b) + \theta^\dagger_\alpha(a^*)^*\Delta(1)\theta_\beta(b). \tag{6.1$'$}$$

where Δ is defined in (1.15).

The following can be proved exactly as in [MoS] or [Mey], noting as in Meyer's account that regularity of θ and $\theta^\dagger$ is a sufficiently strong condition to derive the required estimates. However at a crucial step in the proof each iterate $k^{\{n\}}$ is required to be sequentially strongly continuous, and for this reason the condition of Mohari-Sinha regularity is imposed on θ (see Proposition 3.4). This continuity requirement appears to have been overlooked in Meyer's exposition.

Theorem 6.2. *Let $\theta \in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{MS}}$. If $\theta^\dagger \in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{R}}$ and θ satisfies* (6.1) *then k^θ is weakly multiplicative.*

Here is a converse of the Mohari-Sinha theorem.

Proposition 6.3. *Let $\theta \in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))$ be such that both θ and $\theta^\dagger$ are regular. If k^θ is weakly multiplicative then θ satisfies* (6.1).

Proof. Write k and $k^\dagger$ for k^ϕ where ϕ equals θ and $\theta^\dagger$ respectively. Let $f = w^i e_i \mathbf{1}_{[0,1]}$ and $g = z^i e_i \mathbf{1}_{[0,1]}$ for $w, z \in c_{00}(\mathbb{N})$ then, for $t \in [0,1]$, the fundamental formulae imply that

$$\begin{aligned} 0 =& \langle k^\dagger_t(a)u\varepsilon(f), k_t(b)v\varepsilon(g)\rangle - \langle u\varepsilon(f), k_t(a^*b)v\varepsilon(g)\rangle \\ =& \int_0^t w_\alpha z^\beta \big\{ \langle u\varepsilon(f), k_s(a^*\theta^\alpha_\beta(b) + \theta^\alpha_\beta(a^*)b - \theta^\alpha_\beta(a^*b))v\varepsilon(g)\rangle \\ &+ \textstyle\sum_{i\geq 1} \langle k^\dagger_s({\theta^\dagger}^i_\alpha(a))u\varepsilon(f), k_s(\theta^i_\beta(b))v\varepsilon(g)\rangle \big\}\, ds \end{aligned}$$

By (3.9) the convergence in the infinite sum is uniform on compact intervals of time, therefore the integrand is continuous. Evaluating the integrand at $t = 0$ and then varying w and z gives (6.1), since the operator matrices $\theta^\dagger(a)$ and $\theta(b)$ are regular. □

The **-homomorphic structure relations* are:

$$\theta^i_\alpha(a)^*\theta^i_\beta(a) = \theta^\alpha_\beta(a^*a) - a^*\theta^\alpha_\beta(a) - \theta^\beta_\alpha(a)^*a \tag{6.2}$$

with strong convergence in the sum on the left hand side. When θ is bounded they have the global form

$$\theta(a^*a) = \theta(a^*)\iota(a) + \iota(a^*)\theta(a) + \theta(a)^*\Delta(1)\theta(a) \tag{6.2$'$}$$

where ι and Δ were defined in (1.15). Recall the identifications (1.3).

Lemma 6.4. *Let $\theta \in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))$ satisfy* (6.2). *Then*

(a) *θ is bounded and $\widehat{\theta}$ has block matrix form*

$$\widehat{\theta}(a) = \begin{bmatrix} L^*\nu(a)L - \frac{1}{2}\{L^*L, a\} + i[h,a] & L^*\nu(a) - aL^* \\ \nu(a)L - La & \nu(a) \end{bmatrix} \tag{6.3}$$

where ν is a representation of $\mathcal{A}$ on $\oplus_{i\geq 1}\mathfrak{h} = \mathfrak{h} \otimes \mathsf{k}$, L is an element of $\mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k})$ such that $E^e L \in \mathcal{A}''$ for all $e \in \mathsf{k}$, and $h = h^*$ belongs to $\mathcal{A}''$.

