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Flows of constant mean curvature tori in the 3-sphere: The equivariant case

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Abstract. We present a deformation for constant mean curvature tori in the 3-sphere. We show that the moduli space of equivariant constant mean curvature tori in the 3-sphere is connected, and we classify the minimal, the embedded, and the Alexandrov embedded tori therein.

Introduction

Wente's discovery [40] of constant mean curvature (cmc) tori in $\mathbb{R}^3$ stimulated a systematic search [2, 3, 39, 41] for more examples, and culminated in the classification by Pinkall and Sterling [30], and the algebro-geometric description by Bobenko [5, 6]. Pinkall and Sterling, and independently Hitchin [17] noticed that infinitesimal variations of the structure equation for cmc tori have finite dimensional solution spaces, and thus have an algebraic description. This algebraic description consists of a real algebraic curve and a holomorphic line bundle on it. Despite this rather explicit description, the space of all cmc tori remains bewilderingly complicated. The objective in recent years has been to develop methods to investigate the structure of the moduli space of cmc tori, and in particular to classify the embedded ones among these, as in the recent remarkable achievements by Brendle [7], and Andrews and Li [4]. A particularly promising idea has been to adapt the isoperiodic deformation [15, 25] to study moduli spaces of cmc cylinders in various target spaces [16, 23]. Here we use the isoperiodic deformation to start the investigation of the moduli space of cmc tori in the round unit 3-sphere $\mathbb{S}^3$.

In contrast to flat Euclidean 3-space, the 3-sphere admits compact embedded minimal surfaces of any genus [20, 21, 26]. For simple examples like a great 2-sphere or the (minimal) Clifford torus there is a smooth deformation through cmc surfaces with the same topology, which can be expressed in terms of changing radii. Thus the Clifford torus is one member of a continuous real 1-parameter family of homogeneous tori parameterized by the mean curvature. For other minimal tori, as well as higher genus compact embedded minimal surfaces in $\mathbb{S}^3$ it would be useful to have a similar deformation method to study their moduli. We do this for the case of cmc tori, and present a smooth deformation for cmc tori in $\mathbb{S}^3$ in Theorem 1.2. By this theorem the moduli space of cmc tori in $\mathbb{S}^3$ is locally 1-dimensional. To get an idea of the global structure of the moduli space of all cmc tori in $\mathbb{S}^3$, we here provide a detailed study of

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the moduli space of equivariant cmc tori in $\mathbb{S}^3$, cf. [9, 18, 36]. Equivariance reduces the structure equation to an ordinary differential equation which can be solved by elliptic functions. An equivariant cmc tori is either a homogeneous torus or a finite piece of some member of an associated family of a Delaunay surface [12] that closes up into a torus in $\mathbb{S}^3$. We show that the moduli of equivariant cmc tori in $\mathbb{S}^3$ is a connected infinite graph. The edges are 1-dimensional families parameterized by the mean curvature. By flowing through this moduli space of equivariant cmc tori we classify the minimal, the embedded and the Alexandrov embedded tori therein.

By a theorem of Ruh and Vilms [32] a cmc surface has a harmonic Gauss map. Among such harmonic maps [31, 37, 38] there is an important dense subclass consisting of harmonic maps of finite type [8, 17, 30]. To a harmonic map of finite type there corresponds an associated algebraic curve whose compactification is called the spectral curve. The genus of the curve is called the spectral genus, and denoted by g. What makes it possible to adapt the isoperiodic deformation technique [15, 25, 27] to the case of cmc tori in $\mathbb{S}^3$ is precisely the astonishing fact discovered by Pinkall and Sterling [30] and Hitchin [17] that a cmc torus in $\mathbb{S}^3$ always has finite spectral genus. The spectral curve of a cmc torus is a double cover of the Riemann sphere with $2g + 2$ branch points. Two branch points are fixed, but the remaining $2g$ branch points can be arbitrarily placed, subject only to some additional symmetry. During the isoperiodic deformation the spectral curve changes such that the two unmoveable branch points remain fixed, while the other $2g$ branch points may move around. The closing conditions involve a choice of two double points on the real part of the spectral curve: we call these the sym points, and to ensure that the topology of the surface stays intact during the deformation, the flow of the sym points also has to be controlled.

Equivariance reduces the spectral genus to the two simplest cases $g = 0$ and $g = 1$. Homogeneous tori have spectral genus $g = 0$, while tori in associated families of Delaunay surfaces have spectral genus $g = 1$. First consider the family of embedded homogeneous tori. We show that it contains a $\mathbb{Z}$-family which allows a bifurcation into $g = 1$ rotational embedded cmc tori. Such a bifurcation is possible when in addition to the two sym points there is a further double point on the real part of the spectral curve. By opening up this additional double point and moving the resulting two branch points off the real part, the spectral curve becomes a double cover of the Riemann sphere branched now at four points: it has spectral genus $g = 1$ and is the spectral curve of a Delaunay surface. We show that at the end of the flow the new branch points pairwise coalesce with the two unmoveable branch points. Hence in the limit the coalescing pairs of branch points disappear, and the limit curve is an unbranched double cover of the sphere: the spectral curve of a bouquet of spheres. In the rotational cmc case, see also [19], our deformation corresponds to pinching the neck of a Delaunay surface, starting at a homogeneous torus and continuing through to a bouquet of spheres. Thus the connected component of the Clifford torus is an infinite comb: the spine ($g = 0$) consists of embedded homogeneous tori parameterized by the mean curvature, and each tooth ($g = 1$) of embedded Delaunay tori ends in a bouquet of spheres. By considering covers of homogeneous tori, the moduli space of rotational cmc tori is a $\mathbb{Z}^2$–family of such combs. It turns out that each bouquet of spheres occurs exactly twice in this moduli space (Theorem 4.4), so that we may glue the two families together there. Thus the moduli space of rotational cmc tori in $\mathbb{S}^3$ is an infinite connected graph.

A similar picture transpires in the equivariant non-rotational cmc case. In each isogeny class there is a sequence of spectral genus $g = 0$ cmc tori that can be deformed into spectral genus $g = 1$ cmc tori. In the equivariant non-rotational cmc case a spectral genus $g = 1$ de-

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formation family stays away from bouquets of spheres, and we prove that every spectral genus $g = 1$ family of cmc tori begins and ends at a spectral genus $g = 0$ torus (Theorem 4.5). The above results for both rotational as well as non-rotational combined give that every deformation family of spectral genus $g = 1$ cmc tori ends at a homogeneous torus. The classification of spectral genus $g = 1$ cmc tori is thus reduced to that of spectral curves of homogeneous tori with a double point on the real part; this initial data is classified and interpreted geometrically. We classify the equivariant minimal tori, as well as the embedded and Alexandrov embedded equivariant cmc tori, and prove that the Clifford torus is the only embedded minimal equivariant torus in the 3-sphere (Theorem 5.11). We also show that the spectral curve of an equivariant cmc torus has no double points off the real part (Theorem 2.9), which implies that there can not be a Bianchi-Bäcklund transform of an equivariant cmc torus into a cmc torus. We prove that the moduli space of equivariant cmc tori in $\mathbb{S}^3$ is connected. Throughout the text we provide graphics of some of the surfaces under discussion. More images and videos of deformation families are available on the website [33].

We briefly summarize the organization of the paper: In Section 1 we start by recalling the description of cmc tori in terms of spectral curves and abelian differentials [5, 17, 22, 28]. We then adapt the isoperiodic deformation technique from [15] and prove that any generic cmc torus in $\mathbb{S}^3$ lies in a smooth family of cmc tori in $\mathbb{S}^3$. From Section 2 onwards we specialize to equivariant cmc tori, and apply the isoperiodic deformation to study their moduli space. In the equivariant case the isoperiodic deformation defines a vector field on a 3-parameter space $(-1, 1) \times (-1, 1) \times (-1, 1)$. The three parameters are an elliptic modulus, and the arguments of the two sym points. Integral curves of the vector field in this cube describe all deformation families of spectral genus $g = 1$ cmc tori, and we provide a very detailed qualitative analysis of the global behaviour of all integral curves. To obtain a detailed description of the extrinsic geometry of equivariant cmc tori we compute in the third section the extended frame of any equivariant cmc surface, and discuss the geometry of the extrinsic closing conditions. In the fourth section we determine those homogeneous tori which allow a bifurcation into spectral genus $g = 1$ cmc tori, and we classify these. By also considering all possible covers of homogeneous tori we obtain all possible endpoints of deformation families of spectral genus $g = 1$ cmc tori in terms of three coprime integers (Theorem 4.2). These three integers encode the extrinsic geometry of the corresponding deformation family, and are discussed in section five. There we also reveal the infinite diversity of equivariant minimal tori. We show that non-rotational spectral genus $g = 1$ cmc tori are never embedded. We also classify the families of embedded rotational cmc tori. By the affirmation of the Pinkall-Sterling conjecture by Andrews–Li [4] all embedded cmc tori in $\mathbb{S}^3$ lie in these families. Whilst the Lawson conjecture was recently proven by Brendle [7], we give an independent proof of a very special case of the Lawson conjecture, and show that the Clifford torus is the only equivariant embedded minimal torus in $\mathbb{S}^3$ (Theorem 5.11). In the sixth and final section of the paper we prove that the moduli space of equivariant cmc tori in $\mathbb{S}^3$ is connected.

1. Spectral curve

We start by recalling the description of cmc tori in terms of spectral curves and abelian differentials [5, 17, 22, 28]. We then adapt a deformation technique from [15] and prove that any generic cmc torus in $\mathbb{S}^3$ lies in a smooth family of cmc tori in $\mathbb{S}^3$.

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Let Y be a hyperelliptic Riemann surface with meromorphic function λ of degree two and with branch points over $\lambda = 0\,(y^+)$ and $\lambda = \infty\,(y^-)$. Then Y is the *spectral curve* of an immersed cmc torus in $\mathbb{S}^3$ if and only if the following four conditions hold:

(1) Besides the hyperelliptic involution σ, the surface Y has two further anti-holomorphic involutions η and $\varrho = \eta \circ \sigma = \sigma \circ \eta$, such that η has no fix points and $\eta(y^+) = y^-$.

(2) There exist two non-zero holomorphic functions μ_1, μ_2 on $\mathrm{Y} \setminus \{y^+, y^-\}$ such that for $i = 1, 2$
$$\mu_i \circ \sigma = \mu_i^{-1}, \quad \mu_i \circ \eta = \bar{\mu}_i, \quad \mu_i \circ \varrho = \bar{\mu}_i^{-1}.$$

(3) The forms $d \ln \mu_i$ are meromorphic differentials of the second kind with double poles at $y^{\pm}$. The singular parts at y^+ respectively y^- of these two differentials are linearly independent.

(4) There are four fixed points $y_1, y_2 = \sigma(y_1), y_3, y_4 = \sigma(y_3)$ of ϱ, such that the functions μ_1 and μ_2 are either 1 or -1 there.

Definition 1.1. Given the spectral curve of an immersed cmc torus in $\mathbb{S}^3$, let $\lambda_1 \in \mathbb{S}^1$ denote the value $\lambda(y_1) = \lambda(y_2)$, and $\lambda_2 \in \mathbb{S}^1$ the value $\lambda(y_3) = \lambda(y_4)$ at the four fixed points of ϱ where μ_1 and μ_2 are either 1 or -1. We call λ_1 and λ_2 the *sym points*.

For $\mathbb{i} = \sqrt{-1}$, the mean curvature of the corresponding cmc torus in terms of the sym points is

$$H = \mathbb{i}\,\frac{\lambda_2 + \lambda_1}{\lambda_2 - \lambda_1}. \tag{1.1}$$

We shall describe spectral curves of cmc tori in $\mathbb{S}^3$ via hyperelliptic surfaces of the form

$$\nu^2 = \lambda\, a(\lambda)$$

where $a \in \mathbb{C}^g[\lambda]$ is a polynomial of degree g and

$$\bar{\lambda}^{-2g}\bar{a}(\lambda) = a(\bar{\lambda}^{-1}) \quad \text{and} \quad \lambda^{-g}a(\lambda) \geq 0 \quad \text{for } |\lambda| = 1.$$

The involutions of the spectral curve are

$$\begin{aligned} \sigma\,(\lambda, \nu) &= (\lambda, -\nu), \\ \eta\,(\lambda, \nu) &= (\bar{\lambda}^{-1}, -\bar{\nu}\bar{\lambda}^{-g-1}), \\ \varrho\,(\lambda, \nu) &= (\bar{\lambda}^{-1}, \bar{\nu}\bar{\lambda}^{-g-1}). \end{aligned} \tag{1.2}$$

Making use of a rotation of λ and a rescaling of ν we may assume that a is a polynomial with highest coefficient one. The meromorphic differentials $d \ln \mu_i$ have the form

$$d \ln \mu_i := \pi \frac{b_i(\lambda)}{\nu}\frac{d\lambda}{\lambda} \tag{1.3}$$

with polynomials $b_i \in \mathbb{C}^{g+1}[\lambda]$ of degree $g + 1$ and $\bar{\lambda}^{-g-1}\bar{b}_i(\lambda) = b_i(\bar{\lambda}^{-1})$.

We next describe a one parameter family of deformations of the spectral curve, that depends on a deformation parameter t. We view all functions on the corresponding spectral curves as functions in the variables λ and t.

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Since the path integrals of the differentials $d \ln \mu_i$ along all cycles in $H_1(\mathrm{Y}, \mathbb{Z})$ are integer multiples of $2\pi\,\mathbb{i}$, these differentials do not depend on the deformation parameter t. Further, see [29, Chapter 3, Proposition 1.10], a meromorphic function f on a hyperelliptic Riemann surface given by $\nu^2 = h(\lambda)$ is of the form $f(\lambda) = r(\lambda) + \nu\, s(\lambda)$ with rational functions r, s. Hence both $\partial_t \ln \mu_i$ are global meromorphic functions on Y with only possible poles at the branch points of Y. More precisely these meromorphic functions are of the form

$$\partial_t \ln \mu_i = \pi \frac{\mathbb{i}\, c_i(\lambda)}{\nu} \tag{1.4}$$

with polynomials $c_i \in \mathbb{C}^{g+1}[\lambda]$ of degree $g+1$ and $\bar\lambda^{-g-1}\bar c_i(\lambda) = c_i(\bar\lambda^{-1})$.

Integrability $\partial^2_{t\lambda} \ln \mu_i = \partial^2_{\lambda t} \ln \mu_i$ reads $\partial_t(\lambda^{-1}\nu^{-1} b_i) = \partial_\lambda\, \nu^{-1}\mathbb{i}\, c_i$ which yields

$$2a(\lambda)\dot b_i(\lambda) - \dot a(\lambda) b_i(\lambda) = \mathbb{i}(2\lambda a(\lambda) c_i'(\lambda) - \lambda a'(\lambda) c_i(\lambda) - a(\lambda) c_i(\lambda)). \tag{1.5}$$

Here dash and dot denote the derivatives with respect to λ respectively t.

The differential

$$\Omega = (\partial_t \ln \mu_1)\, d \ln \mu_2 - (\partial_t \ln \mu_2)\, d \ln \mu_1$$

is a meromorphic 1-form on Y with only poles at most of order three at $\lambda = 0, \infty$, and roots at the sym points λ_1, λ_2. Further, since $\eta^*\bar\Omega = \Omega$, $\varrho^*\bar\Omega = \Omega$ and $\sigma^*\Omega = \Omega$, we conclude that

$$\Omega = \pi^2\, \mathrm{C}_1\, \frac{(\lambda-\lambda_1)(\lambda-\lambda_2)}{\lambda\sqrt{\lambda_1\,\lambda_2}}\, \frac{d\lambda}{\mathbb{i}\lambda}$$

with a real function $\mathrm{C}_1 = \mathrm{C}_1(t)$. Using (1.3) and (1.4) we obtain

$$b_1(\lambda)c_2(\lambda) - b_2(\lambda)c_1(\lambda) = \mathrm{C}_1\, a(\lambda) \frac{(\lambda-\lambda_1)(\lambda-\lambda_2)}{\sqrt{\lambda_1\lambda_2}}, \tag{1.6}$$

and combining the above gives

$$(\partial_t \ln \mu_1)\, d \ln \mu_2 - (\partial_t \ln \mu_2)\, d \ln \mu_1 = \pi^2\, \mathrm{C}_1\, \frac{(\lambda-\lambda_1)(\lambda-\lambda_2)}{\lambda\,\sqrt{\lambda_1\,\lambda_2}}\, \frac{d\lambda}{\mathbb{i}\lambda}. \tag{1.7}$$

To preserve the topology during the flow, the values of $\ln \mu_i$ should be fixed at the two sym points $\lambda = \lambda_1$ and $\lambda = \lambda_2$. Consequently $\partial_t \ln \mu_i|_{\lambda=\lambda_j} = ((\partial_\lambda \ln \mu_i)\,\dot\lambda + \partial_t \ln \mu_i)|_{\lambda=\lambda_j} = 0$, which gives

$$\dot\lambda_j = -\left.\frac{\partial_t \ln \mu_i}{\partial_\lambda \ln \mu_i}\right|_{\lambda=\lambda_j}. \tag{1.8}$$

Theorem 1.2. *Let* Y *be a genus* g *spectral curve of a cmc torus in* $\mathbb{S}^3$. *If the two differentials* $d \ln \mu_i$ *for* $i = 1, 2$ *have no common roots, then* Y *is contained in a unique smooth one dimensional family of spectral curves of cmc tori in* $\mathbb{S}^3$.

Proof. If the roots of b_1 and b_2 are pairwise distinct, then the $2g+2$ values of equation (1.6) at these roots uniquely determine the values of c_1 and c_2 there. Therefore (1.6) determines the polynomials c_1 and c_2 uniquely up to a real multiple of b_1 and b_2. The choice $c_1 = b_1$ and $c_2 = b_2$ in (1.4) and (1.5) corresponds to a rotation of λ. For given c_1 and c_2

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equations (1.5) determine uniquely the derivatives of all roots of the polynomials a and b_i. The condition $\bar{\lambda}^{-2g}\bar{a}(\lambda) = a(\bar{\lambda}^{-1})$ together with the assumption that the highest (and lowest) coefficient of the polynomial a has absolute value one determines the polynomial a in terms of its roots up to multiplication with ± 1. We conclude that these conditions on a determine $\dot{a}$, $\dot{b}_1$ and $\dot{b}_2$ in terms of c_1 and c_2. We remark that due to equation (1.6) the solutions $\dot{a}$ of both equations (1.5) coincide. Finally the condition that the highest (and lowest) coefficient of a is qual to one, fixes the rotations. Therefore there exist unique solutions c_1 and c_2 of (1.6). □

We expect that one can pass through common zeroes of the polynomials b_i, therefore making the deformation global, but this will be considered elsewhere. Below we will restrict to the cases of spectral genera $g = 0$, where this is trivially true, and $g = 1$ where the roots of b_1 and b_2 turn out to always be distinct throughout the flow (Corollary 2.3).