(b) *If $\mathcal{A}$ is a von Neumann algebra and each θ^i_i is ultraweakly continuous, then ν is normal.*

Proof. First note that the strong convergence in (6.2) implies that each operator matrix $\theta(a)$ is regular; (6.2) also implies that θ is real. Thus $\widehat{\theta}$ has block matrix form

$$\widehat{\theta} = \begin{bmatrix} \tau & \chi^\dagger \\ \chi & \nu \end{bmatrix},$$

in which $\tau \in \mathcal{B}(\mathcal{A})$ and ν are real, and $\chi \in \mathcal{B}(\mathcal{A}; \mathcal{B}(\mathfrak{h}; \mathfrak{h} \otimes \mathsf{k}))$ is bounded. We must first show that ν is bounded. (6.2) gives the following identities:

$$\nu^i_j(a^*b) = \nu^l_i(a)^* \nu^l_j(b), \tag{6.4}$$

$$\chi^i(a^*b) = \chi^i(a^*)b + \nu^i_l(a^*)\chi^l(b), \;\; \tau(a^*b) - \tau(a)^*b - a^*\tau(b) = \chi(a)^*\chi(b). \tag{6.5}$$

Thus, for $x \in \mathfrak{h} \otimes \mathsf{k} = \oplus_{i \geq 1} \mathfrak{h}$, with components eventually zero,

$$\langle x, \nu(a^*a)x \rangle = \|\nu(a)x\|^2 \geq 0 \tag{6.4$'$}$$

$$\|\nu(1)x\|^2 = \langle x, \nu(1)x \rangle \leq \|x\| \|\nu(1)x\|.$$

Therefore $\nu(1)$ is bounded and ν is 'order preserving,' so that

$$\|\nu(a)x\|^2 = \langle x, \nu(a^*a)x \rangle \leq \langle x, \nu(\|a\|^2 1)x \rangle \leq \|a\|^2 \|x\|^2.$$

Thus ν is bounded; since it is real (6.4)$'$ implies that it is *-homomorphic, and from (6.5) χ is a ν-derivation satisfying $\chi(a)^*\chi(b) \in \mathcal{A}$. By [ChE], Theorem 2.1, χ is of the stated form. Since any real map τ satisfying (6.5) differs from the particular solution $\tau_0 : a \mapsto L^*\nu(a)L - \frac{1}{2}\{L^*L, a\}$ by a bounded real derivation $\mathcal{A} \to \mathcal{A}'' \subset \mathcal{B}(\mathfrak{h})$, another application of [ChE], Theorem 2.1 implies that τ is of the advertised form too.

(b) This follows from Proposition 1.4. □

Combining the results we conclude with

Theorem 6.5. *Let $\theta \in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))$.*

(a) *If θ satisfies* (6.2) *then θ is bounded, and has the form* (6.3).

(b) (i) *If θ satisfies* (6.2) *and is M-S regular, then k^θ is *-homomorphic.*
(ii) *If θ is regular and k^θ is *-homomorphic, then θ satisfies* (6.2).

(c) *Suppose that $\mathcal{A}$ is a von Neumann algebra and each θ^i_i is ultraweakly continuous.*
(iii) *If θ satisfies* (6.2) *then θ is completely regular.*
In this case θ is ultraweakly continuous, and k^θ is normal.

7. H-P Processes

In this section we apply the general machinery developed for QS flows to give an alternative approach that both strengthens and clarifies the results of Fagnola and Mohari ([Fag],[Moh]) concerning the existence, contractivity, isometry and coisometry of solutions to the *right* and *left* linear QSDE's

$$dX_t = F^\alpha_\beta X_t \, d\Lambda^\beta_\alpha(t), \quad X_0 = 1, \tag{7.1}$$

$$dY_t = Y_t F^\alpha_\beta \, d\Lambda^\beta_\alpha(t), \quad Y_0 = 1, \tag{7.2}$$

where $F \in \mathrm{M}_\infty(\mathcal{B}(\mathfrak{h}))$ and the solutions are operator-valued processes. The semigroup representation of QSDE solutions provides a straightforward confirmation that solutions of (7.1) and (7.2) are interchanged by a time reversal operation on processes, greatly simplifying the proofs of Mohari and Parthasarathy ([Moh], [MoP]).

Existence and Uniqueness. Recall the regularity conditions (2.5) and (2.6) for processes on $\mathfrak{h}$. The definitions of $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-weak and $\mathcal{E}_{\mathcal{S}}$-strong solutions to the process equations are analogous to those for the flow equation.

Recall also the definitions of regularity for an operator matrix F, and the associated operators F_β (1.6).

Theorem 7.1. *Let $F \in \mathrm{M}_\infty(\mathcal{B}(\mathfrak{h}))$ and let $\mathcal{S}$ and $\mathcal{S}'$ be admissible sets.*

(a) *The* QSDE*'s* (7.1) *and* (7.2) *each have at most one $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-regular $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-weak solution.*

(b) *If $Z \in \mathbb{P}(\mathfrak{h}, \mathcal{E}_{\mathcal{S}})$ and $\mathcal{S} \cup \mathcal{S}' \subset \mathbb{S}$ then the following are equivalent:*
  (i) *Z is an $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-regular $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-weak solution of* (7.1)*.*
  (ii) *For $f \in \mathcal{S}', g \in \mathcal{S}$ with discontinuities contained in the set $\{0 = t_0 < t_1 < \cdots t_n < t_{n+1} = \infty\}$,*
$$\langle u\varepsilon(f), Z_t v\varepsilon(g)\rangle = \langle u, P^{[i]}_{t-t_i} \cdots P^{[0]}_{t_1-t_0} v\rangle \langle \varepsilon(f), \varepsilon(g)\rangle, \tag{7.3}$$
  for $t \in [t_i, t_{i+1}[$, where $P^{[k]}$ is the one-parameter semigroup on $\mathfrak{h}$ with generator $F_{w,z}$ where $w = f(t_k)$ and $z = g(t_k)$.

Similarly for (7.2) *with* (7.3) *replaced by*
$$\langle u\varepsilon(f), Z_t v\varepsilon(g)\rangle = \langle u, P^{[0]}_{t_1-t_0} \cdots P^{[i]}_{t-t_i} v\rangle \langle \varepsilon(f), \varepsilon(g)\rangle.$$

(c) *Let Z be an $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-regular $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-weak solution of* (7.1)*, then the following are equivalent:*
  (i) *$Z \in \mathbb{P}^{\ddagger}(\mathfrak{h}, \mathcal{E}_{(\mathcal{S}',\mathcal{S})})$.*
  (ii) *$dW_t = W_t (F^\beta_\alpha)^* \, d\Lambda^\beta_\alpha(t)$ has an $\mathcal{E}_{(\mathcal{S},\mathcal{S}')}$-regular $\mathcal{E}_{(\mathcal{S},\mathcal{S}')}$-weak solution.*

In this case $W = Z^\dagger$. Similarly for (7.2)*, with* (ii) *replaced by $dW_t = (F^\beta_\alpha)^* W_t \, d\Lambda^\beta_\alpha(t)$.*

(d) *If $F \in \mathrm{M}_\infty(\mathcal{B}(\mathfrak{h}))_{\mathrm{R}}$ then* (7.1) *and* (7.2) *have $\mathcal{E}_{\mathbb{M}}$-regular $\mathcal{E}_{\mathbb{M}}$-strong solutions.*

Proof. Let $\mathcal{A} = \mathcal{B}(\mathfrak{h})$, the full algebra of all bounded operators on $\mathfrak{h}$, and let ${}^1\theta, {}^2\theta \in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))$ be the mapping matrices defined by
$${}^1\theta^\alpha_\beta(a) = aF^\alpha_\beta; \quad {}^2\theta^\alpha_\beta(a) = F^\alpha_\beta a.$$
There is a bijective correspondence between $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-regular $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-weak solutions of (7.1) and $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-regular $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-weak solutions of (3.1) with $\theta = {}^1\theta$, given by
$${}^1k_t(a) = aX_t; \quad X_t = {}^1k_t(1).$$
In one direction this is an immediate consequence of the integral identity (3.2b) and its equivalent form for processes on $\mathfrak{h}$. In the other direction applying (3.2b) to the difference
$$\langle u\varepsilon(f), {}^1k_t(ab)v\varepsilon(g)\rangle - \langle u\varepsilon(f), a\,{}^1k_t(b)v\varepsilon(g)$$
and iterating shows that the $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-regular $\mathcal{E}_{(\mathcal{S}',\mathcal{S})}$-weak solution ${}^1k$ must satisfy ${}^1k_t(a) = a\,{}^1k_t(1)$. Note that in this correspondence $aF^\alpha_\beta X_t = {}^1k_t({}^1\theta^\alpha_\beta(a))$, so the process matrix $[F^\alpha_\beta X]$ is stochastically integrable if and only if each of the process matrices $[{}^1k({}^1\theta^\alpha_\beta(a))]$ are $(a \in \mathcal{B}(\mathfrak{h}))$, and therefore ${}^1k$ is a strong solution if and only if X is. Now, for all $w, z \in c_{00}(\mathbb{N})$, let ${}^1\mathcal{P}^{w,z}$ (respectively $P^{w,z}$) be the one-parameter semigroup on $\mathcal{A}$ with generator ${}^1\theta_{w,z}$ (resp. on $\mathfrak{h}$ with generator $F_{w,z}$), and define ${}^2\mathcal{P}_{w,z}$ similarly. Then
$${}^1\mathcal{P}^{w,z}_t(a) = aP^{w,z}_t, \text{ and } {}^2\mathcal{P}^{w,z}_t(a) = P^{w,z}_t a, \tag{7.4}$$
and when $F \in \mathrm{M}_\infty(\mathcal{B}(\mathfrak{h}))_{\mathrm{R}}$,
$$\sum_{\alpha \geq 0} {}^1\theta^\alpha_\beta(a)^* {}^1\theta^\alpha_\beta(a) = \sum_{\alpha \geq 0} (F^\alpha_\beta)^* a^* a F^\alpha_\beta = F^*_\beta(a^*a \otimes 1)F_\beta, \tag{7.5}$$