2. Flows of equivariant cmc tori in the 3-sphere

The double cover of the isometry group of $\mathbb{S}^3 \cong \mathrm{SU}_2$ is $\mathrm{SU}_2 \times \mathrm{SU}_2$ via the action $P \mapsto F\,P\,G^{-1}$. A surface is *equivariant* if it is preserved set-wise by a one-parameter family of isometries. To an equivariant surface we associate two *axes*: These are geodesics which are fixed set-wise by the one-parameter family of isometries. An equivariant cmc surface is rotational precisely when one of its axes is fixed point-wise. Equivariant cmc surfaces have spectral genus zero or one [9]. By a theorem of do Carmo and Dajczer [12], an equivariant cmc surface is a member of an associated family of a Delaunay surface, so up to isometry determined by an elliptic modulus and its sym points. Hence an equivariant cmc surface in the 3-sphere is parameterized by its elliptic modulus, its mean curvature, and its associated family parameter. We will express the differential equations (1.7) and (1.8) in terms of the three coordinates

$$(\mathrm{k}, \mathrm{q}, \mathrm{h}) \in [-1, 1]^3 \tag{2.1}$$

where k is the elliptic modulus, and q, h are defined in terms of the sym points as

$$\mathrm{q} := \frac{1}{2}\big(\sqrt{\lambda_1\lambda_2} + 1/\sqrt{\lambda_1\lambda_2}\big) \quad \text{and} \quad \mathrm{h} := \frac{1}{2}\left(\sqrt{\frac{\lambda_1}{\lambda_2}} + \sqrt{\frac{\lambda_2}{\lambda_1}}\right). \tag{2.2}$$

The mean curvature H in (1.1) can then be expressed as

$$H = \frac{\mathrm{h}}{\sqrt{1-\mathrm{h}^2}}. \tag{2.3}$$

The following identities will be used below:

$$\begin{aligned}
&2\sqrt{\mathrm{q}^2-1} = \sqrt{\lambda_1\lambda_2} - 1/\sqrt{\lambda_1\lambda_2}, && 2\sqrt{\mathrm{h}^2-1} = \sqrt{\lambda_1/\lambda_2} - \sqrt{\lambda_2/\lambda_1},\\
&4(\mathrm{q}^2-\mathrm{h}^2) = (\lambda_1-\lambda_1^{-1})(\lambda_2-\lambda_2^{-1}), && 4\,\mathrm{q}\sqrt{\mathrm{q}^2-1} = \lambda_1\lambda_2 - \lambda_1^{-1}\lambda_2^{-1},\\
&4\,\mathrm{h}\sqrt{\mathrm{q}^2-1} = \lambda_1 - \lambda_1^{-1} + \lambda_2 - \lambda_2^{-}1, && 4\,\mathrm{q}\sqrt{\mathrm{h}^2-1} = \lambda_1 - \lambda_1^{-1} - \lambda_2 + \lambda_2^{-1},\\
&4\,\mathrm{h}\sqrt{\mathrm{h}^2-1} = \lambda_1\lambda_2^{-1} - \lambda_1^{-1}\lambda_2.
\end{aligned} \tag{2.4}$$

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The derivatives of q, h with respect to the flow parameter are

$$\dot{\mathrm{q}} = \frac{\sqrt{\mathrm{q}^2-1}}{2}\left(\frac{\dot\lambda_1}{\lambda_1}+\frac{\dot\lambda_2}{\lambda_2}\right), \quad \dot{\mathrm{h}} = \frac{\sqrt{\mathrm{h}^2-1}}{2}\left(\frac{\dot\lambda_1}{\lambda_1}-\frac{\dot\lambda_2}{\lambda_2}\right). \tag{2.5}$$

2.1. Spectral genus zero. In the spectral genus zero case, $a \equiv 1$, the spectral curve is $\nu^2 = \lambda$, and the elliptic modulus $\mathrm{k} \equiv \pm 1$. The functions b_i in (1.3) for $i = 1, 2$ are polynomial in λ of degree one, and from $\bar\lambda^{-1}\bar b_i(\lambda) = b_i(\bar\lambda^{-1})$ we conclude that $b_i = \beta_i\,\lambda + \bar\beta_i$ for some smooth complex-valued functions $t \mapsto \beta_i(t)$. Thus $d\ln\mu_i$ integrates to

$$\ln\mu_j = \frac{2\pi(\beta_j\,\lambda - \bar\beta_j)}{\nu},$$

and consequently the functions c_i, $i = 1, 2$, in (1.4) are $c_i = 2\mathbb{i}(\dot{\bar\beta}_i - \dot\beta_i\lambda)$. We may fix one of the sym points, and assume without loss of generality that $\lambda_1 \equiv 1$ during the flow. Then $\partial_t \ln\mu_j|_{\lambda=1} = 0$, which is equivalent to $c_i(\lambda = 1) = 0$, or equivalently $\dot\beta_i \in \mathbb{R}$. Thus equation (1.7) reads

$$2\mathbb{i}\dot\beta_2(\beta_1\lambda + \bar\beta_1) - 2\mathbb{i}\dot\beta_1(\beta_2\lambda + \bar\beta_2) = \frac{C_1}{\sqrt{\lambda_2}}\,(\lambda_2 - \lambda).$$

Evaluating this at $\lambda = -\bar\beta_i/\beta_i$ gives

$$\dot\beta_i = \frac{C_1(\beta_i\lambda_2^{1/2} + \bar\beta_i\lambda_2^{-1/2})}{2\mathbb{i}\,(\bar\beta_1\beta_2 - \beta_1\bar\beta_2)}.$$

From (1.8) we get

$$\frac{\dot\lambda_2}{\lambda_2} = \frac{c_1}{\mathbb{i}b_1} = \frac{2\dot\beta_1(1-\lambda_2)}{\beta_1\lambda_2 + \bar\beta_1} = C_1\frac{\lambda_2^{1/2} - \lambda_2^{-1/2}}{\mathbb{i}(\beta_1\bar\beta_2 - \bar\beta_1\beta_2)}.$$

Set $C_1 := \mathbb{i}(\beta_1\bar\beta_2 - \bar\beta_1\beta_2)$. Then together with (2.4) and the fact that $\mathrm{q} = \mathrm{h}$ when $\lambda_1 \equiv 1$, the flow equations (1.7) and (1.8) reduce to the single equation $\dot{\mathrm{h}} = \mathrm{h}^2 - 1$. The solution to this equation is given by $\mathrm{h}(t) = -\tanh(t + C)$ with some constant of integration $C \in \mathbb{R}$. Consequently, the argument of the sym point $\arg[\lambda_2] = 2\arccos(-\tanh(t))$ can vary over all of the interval $(0, 2\pi)$, and is strictly monotonic. Hence also the mean curvature H by equation (2.3) is strictly monotonic, and given by $H(t) = -\sinh(t)$. In summary, we have proven the following

Theorem 2.1. *Every homogeneous torus in the* 3*-sphere lies in a smooth* $\mathbb{R}$*-family of homogeneous tori. Each family is parameterized by the mean curvature.*

2.2. Spectral genus one. In the spectral genus one case, the meromorphic differentials $d\ln\mu_i$ are linear combinations of the derivative of a meromorphic function and an elliptic integral. We write these meromorphic differentials in terms of Jacobi's elliptic functions. We define an elliptic curve by

$$4\nu^2 = (\lambda - \mathrm{k})(\lambda^{-1} - \mathrm{k}). \tag{2.6}$$

Here $\mathrm{k} \in [-1, 1]$ is the real modulus. Then

$$d\nu = \mathrm{k}\,\frac{\lambda^{-1} - \lambda}{8\nu}\,\frac{d\lambda}{\lambda}, \tag{2.7}$$

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and the only roots of dv are at $\lambda = \pm 1$. In addition to the three involutions in (1.2), the elliptic curve (2.6) has a further holomorphic involution

$$\tau(\lambda, v) = (\lambda^{-1}, v). \tag{2.8}$$

We decompose $\frac{\ln \mu_i}{\pi \mathbb{i}}$ into the symmetric and skew-symmetric part with respect to τ. The symmetric part is a real multiple of the single-valued meromorphic function v, and the skew-symmetric part is a real multiple of a multi-valued function ω with real periods. The first homology group of the elliptic curve (2.6) is generated by a cycle around the two branch points at $\lambda = \mathrm{k}^{\pm 1}$ and the cycle $\mathbb{S}^1 = \{\lambda \mid |\lambda| = 1\}$. The first cycle is symmetric with respect to τ and the second skew-symmetric. Therefore the integral of $d\omega$ along the first cycle vanishes. We assume that the integral of $d\omega$ along the second cycle is equal to 2. Together with the first order poles of ω at $y^{\pm}$ and the skew-symmetry with respect to τ this normalization determines ω uniquely. Further, since $\int_{\mathbb{S}^1} d \ln \mu_i \in 2\pi \mathbb{i}\mathbb{Z}$, we conclude that the functions $\ln \mu_i$ are linear combinations of v and ω, and given by

$$\ln \mu_1 = \pi \mathbb{i}\,(x_1\, v + p_{10}\, \omega), \quad \ln \mu_2 = \pi \mathbb{i}\,(x_2\, v + p_{20}\, \omega) \quad \text{for } p_{10}, p_{20} \in \mathbb{Z}. \tag{2.9}$$