so ${}^1\theta \in \mathrm{M}_\infty(\mathcal{B}(\mathcal{A}))_{\mathrm{MS}}$. Half of the parts (a), (b) and (d) therefore follow from Theorems 3.1 and 3.3.

The other halves follow by repeating the above argument, starting from the correspondence between solutions of (7.2) and solutions of (3.1) with $\theta = {}^2\theta$ (${}^2k(a) = Ya; Y = {}^2k(1)$), using (7.4), and replacing (7.5) with

$$\sum_{\alpha \geq 0} {}^2\theta^\alpha_\beta(a)^* \, {}^2\theta^\alpha_\beta(a) \leq \|F_\beta\|^2 a^* a.$$

These two correspondences now combine with Proposition 3.2 to prove (c). □

Remarks. (i) For contraction processes, considerable improvement will be revealed in Proposition 7.6.

(ii) The regularity assumption is not used for (ci ⇒ cii).

We denote the $\mathcal{E}_{\mathbb{M}}$-regular solutions of (7.1) and (7.2) obtained in Theorem 7.1(d) for a regular operator matrix F by ${}^F X$ and X^F respectively. By (c), if F and $F^\dagger$ are both regular then X^G and ${}^G X$, with $G = F^\dagger$, are restrictions to $\mathfrak{h} \odot \mathcal{E}_{\mathbb{M}}$ of $({}^F X)^*$ and $(X^F)^*$ respectively.

Time Reversal. For each $t \geq 0$ let r_t be the operator on $L^2(\mathbb{R}_+; \mathsf{k})$ given by

$$(r_t f)(s) = \begin{cases} f(t-s), & s \leq t \\ f(s), & s > t \end{cases},$$

and let R_t be its second quantisation; thus $R_t u\varepsilon(f) = u\varepsilon(r_t f)$. The self-adjoint unitary process $R = (R_t)_{t \geq 0}$ is called the *time reflection process.* For any $\mathcal{E}_{\mathcal{S}}$-process Z, if the admissible set $\mathcal{S}$ is invariant under each map r_t, then

$$\widetilde{Z}_t := R_t Z_t R_t$$

defines another $\mathcal{E}_{\mathcal{S}}$-process $\widetilde{Z}$, which we shall call the *time reversed process* of Z. Obviously time reversal is involutive: if $Y = \widetilde{X}$ then $\widetilde{Y} = X$. We next show how time reversal interchanges solutions of (7.1) and (7.2).

Theorem 7.2. *Let $F \in \mathrm{M}_\infty(\mathcal{B}(\mathfrak{h}))$.*

(a) *For an $\mathcal{E}_{(\mathbb{S},\mathbb{S})}$-regular $\mathcal{E}_{\mathbb{S}}$-process Z the following are equivalent:*
  (i) *Z is an $\mathcal{E}_{(\mathbb{S},\mathbb{S})}$-weak solution of* (7.1).
  (ii) *$\widetilde{Z}$ is an $\mathcal{E}_{(\mathbb{S},\mathbb{S})}$-weak solution of* (7.2).

(b) *If $F \in \mathrm{M}_\infty(\mathcal{B}(\mathfrak{h}))_{\mathrm{R}}$ then $X^F = \widetilde{{}^F X}$.*

Proof. (a) Fix $f, g \in \mathbb{S}$ and let $\{0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = \infty\}$ contain their points of discontinuity. Let Z be an $\mathcal{E}_{(\mathbb{S},\mathbb{S})}$-regular $\mathcal{E}_{\mathbb{S}}$-weak solution of (7.1). Bearing in mind that the semigroup representation refers to right continuous versions of the test functions, so that for $t \in [t_i, t_{i+1}[$, $r_t f$ and $r_t g$ must be considered constant on the intervals of the form $[0, t - t_i[, [t - t_i, t - t_{i-1}[, \ldots, [t - t_1, t[$, (7.3) gives

$$\langle u\varepsilon(r_t f), Z_t v\varepsilon(r_t g)\rangle = \langle u, P^{[0]}_{t_1 - t_0} \cdots P^{[i]}_{t - t_i} v\rangle \langle \varepsilon(r_t f), \varepsilon(r_t g)\rangle.$$

where $P^{[k]}$ is as in Theorem 7.1. Since r_t is unitary, Theorem 7.1(b) implies that $\widetilde{Z}$ is an $\mathcal{E}_{(\mathbb{S},\mathbb{S})}$-weak solution of (7.2). The converse is established by a symmetrical argument.

(b) By (a), and Theorem 7.1(a), X^F and $\widetilde{{}^F X}$ agree on $\mathfrak{h} \odot \mathcal{E}_{\mathbb{S}}$. Their equality on $\mathfrak{h} \odot \mathcal{E}_{\mathbb{M}}$ follows from Proposition 3.5. □

Remarks. (i) When Z is an $\mathcal{E}_{(\mathbb{S},\mathbb{S})}$-weak solution of (7.1), $\widetilde{Z}$ is also an $\mathcal{E}_{(\mathbb{S},\mathbb{S})}$-weak solution of a time reversed version of the differential equation (7.1). This is better described by means of two time parameters (see [LW2]).

(ii) The time reversed process is different from Journé's *dual process* ([Jou]). For a bounded operator process V, the Journé dual is given by $R_t V_t^* R_t$; it is therefore the *adjoint* process of our time reversed process.

Conjugation. For a regular operator matrix F we may define a mapping matrix ${}^3\theta$ on $\mathcal{A} = \mathcal{B}(\mathfrak{h})$ by

$$\begin{aligned} {}^3\theta^\alpha_\beta(a) =& aF^\alpha_\beta + (F^\beta_\alpha)^* a + (F^i_\alpha)^* a F^i_\beta \\ =& (F^\beta_\alpha)^* a + aF^\alpha_\beta + (F_\alpha)^* \Delta(a) F_\beta. \end{aligned} \tag{7.6}$$

When F is bounded, ${}^3\theta$ is given by

$$ {}^3\theta(a) = F^* \iota(a) + \iota(a) F + F^* \Delta(a) F. \qquad (7.6)'$$

Proposition 7.3. *Let $F \in \mathrm{M}_\infty(\mathcal{B}(\mathfrak{h}))_{\mathrm{R}}$ and let ${}^3\theta$ be defined by* (7.6). *Then*

(a) *${}^3\theta$ satisfies the CP structure equations* (4.1)$'$.
(b) *${}^3\theta$ is bounded if and only if F is bounded.*
(c) *If F is bounded then ${}^3\theta$ is completely regular.*

Proof. (a) Defining $(\pi, \mathcal{K}), \{W_\gamma\} \subset \mathcal{B}(\mathfrak{h}; \mathcal{K})$ and $\{K_\gamma\} \subset \mathcal{B}(\mathfrak{h})$ by

$$\mathcal{K} = \oplus_{i \geq 1} \mathfrak{h} = \mathfrak{h} \otimes \mathsf{k}, \;\; \pi(a) = P\Delta(a)|_{\mathcal{K}}, \;\; W_\gamma = P\widehat{F}_\gamma \text{ and } K_\gamma = F^0_\gamma$$

where $P = \Delta(1)$, the orthogonal projection of $\oplus_{\gamma \geq 0} \mathfrak{h} = \mathfrak{h} \oplus \mathcal{K}$ onto $\mathcal{K}$, it is easily verified that (4.1)$'$ holds.