We shall express ω as a linear combination of complete elliptic integrals of the first and second kind. First we shall relate the curve (2.6) to the elliptic curve Y in Legendre's form with elliptic modulus $\frac{1-\mathrm{k}}{1+\mathrm{k}}$ given by

$$\mathbf{y}^2 = (1 - \mathbf{x}^2)\left(1 - \left(\frac{1-\mathrm{k}}{1+\mathrm{k}}\right)^2 \mathbf{x}^2\right).$$

Let $\mathbf{K}$ and $\mathbf{E}$ denote the complete integrals of the first and second kind

$$\mathbf{K}(\mathrm{k}) := \int_0^1 \frac{d\mathbf{x}}{\sqrt{(1-\mathbf{x}^2)(1-\mathrm{k}^2\,\mathbf{x}^2)}}, \quad \mathbf{E}(\mathrm{k}) := \int_0^1 \sqrt{\frac{1-\mathrm{k}^2\,\mathbf{x}^2}{1-\mathbf{x}^2}}\, d\mathbf{x}.$$

Since $\mathbf{E}$ has first order poles at the two points over $\mathbf{x} = \infty$, these two points correspond to $y^{\pm}$. The involution τ corresponds to $(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}, -\mathbf{y})$. Therefore we set

$$\lambda = \frac{1 + (\frac{1-\mathrm{k}}{1+\mathrm{k}})^2(1 - 2\,\mathbf{x}^2) - 2\frac{1-\mathrm{k}}{1+\mathrm{k}}\,\mathbf{y}}{1 - (\frac{1-\mathrm{k}}{1+\mathrm{k}})^2},$$

$$\lambda^{-1} = \frac{1 + (\frac{1-\mathrm{k}}{1+\mathrm{k}})^2(1 - 2\,\mathbf{x}^2) + 2\frac{1-\mathrm{k}}{1+\mathrm{k}}\,\mathbf{y}}{1 - (\frac{1-\mathrm{k}}{1+\mathrm{k}})^2},$$

$$v = \frac{\frac{1-\mathrm{k}}{1+\mathrm{k}}\,\mathbf{x}}{1 + \frac{1-\mathrm{k}}{1+\mathrm{k}}}.$$

The integrals of the meromorphic differentials $d \ln \mu_i$ along all cycles of Y are purely imaginary. Define

$$d\omega := \left(\mathbf{E}\left(\frac{1-\mathrm{k}}{1+\mathrm{k}}\right) - \mathbf{K}\left(\frac{1-\mathrm{k}}{1+\mathrm{k}}\right)\left(1 - \left(\frac{1-\mathrm{k}}{1+\mathrm{k}}\right)^2 \mathbf{x}^2\right)\right)\frac{2d\mathbf{x}}{\pi \mathbb{i}\,\mathbf{y}}. \tag{2.10}$$

The cycle around the two branch points $\lambda = \mathrm{k}^{\pm 1}$ corresponds to the real period and the cycle $\mathbb{S}^1$ to the imaginary period of Jacobi's elliptic functions. The integral along the real period vanishes and due to Legendre's relations [1, equation 17.3.13] the integral along the imaginary period is equal to 2. In summary, we have found a linear combination of elliptic integrals of the first and second kind which obeys the conditions that characterize ω.

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In terms of the complete elliptic integrals

$$\mathbf{K}' = \mathbf{K}'(\mathrm{k}) = \mathbf{K}(\sqrt{1-\mathrm{k}^2}) \quad \text{and} \quad \mathbf{E}' = \mathbf{E}'(\mathrm{k}) = \mathbf{E}(\sqrt{1-\mathrm{k}^2})$$

and the functions λ and ν, and the formulas [1, equations 17.3.29 and 17.3.30]

$$\mathbf{E}\left(\frac{1-\mathrm{k}}{1+\mathrm{k}}\right) = \left(1+\frac{1-\mathrm{k}}{1+\mathrm{k}}\right)\mathbf{E}' + \left(1-\frac{1-\mathrm{k}}{1+\mathrm{k}}\right)\mathbf{K}', \quad \mathbf{K}\left(\frac{1-\mathrm{k}}{1+\mathrm{k}}\right) = \frac{2}{1+\frac{1-\mathrm{k}}{1+\mathrm{k}}}\mathbf{K}',$$

the differential $d\omega$ in (2.10) simplifies to

$$d\omega = \frac{2\mathbf{E}' - \mathrm{k}\,\mathbf{K}'(\lambda+\lambda^{-1})}{4\pi\nu}\frac{d\lambda}{\mathbb{i}\lambda}. \tag{2.11}$$

Proposition 2.2. *The complete elliptic integrals* $\mathbf{K}'$ *and* $\mathbf{E}'$ *satisfy*

$$1 \le \frac{2\mathbf{E}'}{1+\mathrm{k}^2} < \mathbf{K}' < \frac{\mathbf{E}'}{|\mathrm{k}|} \quad \textit{for } 0 < |\mathrm{k}| < 1.$$

Proof. Assume first that $\mathrm{k} \in (0,1)$. Since $\tau^* d\omega = -d\omega$, we have that

$$\int_{\lambda=\mathrm{k}}^{\lambda=\mathrm{k}^{-1}} d\omega = 2\int_{\lambda=\mathrm{k}}^{\lambda=1} d\omega = 0.$$

Hence the function $2\mathbf{E}' - \mathrm{k}\,\mathbf{K}'(\lambda+\lambda^{-1})$ has a real root in $\lambda \in (\mathrm{k},1)$, say at $r \in (\mathrm{k},1)$, and thus also at $r^{-1} \in (1,\mathrm{k}^{-1})$. Then $2 < r+1/r < k+1/k$, and together with $2\mathbf{E}'-\mathrm{k}\,\mathbf{K}'(r+r^{-1}) = 0$ gives $2\mathbf{E}' > 2\,\mathrm{k}\,\mathbf{K}'$ and $2\mathbf{E}' < \mathbf{K}'(1+\mathrm{k}^2)$. Finally, from $2 > \mathrm{k}^2+1$ and $\mathbf{E}' \ge 1$ ([14, Section 13.8.11]), we obtain $2\mathbf{E}' \ge 1+\mathrm{k}^2$.

For $\mathrm{k} \in (-1,0)$, the function $2\mathbf{E}' - \mathrm{k}\,\mathbf{K}'(\lambda+\lambda^{-1})$ has a pair of reciprocal real roots s, s^{-1} with $-1 < s < \mathrm{k}^{-1}$, and analogous arguments prove the assertion in this case. □

Corollary 2.3. *The differentials* $d\nu$ *and* $d\omega$ *have no common zeroes.*

Proof. While $d\nu$ in (2.7) has only roots at $\lambda = \pm 1$, we saw in the proof of Proposition 2.2 that $d\omega$ has only roots $\lambda \in (\mathrm{k},1)\cup(1,\mathrm{k}^{-1})$ when $\mathrm{k} \in (0,1)$, and $\lambda \in (-1,\mathrm{k})\cup(\mathrm{k}^{-1},-1)$ when $\mathrm{k} \in (-1,0)$. □

Theorem 2.4. *Every spectral genus one cmc torus in* $\mathbb{S}^3$ *lies on an integral curve of the vector field*

$$\begin{pmatrix}\dot{\mathrm{k}}\\ \dot{\mathrm{q}}\\ \dot{\mathrm{h}}\end{pmatrix} = \begin{pmatrix}\mathrm{k}\,(\mathbf{E}'\,\mathrm{q} - \mathrm{k}\,\mathbf{K}'\,\mathrm{h})\\ \frac{1-\mathrm{q}^2}{1-\mathrm{k}^2}\,((1+\mathrm{k}^2)\mathbf{E}' - 2\,\mathrm{k}^2\,\mathbf{K}')\\ \mathrm{k}\,\frac{1-\mathrm{h}^2}{1-\mathrm{k}^2}\,(2\mathbf{E}' - (1+\mathrm{k}^2)\mathbf{K}'))\end{pmatrix}. \tag{2.12}$$

The vector field $(\dot{\mathrm{k}},\dot{\mathrm{q}},\dot{\mathrm{h}})$ *is analytic on the set* $\mathcal{D} := \{(\mathrm{k},\mathrm{q},\mathrm{h}) \in \mathbb{R}^3 \mid \mathrm{k} \neq 0\}$*, and its zero-set is* $\{\mathrm{k}^2 = 1\} \cap \{\mathrm{q} = \mathrm{k}\,\mathrm{h}\} \cap \mathcal{D}$.

Proof. We shall calculate the differential equations (1.7) and (1.8) in terms of the coordinates (2.1). From (2.6) we compute the derivative of ν with respect to t, and obtain

$$\dot{\nu} = \dot{\mathrm{k}}\,\frac{2\,\mathrm{k} - \lambda - \lambda^{-1}}{8\nu}. \tag{2.13}$$

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In agreement with the skew-symmetry of ω with respect to τ we set

$$\dot{\omega} = \mathrm{C}_2 \, \frac{\lambda - \lambda^{-1}}{4\pi \, \mathbb{i} \, \nu}.$$

From (2.9) we obtain $\partial_t \ln \mu_j = \pi \, \mathbb{i} \, (\dot{x}_j \, \nu + x_j \, \dot{\nu} + p_{j0} \, \dot{\omega})$ and $d \ln \mu_j = \pi \, \mathbb{i} \, (x_j \, d\nu + p_{j0} \, d\omega)$ for $j = 1, 2$. Putting all this together, the differential $\Omega = (\partial_t \ln \mu_1) \, d \ln \mu_2 - (\partial_t \ln \mu_2) \, d \ln \mu_1$ reads

$$\Omega = \pi^2 \big((\dot{x}_1 x_2 - \dot{x}_2 x_1) \, \nu \, \nu' + (\dot{x}_2 p_{10} - \dot{x}_1 p_{20}) \, \nu \, \omega' + (p_{10} x_2 - p_{20} x_1)(\dot{\nu} \, \omega' - \nu' \dot{\omega}) \big)$$

with

$$\begin{aligned}
\nu \, \nu' &= \frac{\mathrm{k} \, (\lambda^{-1} - \lambda)}{8 \, \lambda}, \\
\nu \, \omega' &= \frac{2 \, \mathbf{E}' - \mathrm{k} \, (\lambda + \lambda^{-1}) \, \mathbf{K}'}{4 \, \pi \, \mathbb{i} \, \lambda}, \\
\dot{\nu} \, \omega' - \nu' \dot{\omega} &= \frac{\mathrm{C}_2 \, \mathrm{k} (\lambda - \lambda^{-1})^2 + \dot{\mathrm{k}} \, (\lambda - 2 \, \mathrm{k} + \lambda^{-1}) (\mathrm{k} \, (\lambda + \lambda^{-1}) \, \mathbf{K}' - 2 \, \mathbf{E}')}{8 \, \pi \, \mathbb{i} \, \lambda \, (\lambda - \mathrm{k}) (\lambda^{-1} - \mathrm{k})}.
\end{aligned}$$

Note that while $\dot{\nu} \, \omega' - \nu' \dot{\omega}$ has simple poles at $\lambda = \mathrm{k}, \mathrm{k}^{-1}$, but Ω does not, we conclude that the numerator of $\dot{\nu} \, \omega' - \nu' \dot{\omega}$ must vanish at $\lambda = \mathrm{k}, \mathrm{k}^{-1}$, giving

$$\mathrm{C}_2 = \dot{\mathrm{k}} \, \frac{(\mathrm{k}^2 + 1) \, \mathbf{K}' - 2 \, \mathbf{E}'}{\mathrm{k}^2 - 1}.$$

The unique solution of equation (1.7) can now be computed, and is given by

$$\begin{aligned}
\dot{\mathrm{k}} &= \frac{4\pi \, \mathrm{C}_1 (\mathrm{k}^2 - 1)(\mathrm{k} \, \mathrm{h} \, \mathbf{K}' - \mathrm{q} \, \mathbf{E}')}{(x_1 \, p_{20} - x_2 \, p_{10})(\mathbf{E}'^2 - \mathrm{k}^2 \, \mathbf{K}'^2)}, \\
\dot{x}_1 x_2 - \dot{x}_2 x_1 &= \frac{8 \, \mathrm{C}_1 \, \sqrt{1 - \mathrm{q}^2}}{\mathrm{k}}, \\
\dot{x}_1 \, p_{20} - \dot{x}_2 \, p_{10} &= \frac{4\pi \, \mathrm{C}_1 (\mathrm{k} \, \mathrm{q} (\mathbf{E}' - \mathbf{K}') + \mathrm{h} (\mathbf{E}' - \mathrm{k}^2 \, \mathbf{K}'))}{\mathbf{E}'^2 - \mathrm{k}^2 \, \mathbf{K}'^2}.
\end{aligned}$$

Since $\dot{\omega} = \partial_{\mathrm{k}} \omega \, \dot{\mathrm{k}}$, the above imply that

$$\frac{\partial \omega}{\partial \mathrm{k}} = \frac{(1 + \mathrm{k}^2) \mathbf{K}' - 2 \mathbf{E}'}{4\pi \, \mathbb{i} (1 - \mathrm{k}^2) \nu} \, (\lambda - \lambda^{-1}). \tag{2.14}$$

Now setting

$$\mathrm{C}_1 := - \frac{\mathrm{k} \, (x_1 \, p_{20} - x_2 \, p_{10})(\mathbf{E}'^2 - \mathrm{k}^2 \, \mathbf{K}'^2)}{4\pi (\mathrm{k}^2 - 1)}$$

gives $\dot{\mathrm{k}} = \mathrm{k}(\mathbf{E}' \, \mathrm{q} - \mathrm{k} \, \mathbf{K}' \, \mathrm{h})$ as required.

The deformation equations of the sym points (1.8) read

$$\begin{aligned}
\dot{\lambda}_j \, (x_1 \, \nu' + p_{10} \, \omega') &= -(\dot{x}_1 \, \nu + x_1 \, \dot{\nu} + p_{10} \, \dot{\omega}), \\
\dot{\lambda}_j \, (x_2 \, \nu' + p_{20} \, \omega') &= -(\dot{x}_2 \, \nu + x_2 \, \dot{\nu} + p_{20} \, \dot{\omega}).
\end{aligned}$$

Multiplying the first equation by p_{20} and the second equation by p_{10} and subtracting gives

$$\dot{\lambda}_j = - \left. \frac{(\dot{x}_1 \, p_{20} - \dot{x}_2 \, p_{10}) \, \nu + (x_1 \, p_{20} - x_2 \, p_{10}) \, \dot{\nu}}{(x_1 \, p_{20} - x_2 \, p_{10}) \, \nu'} \right|_{\lambda = \lambda_j}.$$

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Using the above formulas as well as (2.4) we obtain

$$\begin{aligned}
\frac{\dot\lambda_1}{\lambda_1}+\frac{\dot\lambda_2}{\lambda_2} &= \frac{2\sqrt{q^2-1}}{h^2-q^2}\bigg(\frac{k\,q(\mathbf{E}'-\mathbf{K}')+h(\mathbf{E}'-k^2\,\mathbf{K}')}{1-k^2}(2\,k\,q-h(k^2+1))\\
&\qquad\qquad +(q\,\mathbf{E}'-h\,k\,\mathbf{E}')(q-h\,k)\bigg)\\
&= \frac{2\sqrt{(q^2-1)}}{k^2-1}((1+k^2)\mathbf{E}'-2\,k^2\,\mathbf{K}'),\\
\frac{\dot\lambda_1}{\lambda_1}-\frac{\dot\lambda_2}{\lambda_2} &= \frac{2\sqrt{h^2-1}}{h^2-q^2}\bigg(\frac{k\,q(\mathbf{E}'-\mathbf{K}')+h(\mathbf{E}'-k^2\,\mathbf{K}')}{1-k^2}((k^2+1)\,q-2\,h\,k)\\
&\qquad\qquad +(q\,\mathbf{E}'-h\,k\,\mathbf{E}')(k\,q-h)\bigg)\\
&= \frac{2\sqrt{h^2-1}}{k^2-1}\;k(2\mathbf{E}'-(1+k^2)\mathbf{K}')
\end{aligned}$$

and putting these into (2.5) gives the equations for $\dot q$ and $\dot h$, and concludes the proof of (2.12).

The elliptic integrals $\mathbf{K}'$ and $\mathbf{E}'$ are analytic on $k\in\mathbb{R}^\times$, and at $k=\pm1$ equal to $\frac{\pi}{2}$. Therefore the right hand sides of (2.12) extend analytically to $k\in\mathbb{R}^\times$. Due to Proposition 2.2, for $q,h\in(-1,1)$ we have

$$\dot q>0 \;\text{ for } 0<|k|<1,\quad \dot h>0 \;\text{ for }\; -1<k<0,\quad \dot h<0 \;\text{ for } 0<k<1. \tag{2.15}$$

By the properties of $\mathbf{K}'$ and $\mathbf{E}'$ the vector field $(\dot k,\dot q,\dot h)$ is analytic in k on $\mathbb{R}^\times$ and has simple zeros at $k=\pm1$. Thus the vector field is analytic on $\mathcal{D}$. The zero-set statement follows from the fact that on $\{k\neq0\}$, the functions $(1+k^2)\mathbf{E}'-2\,k^2\,\mathbf{K}'$ and $2\mathbf{E}'-(1+k^2)\mathbf{K}'$ have zeros only at $k=\pm1$, and $\mathbf{E}'=k\,\mathbf{K}'$ holds only at $k=1$. That $k=\pm1$ is a simple root follows from the series expansions at $k=1$ (similarly at $k=-1$), given by

$$\begin{aligned}
\mathbf{K}'(k) &= \pi\bigg(\frac12-\frac14(k-1)+\frac{5}{32}(k-1)^2-\frac{7}{64}(k-1)^3+O(k-1)^4\bigg),\\
\mathbf{E}'(k) &= \pi\bigg(\frac12+\frac14(k-1)+\frac{1}{32}(k-1)^2-\frac{1}{64}(k-1)^3+O(k-1)^4\bigg).
\end{aligned}$$

□

2.3. Global behaviour of the integral curves. The proof of the following Proposition 2.5 is deferred to Section 3.

Proposition 2.5. *An equivariant cmc torus in $\mathbb{S}^3$ is rotational if and only if the sym points are reciprocal.*

Special values of q and h include

$$\begin{aligned}
q^2=1 &\iff \lambda_1=\lambda_2^{-1}\text{ (rotational)}, & h^2=1 &\iff \lambda_1=\lambda_2\ (H=\infty),\\
q=0 &\iff \lambda_1=-\lambda_2^{-1}, & h=0 &\iff \lambda_1=-\lambda_2\ (H=0),\\
q=h &\iff \lambda_1=1\text{ or }\lambda_2=1, & q=-h &\iff \lambda_1=-1\text{ or }\lambda_2=-1.
\end{aligned}$$

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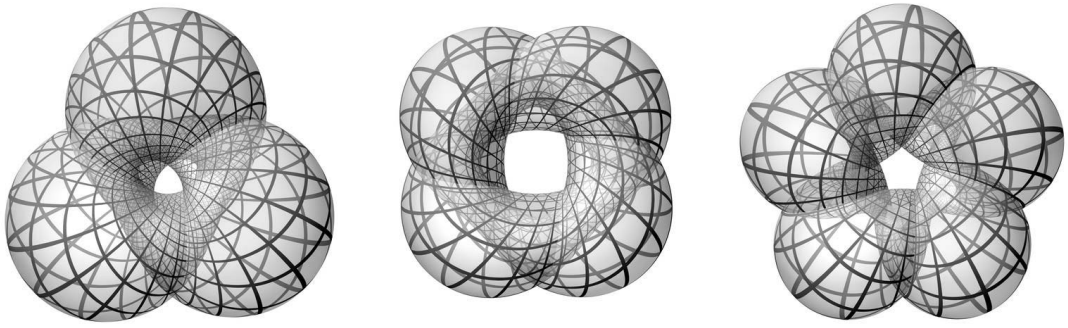

Figure 1. Equivariant $(2, 1, n)$ cmc tori ($n = 3, 4, 5$). By Proposition 5.3, there are no twizzled tori with one or two major lobes.