(b) If ${}^3\theta$ is bounded then let $G = {}^3\theta(1)$ and let $P_0 = P^\perp$, the orthogonal projection of $\mathfrak{h} \oplus (\mathfrak{h} \otimes \mathsf{k})$ onto $\mathfrak{h}$. By the regularity of F, FP_0 is everywhere defined and bounded. Thus $F = FP_0 + PFP + P_0FP$ where F, PFP and P_0FP all have domain $\mathfrak{h}_{00}$, and we have

$$\langle x, Gy \rangle = \langle Fx, y \rangle + \langle x, Fy \rangle + \langle PFx, PFy \rangle \;\; \forall x, y \in \mathfrak{h}_{00} \tag{7.7}$$

Let $w, z \in \mathfrak{h}_{00}$. Putting $x = y = Pw$, (7.7) implies that

$$\|PFPw\|^2 \leq \|G\| \|w\|^2 + 2\|PFPw\| \|w\|,$$

and so PFP is bounded. Putting $x = P_0 w$ and $y = Pz$, (7.7) implies

$$|\langle w, P_0FPz \rangle| \leq \{\|G\| + \|FP_0\| + \|FP_0\| \|PFP\|\} \|w\| \|z\|,$$

and so P_0FP is bounded too. Thus F is bounded.

(c) If F is bounded then it is clear from (7.6)$'$ that ${}^3\theta$ is completely bounded and ultraweakly continuous, and so completely regular by Proposition 1.4. □

The relevance of the mapping matrix ${}^3\theta$ is that under suitable circumstances it generates the flow $X_t^* a X_t$ where $X = {}^F X$. We note one consequence of the second fundamental formula (2.9): for $\xi = \sum_p u_p \varepsilon(f_{(p)})$,

$$\begin{aligned} \langle X_t \xi, a X_t \xi \rangle =& \langle {}^1k_t(1)\xi, {}^1k_t(a)\xi \rangle \\ =& \langle \xi, a\xi \rangle + \int_0^t \langle x(s), {}^3\theta(a) x(s) \rangle \, ds \end{aligned}$$

where x is the $\mathcal{H}_{00} := \mathcal{H} \odot \mathrm{Lin}\{e_\gamma\}$-valued function given by

$$x^\alpha(s) = \textstyle\sum_p f^\alpha_{(p)}(s) X_s u_p \varepsilon(f_{(p)})$$

and ${}^3\theta$ is identified with its algebraic amplification.

Theorem 7.4. *Let $F \in \mathrm{M}_\infty(\mathcal{B}(\mathfrak{h}))_{\mathrm{R}}$, let $\mathcal{S}_1$ and $\mathcal{S}_2$ be admissible sets, and let ${}^3\theta$ be defined by* (7.6)*. Then, putting $X^{(i)}_t = {}^F X_t|_{\mathfrak{h} \odot \mathcal{E}_{\mathcal{S}_i}}$, the following are equivalent:*

(i) *${}^3\theta$ weakly generates an $\mathcal{E}_{(\mathcal{S}_2, \mathcal{S}_1)}$-flow ${}^3k$.*
(ii) $\mathrm{Ran}\, a X^{(1)}_t \subset \mathrm{Dom}\, X^{(2)*}_t \;\; \forall a, t.$

In this case, ${}^3k_t(a) = X^{(2)*}_t a X^{(1)}_t$.

Proof. (i $\Rightarrow$ ii) Let $T > 0, f \in \mathcal{S}_2, g \in \mathcal{S}_1, u, v \in \mathfrak{h}$ and $a \in \mathcal{A}$. Put $d = \max\{\dim f, \dim g\}$ and define maps $\Gamma_t : \mathcal{A} \to \mathbb{C}, t \in [0,T]$ by

$$\Gamma_t(a) = \langle u\varepsilon(f), {}^3k_t(a)v\varepsilon(g)\rangle - \langle X_t^{(2)}u\varepsilon(f), aX_t^{(1)}v\varepsilon(g)\rangle$$

Applying (3.2b) and (2.9) gives

$$\Gamma_t(a) = \int_0^t f_\alpha(s)g^\beta(s)\Gamma_s({}^3\theta^\alpha_\beta(a))\,ds.$$

Iterating this and using the $\mathcal{E}_{\mathbb{M}}$-regularity of X and the $\mathcal{E}_{(\mathcal{S}_2,\mathcal{S}_1)}$-regularity of ${}^3k$ gives a constant C such that, for $t \le T$,

$$|\Gamma_t(a)| \le C\|u\|\|a\|\|v\|(n!)^{-1}(\|p_{[0,t]}\|_1)^n$$

where p is the locally integrable function used in (3.11). Thus Γ is identically zero, and so (ii) holds.

(ii $\Rightarrow$ i) Putting ${}^3k_t(a) = X_t^{(2)*}aX_t^{(1)}$, it is an immediate consequence of the fundamental formulae that ${}^3k$ is an $\mathcal{E}_{(\mathcal{S}_2,\mathcal{S}_1)}$-weak solution of (3.1) for $\theta = {}^3\theta$; moreover the $\mathcal{E}_{\mathbb{M}}$-regularity of X implies that ${}^3k$ is $\mathcal{E}_{(\mathcal{S}_2,\mathcal{S}_1)}$-regular. □

Thus, when F is a bounded operator matrix, or F is regular and ${}^FX$ is a bounded operator-valued process, we have

$$k_t(a) = X_t^* a X_t$$

where $X = {}^FX, k = k^\theta$ and $\theta = {}^3\theta$.

Contractivity, Isometry and Coisometry. We now apply our results to obtain characterisations of solutions of (7.1) and (7.2) which are contraction, isometry or coisometry processes. Note the following identity: when $F \in \mathrm{M}_\infty(\mathcal{B}(\mathfrak{h}))$ is bounded and ${}^3\theta$ is the mapping matrix given by (7.6),

$${}^3\theta(1) = F + F^* + F^*\Delta(1)F \tag{7.8}$$

Theorem 7.5. *For $F \in \mathrm{M}_\infty(\mathcal{B}(\mathfrak{h}))_{\mathrm{R}}$ the following equivalences hold:*

(ai) *X^F is a contraction process.*
(aii) *${}^FX$ is a contraction process.*
(aiii) *F is bounded and satisfies the operator inequality $F + F^* + F^*\Delta(1)F \le 0$.*
(aiv) *F is bounded and satisfies the operator inequality $F + F^* + F\Delta(1)F^* \le 0$.*

(bi) *X^F is an isometry process.*
(bii) *${}^FX$ is an isometry process.*
(biii) *F is bounded and satisfies $F + F^* + F^*\Delta(1)F = 0$.*

(ci) *X^F is a coisometry process.*
(cii) *${}^FX$ is a coisometry process.*
(ciii) *F is bounded and satisfies $F + F^* + F\Delta(1)F^* = 0$.*

Proof. Let $X = {}^FX$ and $Y = X^F$. If X (equivalently Y) is a contraction process then, by Theorem 7.4, ${}^3k_t(a) = X_t^* a X_t$ is an $\mathcal{E}_{(\mathbb{M},\mathbb{M})}$-weak solution of (3.1) for $\theta = {}^3\theta$ which is a CP contraction process. By Theorem 5.2 ${}^3\theta$ is bounded, and therefore by Proposition 7.3, F is bounded in all cases.

(i $\Leftrightarrow$ ii a, b & c) These are immediate consequences of Theorem 7.2 and the unitarity of the time reflection process.

(ai $\Leftrightarrow$ aiii, bi $\Leftrightarrow$ biii) Since X is a contraction (respectively isometry) process if and only if ${}^3k$ is a contraction (resp. unital) process these equivalences follow from (7.8) and Proposition 5.1.

(aii $\Leftrightarrow$ aiv, cii $\Leftrightarrow$ ciii) Since $F^\dagger$ is bounded, these equivalences follow from Theorem 7.1(c) and the above equivalences applied to the adjoint equation to (7.2). □

Remark. In fact the equivalence of (aiii) and (aiv) is valid with $\Delta(1)$ replaced by any positive self-adjoint operator. A purely algebraic proof of this fact uses the identities at the start of the proof of Theorem 5.3 in [LP2].

The final result closes a loop for contraction processes and H-P equations.