By the deformation equations (2.12) if $\mathrm{q}^2(t_0) = 1$, for some t_0, then $\mathrm{q}^2 \equiv 1$ throughout the flow. Hence rotational cmc tori stay rotational during the flow. We begin the qualitative analysis of the flow by first considering equivariant cmc tori which are not rotational ($\mathrm{q}^2 < 1$): we call these *twizzled* tori. The cmc tori of revolution ($\mathrm{q}^2 = 1$) are treated subsequently in Proposition 2.8.

The flow will be investigated in the open solid cuboid

$$\mathcal{B} := \{(\mathrm{k}, \mathrm{q}, \mathrm{h}) \in (-1, 1)^3 \mid \mathrm{k} \neq 0\}.$$

Due to (2.15), $\mathrm{q} - \mathrm{sign}(\mathrm{k})\,\mathrm{h}$ is strictly monotonic on $\mathcal{B}$ with the locally constant function sign(k).

Proposition 2.6. *Define the set*

$$\mathcal{L}_c := \left\{(\mathrm{k}, \mathrm{q}, \mathrm{h}) \in \mathcal{B} \mid (1-\mathrm{q}^2)(1-\mathrm{h}^2) = c\left(\frac{1+\mathrm{k}^2}{2\,\mathrm{k}} - \mathrm{q}\,\mathrm{h}\right)^2\right\}, \quad c \in \mathbb{R}_+.$$

Then the following statements hold:

(1) *Every integral curve in $\mathcal{B}$ lies in $\mathcal{L}_c \cap \mathcal{B}$ for some $c \in (0, 1)$.*

(2) *The following uniform estimates hold on the integral curve through $(\mathrm{k}_0, \mathrm{q}_0, \mathrm{h}_0) \in \mathcal{L}_c$:*

$$|\mathrm{k}| \geq \frac{\sqrt{c}}{2 + 2\sqrt{c}} \quad \textit{and} \quad \min\{1-\mathrm{q}^2, 1-\mathrm{h}^2\} \geq c\,(\min\{1 - |\,\mathrm{q}_0\,|, 1 - |\,\mathrm{h}_0\,|\})^2.$$

(3) *The function $\dot{\mathrm{k}}$ has at most one zero along any integral curve in $\mathcal{B}$.*

Proof. (1) From (2.12) we compute $\mathcal{W}_t[(1-\mathrm{q}^2)(1-\mathrm{h}^2), (\frac{1+\mathrm{k}^2}{2\,\mathrm{k}} - \mathrm{q}\,\mathrm{h})^2] = 0$, where $\mathcal{W}_t[X, Y] := X\dot{Y} - \dot{X}Y$ is the Wronskian with respect to the flow parameter t.

Since $(1-\mathrm{q}^2)(1-\mathrm{h}^2)$ and $(\frac{1+\mathrm{k}^2}{2\,\mathrm{k}} - \mathrm{q}\,\mathrm{h})^2$ are strictly positive in $\mathcal{B}$, every integral curve in $\mathcal{B}$ lies in $\mathcal{L}_c$ for some $c \in \mathbb{R}_+$. In $\mathcal{B}$, we have

$$(1-\mathrm{q}^2)(1-\mathrm{h}^2) \leq (1-\mathrm{q}\,\mathrm{h})^2 < \left(\frac{1+\mathrm{k}^2}{2\,\mathrm{k}} - \mathrm{q}\,\mathrm{h}\right)^2,$$

so $c \in (0, 1)$.

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(2) For $(\mathrm{k},\mathrm{q},\mathrm{h}) \in \mathscr{L}_c$ the first inequality follows from

$$\frac{1}{|\mathrm{k}|} \leq \frac{1+\mathrm{k}^2}{|\mathrm{k}|} \leq 2\sqrt{\frac{(1-\mathrm{q}^2)(1-\mathrm{h}^2)}{c}} + 2|\,\mathrm{q}\,\mathrm{h}\,| \leq \frac{2+2\sqrt{c}}{\sqrt{c}}.$$

On each integral curve k is either positive or negative. The second inequality follows from

$$\min\{1-\mathrm{q}^2, 1-\mathrm{h}^2\} \geq (1-\mathrm{q}^2)(1-\mathrm{h}^2) \geq c\,(1-\operatorname{sign}(\mathrm{k})\,\mathrm{q}\,\mathrm{h})^2 \geq c\,(\max\{1-|\,\mathrm{q}\,|, 1-|\,\mathrm{h}\,|\})^2.$$

In fact, due to (2.15) either $\operatorname{sign}(\mathrm{k})\,\mathrm{q}\,\mathrm{h} \leq 0$ or one of the functions $1-|\,\mathrm{q}\,|$ and $1-|\,\mathrm{h}\,|$ is increasing and the other one decreasing. Furthermore, if at some point t with $\operatorname{sign}(\mathrm{k})\,\mathrm{q}\,\mathrm{h} > 0$ one of these functions is increasing, it stays increasing for all $t_0 \leq t$, and if it is decreasing, it stays decreasing for all $t_0 \geq t$, since the derivative of these functions can change sign only at the maximal value 1.

(3) If $\dot{\mathrm{k}}(t_0) = 0$ for some value of the flow parameter $t = t_0$, then due to (2.12) and (2.15)

$$\operatorname{sign}(\mathrm{k})\ddot{\mathrm{k}} = \operatorname{sign}(\mathrm{k})\frac{d}{dt}\,\mathrm{k}(\mathbf{E}'\,\mathrm{q} - \mathrm{k}\,\mathbf{K}'\,\mathrm{h}) = |\mathrm{k}|(\mathbf{E}'\dot{\mathrm{q}} - \mathrm{k}\,\mathbf{K}'\dot{\mathrm{h}})) > 0$$

for $t = t_0$. Thus $\operatorname{sign}(\mathrm{k})\dot{\mathrm{k}}$ is increasing at each of its zeros, and hence can have at most one zero. □

The next result shows that one endpoint of each integral curve corresponds to a homogeneous torus.

Proposition 2.7. *Every maximal integral curve of the vector yield* (2.12) *in $\mathscr{B}$ is defined on a finite interval and passes from a point in $\{\mathrm{k} = \pm 1\} \cap \{\mathrm{q} - \mathrm{k}\,\mathrm{h} < 0\} \cap \partial\mathscr{B}$ to a point in $\{\mathrm{k} = \pm 1\} \cap \{\mathrm{q} - \mathrm{k}\,\mathrm{h} > 0\} \cap \partial\mathscr{B}$ with the same sign of* k. *Furthermore, the mean curvature $t \mapsto H(t)$ is a diffeomorphism of the maximal interval of definition onto a finite interval.*

Proof. Due to (2.15) both functions q and h are strictly monotonic. Since $\dot{\mathrm{q}}$ and $\dot{\mathrm{h}}$ have no roots in $\mathscr{B}$, the integral curve must hit the boundary of $\mathscr{B}$ at the end points (either infinite or finite) of the maximal interval of definition. Due to Proposition 2.6 (2), k takes the same value ± 1 at both endpoints and the vector field (2.12) does not vanish there. Hence the maximal interval of definition is finite. Due to Proposition 2.6 (3) the sign of $\dot{\mathrm{k}}$ changes once and is proportional to $\mathrm{q} - \operatorname{sign}(\mathrm{k})\,\mathrm{h}$ at the end points. Due to (2.15), $\mathrm{q} - \operatorname{sign}(\mathrm{k})\,\mathrm{h}$ is strictly increasing and the first claim follows. Since $\mathrm{h} \mapsto \mathrm{h}/\sqrt{1-\mathrm{h}^2}$ is strictly increasing on $(-1, 1)$, the mean curvature (2.3) is strictly monotonic, and by Proposition 2.6 (2) it is bounded. □

2.4. Global behaviour of the integral curves on the boundary. The boundary $\partial\mathscr{B}$ consists of the three parts with $\mathrm{k} = \pm 1$ or $\mathrm{k} = 0$, $\mathrm{q} = \pm 1$ and $\mathrm{h} = \pm 1$. The first set contains the homogeneous tori and the bouquets of spheres, which we consider later. The third set corresponds to the infinite mean curvature limit, that is cmc tori in $\mathbb{R}^3$. In this limit the two sym points coalesce and the differentials $d \ln \mu_i$ have zeroes there. Since dv and $d\omega$ do not have common zeroes by Corollary 2.3, there are no such examples with spectral genus zero or one. Therefore we treat only $\mathrm{q}^2 = 1$ and $\mathrm{h} \in (-1, 1)$.