Proposition 7.6. *Let $X \in \mathbb{P}(\mathfrak{h}, \mathcal{E}_{\mathcal{S}})$ be a contraction process, with $\mathcal{S}$ fully admissible (see (2.2)). If X weakly satisfies (7.1) (or (7.2)) for some $F \in \mathrm{M}_\infty(\mathcal{B}(\mathfrak{h}))$, then F is bounded. In particular X strongly satisfies (7.1) and $X = {}^F X|_{\mathfrak{h}\odot\mathcal{E}_{\mathcal{S}}}$ (resp. (7.2) and $X = X^F|_{\mathfrak{h}\odot\mathcal{E}_{\mathcal{S}}}$).*

Proof. Let X weakly satisfy (7.1) for some $F \in \mathrm{M}_\infty(\mathcal{B}(\mathfrak{h}))$, and let J_d be the isometry $\mathfrak{h} \otimes \Gamma(L^2(\mathbb{R}_+; \mathbb{C}^d)) \to \mathcal{H}$ defined by (2.10). In the spirit of the proof of Proposition 5.1, let ${}^d X = {}^G X$ for $G = F^{[d]}$, viewed as an element of $\mathrm{M}_{d+1}(\mathcal{B}(\mathfrak{h}))$. Then, by the uniqueness of solutions ([LP2], Theorem 3.1(b)), and the finite dimensional analogue of Theorem 7.1(c),

$$J_d^* X_t J_d \subset {}^d X_t \subset {}^d Y_t^*,$$

where ${}^d Y = X^H$ for $H = F^{[d]\dagger} = F^{\dagger[d]}$. Thus ${}^d X_t$ is a densely defined closable extension of a contraction, and so is a contraction itself. Since ${}^d X$ is a contraction process, Theorem 5.3 of [LP2] implies that $F^{[d]}$ satisfies $F^{[d]} + F^{[d]*} + F^{[d]*}\Delta(1)F^{[d]} \leq 0$. In particular,

$$\sum_{i \leq d} \|F^i_\beta v\|^2 = \|\Delta(1)F^{[d]}E_\beta v\|^2 \leq -2\mathrm{Re}\langle E_\beta v, F^{[d]}E_\beta v\rangle = -2\mathrm{Re}\langle v, F^\beta_\beta v\rangle$$

for $d \geq \beta$. Letting $d \to \infty$ we see that F is regular, so (7.1) has a unique $\mathcal{E}_{\mathbb{M}}$-strong solution ${}^F X$. Since $J_d^* {}^F X J_d = {}^d X$ for all d, ${}^F X$ is a contraction process, and hence F is bounded by Theorem 7.5. Finally another application of the uniqueness of solutions shows that X is the restriction of ${}^F X$. The case when X is a weak solution of (7.2) is dealt with by a symmetrical argument. □

Remarks. (i) The existence of a solution X to the equation (7.1), when F is Mohari-Sinha regular, is proved directly in [Moh],[MoS] by using a Picard iteration scheme, and a similar scheme can be used to construct the solution Y of (7.2). A necessary and sufficient condition for X to be a contraction process is then derived in [Moh], however since F is not shown to be bounded the condition has to be stated in terms of finite dimensional cut-offs. Another consequence of not knowing that F is necessarily bounded when X is unitary (so that the related flow ${}^3 k$ is unital and *-homomorphic) was that the relationship between M-S regularity of F and M-S regularity of ${}^3\theta$ was unclear (see [Mey], p.185). This is now clarified by Proposition 7.3(c) and Theorem 7.5.

(ii) Since Fagnola assumes boundedness of F (in [Fag], Proposition 3.1), our results strengthen and frame his characterisations. He focuses on the left equation (7.2) and goes on to apply the results to study the QSDE when the individual operators F^α_β are no longer bounded — the analytical difficulties being considerably less for the left equation (7.2) than the right. In fact he starts with a matrix $[F^\alpha_\beta]$ of operators with common dense domain $\mathcal{D}$ that satisfies $\sum_{\alpha\geq 0} \|F^\alpha_\beta u\|^2 < \infty$ for each $\beta \geq 0$ and $u \in \mathcal{D}$, and which thus defines an operator F with domain $\{(u^\gamma) \in \mathfrak{h}_{00} : u^\alpha \in \mathcal{D}\ \forall \alpha\}$. He then regularises the matrix F to obtain a sequence of bounded operator matrices that approximate it, and it is his proof that these regularisations are bounded that inspired our Proposition 7.3(b).

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NOTE ADDED IN PROOF: It has now been shown that completely bounded mapping matrices are strongly regular ([LW3]). In view of Theorem 5.2, this entails an existence theorem for CP contraction flows (in particular *-homomorphic flows) on a C^*-algebra, driven by infinite dimensional quantum noise.

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