By Proposition 2.5 the cmc tori of revolution appear in the 2-dimensional boundary $\{\mathrm{q}^2 = 1\}$ of the moduli space of equivariant cmc surfaces in $\mathbb{S}^3$. Since $\dot{\mathrm{q}} \equiv 0$ along $\mathrm{q}^2 = 1$, cmc tori of revolution stay cmc tori of revolution throughout the flow. The flow (2.12) thus

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describes one parameter families of cmc tori of revolution. Later in Theorem 5.9 we describe the range of mean curvature for the families in the flow, and exhibit those that contain a minimal torus of revolution. A consequence of the next result is that spectral genus $g = 1$ cmc tori of revolution lie in 1-parameter families with one endpoint at a homogeneous torus and the other endpoint at a sphere bouquet. In the process we also show that the mean curvature stays bounded during the flow.

Since we often need to evaluate the functions ν and ω at the two sym points, we set

$$\nu_k = \nu(\lambda_k) \quad \text{and} \quad \omega_k = \omega(\lambda_k) \quad \text{for } k = 1, 2. \tag{2.16}$$

Proposition 2.8. *On the integral curve of* (2.12) *through the point* $(\mathrm{k}_0, \mathrm{q}_0, \mathrm{h}_0) \in \partial\mathcal{B}$ *with* $\mathrm{q}_0 = \pm 1$ *and* $\mathrm{k}_0, \mathrm{h}_0 \in (-1, 1)$ *the function* q *is equal to* ± 1 *and* $1 - \mathrm{h}^2$ *is bounded away from zero. The maximal interval of definition is of the form* $(-\infty, t_{\max})$ *with*

$$\lim_{t \downarrow -\infty} \mathrm{k} = 0 \quad \textit{and} \quad \lim_{t \uparrow t_{\max}} \mathrm{k} = \pm 1.$$

The mean curvature $t \mapsto H(t)$ *is a diffeomorphism from* $(-\infty, t_{\max})$ *onto a finite interval.*

Proof. By (2.12), $\mathrm{q} \equiv \mathrm{q}_0$ is constant throughout the flow. Let $\lambda_i = e^{2\mathrm{i}\theta_i}$ for $i = 1, 2$ be the two sym points. If $|\theta_2 - \theta_1|$ is bounded away form zero, then $1 - \mathrm{h}^2$ is bounded away from zero. For $\mathrm{q} = \pm 1$ the values of ν at the sym points coincide.

By (1.8) we have that

$$\omega_2 - \omega_1 = \int_{\theta_1}^{\theta_2} d\omega$$

is constant throughout the flow. We claim that

$$|d\omega| \leq 2|d\theta| \quad \text{for } \theta \in \mathbb{R} \text{ and } \mathrm{k} \in [-1, 1].$$

We will need to consider the two cases $\mathrm{k}\cos(2\theta) \leq 0$ and $\mathrm{k}\cos(2\theta) \geq 0$ separately. In the first case we use $\mathbf{E}' \leq \frac{\pi}{2}$ (see [14, Section 13.8.(11)]), Proposition 2.2 and (2.6) to obtain

$$|d\omega| = \left|\frac{\mathbf{E}' - \mathrm{k}\,\mathbf{K}'\cos(2\theta)}{\pi\,\nu}\right| |d\theta| \leq \left|\frac{\mathbf{E}'(1 - \frac{2\,\mathrm{k}}{1+\mathrm{k}^2}\cos(2\theta))}{\pi\,\nu}\right| |d\theta| \leq \left|\frac{2\nu}{1+\mathrm{k}^2}\right| |d\theta| \leq 2|d\theta|.$$

When $\mathrm{k}\cos(2\theta) \leq 0$, then again by $\mathbf{E}' \leq \frac{\pi}{2}$, Proposition 2.2, and $2|\nu| \geq \sqrt{1+\mathrm{k}^2} \geq 1$ (2.6) we get

$$|d\omega| = \left|\frac{\mathbf{E}' - \mathrm{k}\,\mathbf{K}'\cos(2\theta)}{\pi\,\nu}\right| |d\theta| \leq \left|\frac{4\mathbf{E}'}{\pi\sqrt{1+\mathrm{k}^2}}\right| |d\theta| \leq 2|d\theta|.$$

Thus $|\theta_2 - \theta_1| \geq \frac{1}{2}|\omega_2 - \omega_1|$. Due to (2.11) and Proposition 2.2, the differential $d\omega$ has no root between θ_1 and θ_2. Therefore $|\theta_2 - \theta_1|$ and $1 - \mathrm{h}^2$ are bounded away from zero.

Due to Proposition 2.2, the function k^2 is strictly increasing for $0 < |\mathrm{k}| < 1$. Therefore the maximal integral curve passes from $\mathrm{k} = 0$ to $\mathrm{k} = \pm 1$. The vector field is at the left end point of order O(k) and at the right end point not zero. Therefore the maximal interval of definition has the form $(-\infty, t_{\max})$.

Since h is strictly monotonic, and $1 - \mathrm{h}^2$ is bounded away from zero, the mean curvature (2.3) is a diffeomorphism from $(-\infty, t_{\max})$ onto a finite interval. □

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2.5. Regularity of spectral curve. A corollary of Theorem 2.9 below is that equivariant cmc tori cannot be dressed to cmc tori by simple factors [24, 35]. While there exist large families of cmc cylinders which can be dressed to cylinders with bubbletons [34], it is still an open question raised by Bobenko [5], whether there are cmc tori with bubbletons. A *double point* is a point on the spectral curve at which both logarithms μ_1 and μ_2 are unimodular, but the spectral curve is not branched.

Theorem 2.9. *The spectral curve of an equivariant cmc torus has no double points in* $\mathbb{C}^\times \setminus \mathbb{S}^1$.

Proof. At a double point both logarithms μ_1 and μ_2 are unimodular. Therefore it suffices to prove that the set of all $\lambda \in \mathbb{C}^\times$ where μ_1 and μ_2 are unimodular is the set $\mathbb{S}^1 \cup \{\mathrm{k}, \mathrm{k}^{-1}\}$. This set coincides with the subset of $(\lambda, \nu) \in \mathrm{Y}$, such that ν and ω are real. Due to (2.6) and (2.11) we have for $\mathrm{k} \in (0, 1]$: $\nu \in \mathbb{R}$ if and only if $\lambda \in \mathbb{S}^1 \cup [\mathrm{k}, \mathrm{k}^{-1}] \cup \mathbb{R}_-$, and $\lambda \in \mathbb{R}$ and $\omega \in \mathbb{R}$ if and only if $\lambda \in [0, \mathrm{k}] \cup [\mathrm{k}^{-1}, \infty)$. For $\mathrm{k} \in [-1, 0)$ we have $\nu \in \mathbb{R}$ if and only if $\lambda \in \mathbb{S}^1 \cup [\mathrm{k}^{-1}, \mathrm{k}] \cup \mathbb{R}_+$; $\lambda \in \mathbb{R}$ and $\omega \in \mathbb{R}$ if and only if $\lambda \in (-\infty, \mathrm{k}^{-1}] \cup [\mathrm{k}, 0]$. With $\mathrm{k} = \pm 1$ we cover the genus zero case. □

3. Equivariant extended frames

To obtain a more geometric description of the above deformation we compute the frames of the surfaces. We shall do this in a slightly broader context by studying equivariant maps $\mathbb{C}^2 \to \mathrm{SL}_2\mathbb{C}$ with complex mean curvature as in [13]. This generalized framework for complex equivariant surfaces is then applicable to not only cmc immersions into 3-dimensional space forms, but also pseudospherical surfaces in $\mathbb{R}^3$, cmc surfaces in Minkowski space, and other integrable surfaces in many others spaces by performing an appropriate reduction.

Let $\Sigma \subset \mathbb{C}^2$ be an open and simply-connected domain with complex coordinates (z, w), and define $*$ on 1-forms on Σ by $*dz = \mathbb{i}dz$, $*dw = -\mathbb{i}dw$ extended complex linearly.

Let $\mathrm{sl}_2\mathbb{C} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition. For a 1-form α on Σ, we will write $\alpha = \alpha_\mathfrak{k} + \alpha_\mathfrak{p}$ where with $\alpha_\mathfrak{k} \in \mathfrak{k}$ and $\alpha_\mathfrak{p} \in \mathfrak{p}$. If we equip the matrix Lie algebra $\mathrm{sl}_2\mathbb{C}$ with a multiple of the Killing form $\langle X, Y\rangle := -\frac{1}{2}\operatorname{tr}(XY)$, then we may choose a basis e_0, e_1, e_2 for $\mathrm{sl}_2\mathbb{C}$ satisfying $e_0 \in \mathfrak{k}$, $\langle e_0, e_0\rangle = 1$ and $[e_0, e_1] = e_2$, $[e_1, e_2] = e_0$ and $[e_2, e_0] = e_1$, and set

$$\epsilon_1 = \frac{1}{2}(e_1 - \mathbb{i}e_2), \quad \epsilon_2 = \frac{1}{2}(e_1 + \mathbb{i}e_2).$$

Let $f : \Sigma \to \mathrm{SL}_2\mathbb{C}$ be a conformal immersion such that for the smooth function $v : \Sigma \to \mathbb{C}^*$ we have

$$\langle f_z, f_z\rangle = 0 = \langle f_w, f_w\rangle \quad \text{and} \quad 2\langle f_z, f_w\rangle = v^2.$$

The remaining invariants of f are smooth functions $H, Q, R : \Sigma \to \mathbb{C}$ such that

$$2\langle f_{zz}, N\rangle = Q, \quad 2\langle f_{zw}, N\rangle = v^2 H \quad \text{and} \quad 2\langle f_{ww}, N\rangle = R,$$

and the normal map is $N = -\mathbb{i}v^{-2}[f_z, f_w]$. Further, there exists a unique pair of maps $F, G : \Sigma \to \mathrm{SL}_2\mathbb{C}$ which frame f in the sense that

$$f = FG^{-1}, \quad f_z = vF\epsilon_1 G^{-1}, \quad f_w = vF\epsilon_2 G^{-1}, \quad N = Fe_0 G^{-1}. \tag{3.1}$$

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Then $\alpha_{+1} := F^{-1}dF$ and $\alpha_{-1} := G^{-1}dG$, are smooth 1-forms on Σ with values in $\mathfrak{sl}_2\mathbb{C}$ given by

$$\alpha_\sigma = A_\sigma\, dz + B_\sigma\, dw \tag{3.2}$$

where $\sigma \in \{\pm 1\}$ and

$$\begin{aligned} 2\mathbb{i}\, A_\sigma &:= v^{-1}v_z e_0 + v(H + \mathbb{i}\sigma)\epsilon_1 - v^{-1}Q\epsilon_2, \\ 2\mathbb{i}\, B_\sigma &:= -v^{-1}v_w e_0 + v^{-1}R\epsilon_1 - v(H - \mathbb{i}\sigma)\epsilon_2. \end{aligned}$$

Integrability $2d\alpha_\sigma + [\alpha_\sigma \wedge \alpha_\sigma] = 0$ is equivalent to the Gauss–Codazzi equations

$$(\log v)_{zw} - v^2(H^2 + \sigma^{-2}) + v^{-2}QR = 0, \tag{3.3a}$$

$$v^2 H_w = R_z, \quad v^2 H_z = Q_w. \tag{3.3b}$$

Conversely, given smooth functions $v, H, Q, R : \Sigma \to \mathbb{C}$ with v non-vanishing, satisfying the integrability equations (3.3), a conformal immersion f with these invariants can be recovered by integrating (3.2) to obtain maps $F, G : \Sigma \to \mathrm{SL}_2\mathbb{C}$ which are unique up to transforms $(F, G) \mapsto (AF, BG)$, $A, B \in \mathrm{SL}_2\mathbb{C}$. By the form of α_σ, the maps F and G frame $f := FG^{-1}$ in the sense of (3.1), and f has the specified invariants. The map f is unique up to $f \mapsto AfB^{-1}$, $A, B \in \mathrm{SL}_2\mathbb{C}$.

3.1. Equivariance. Let us first introduce new coordinates (x, y) on Σ such that

$$z = x + \mathbb{i}\, y, \quad w = x - \mathbb{i}\, y. \tag{3.4}$$

We say a smooth map $F = F(x, y) : \mathbb{C}^2 \to \mathrm{SL}_2\mathbb{C}$ is $\mathbb{C}$-*equivariant*, or simply *equivariant*, if it is of the form $F(x, y) = \exp(x\, A)\, P(y)$ for some $A \in \mathfrak{sl}_2\mathbb{C}$ and smooth map $P : \mathbb{C} \to \mathrm{SL}_2\mathbb{C}$. What characterizes an equivariant map is that $F^{-1}dF = P^{-1}A\, P\, dx + P^{-1}P'\, dy$ is x-independent [9]. We first compute the logarithmic derivative of an equivariant map, and then integrate this to obtain the equivariant map itself.

Theorem 3.1. (1) *The x-independent solution $\alpha = \alpha_\mathfrak{k} + \alpha_\mathfrak{p}$ to*

$$\begin{aligned} d\alpha_\mathfrak{k} + \frac{1}{2}[\alpha_\mathfrak{p} \wedge \alpha_\mathfrak{p}] &= 0, \\ d\alpha_\mathfrak{p} + [\alpha_\mathfrak{k} \wedge \alpha_\mathfrak{p}] &= 0, \\ d(*\alpha_\mathfrak{p}) + [\alpha_\mathfrak{k} \wedge (*\alpha_\mathfrak{p})] &= 0 \end{aligned} \tag{3.5}$$

are $\alpha = g^{-1}\Omega\, g + g^{-1}dg$, where g is a smooth x-independent map from Σ into the Lie group of $\mathfrak{k}$, and $\Omega = \Omega_1\, dz + \Omega_2\, dw$, where

$$\begin{aligned} 2\mathbb{i}\,\Omega_1 &:= -\frac{\mathbb{i}}{2}v^{-1}v' e_0 + a_2 v\epsilon_1 - b_1 v^{-1}\epsilon_2, \\ 2\mathbb{i}\,\Omega_2 &:= -\frac{\mathbb{i}}{2}v^{-1}v' e_0 + b_2 v^{-1}\epsilon_1 - a_1 v\epsilon_2, \end{aligned} \tag{3.6}$$

with $a_1, a_2, b_1, b_2, v \in \mathbb{C}$, and $v = v(y) : \mathbb{C} \to \mathbb{C}$ is defined by

$$\begin{aligned} (v^{-1}v')^2 + (a_1 v + b_1 v^{-1})(a_2 v + b_2 v^{-1}) &= 4v^2, \\ (v^{-1}v')' + a_1 a_2 v^2 - b_1 b_2 v^{-2} &= 0. \end{aligned} \tag{3.7}$$

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(2) *A solution to $dF = F\Omega$ is given by*

$$
\begin{aligned}
F(x,y) &:= \exp\left(\left(x\,\nu + \frac{1}{2}\,\chi_0\right)e_0\right)\exp\left(\frac{1}{2}\chi_1 e_1\right)\exp\left(\frac{1}{2}\chi_2 e_0\right), \quad \textit{with}\\
\chi_0 &:= 2\mathbb{i}\,\nu\int_0^y (J_1(t)-J_2(t))dt,\\
\chi_1 &:= \arccos\left(-\frac{1}{2}\nu^{-1}v^{-1}v'\right), \qquad \chi_2 := \frac{\mathbb{i}}{2}\log(X_1^{-1}X_2),\\
X_1 &:= a_1 v + b_1 v^{-1}, \qquad X_2 := a_2 v + b_2 v^{-1},\\
J_1 &:= b_1 v^{-1} X_1^{-1}, \qquad J_2 := b_2 v^{-1} X_2^{-1}.
\end{aligned}
\tag{3.8}
$$

Remark 3.2. The second equation in (3.7) is the derivative of the first, and eliminates spurious constant solutions which would appear if only the first equation were present.

Proof. (1) Write $\alpha_{\mathfrak{k}} = ae_0\,dx + be_0\,dy$ for some functions $a = a(y)$ and $b = b(y)$, and let

$$g = \exp\left(-\left(\int^y b(t)\,dt\right)e_0\right).$$

Then $dg\,g^{-1} = -be_0\,dy$, so the form $g^{-1}\alpha g + g^{-1}dg = ae_0\,dx + g^{-1}\alpha_{\mathfrak{p}}g$ is a multiple of $dx = \frac{1}{2}(dz+dw)$. Then $\alpha_{\mathfrak{k}} = \frac{\mathbb{i}}{4}f(dz+dw)e_0$ for some function $f = f(y)$. Let

$$v = \exp\left(\int^y f(t)dt\right),$$

so $\alpha_{\mathfrak{k}} = \frac{\mathbb{i}}{4}v^{-1}v'e_0(dz+dw)$, where prime is differentiation with respect to y. Then there exist functions a_1, b_1, a_2, b_2 of y such that

$$2\alpha_{\mathfrak{p}} = (a_2 v\epsilon_1 - b_1 v^{-1}\epsilon_2)\,dz + (b_2 v\epsilon_1 - a_1 v^{-1}\epsilon_2)\,dw.$$

By a calculation, (3.5) is equivalent to the second equation of (3.7), along with

$$a_1' = b_1' = a_2' = b_2' = 0.$$

Hence α is of the required form, and concludes the proof of (1).

To prove (2), we have $\Omega = \Omega_x dx + \Omega_y dy$ in (3.6), where

$$2\mathbb{i}\,\Omega_x := -\mathbb{i}\,v^{-1}v'e_0 + (a_2 v + b_2 v^{-1})\epsilon_1 - (a_1 v + b_1 v^{-1})\epsilon_2$$

and

$$2\Omega_y := (a_2 v - b_2 v^{-1})\epsilon_1 + (a_1 v - b_1 v^{-1})\epsilon_2.$$

To compute $F^{-1}dF$, we will first show that $P(x) := \exp(\frac{1}{2}\chi_0 e_0)\exp(\frac{1}{2}\chi_1 e_1)\exp(\frac{1}{2}\chi_2 e_0)$ satisfies

$$\nu P^{-1}e_0 P = \Omega_x, \tag{3.9a}$$

$$P^{-1}P' = \Omega_y. \tag{3.9b}$$

We have

$$P^{-1}e_0 P = \cos\chi_1 e_0 + \mathbb{i}e^{-\mathbb{i}\chi_2}\sin\chi_1\epsilon_1 - \mathbb{i}e^{\mathbb{i}\chi_2}\sin\chi_1\epsilon_2$$

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and $\cos\chi_1 = -\frac{1}{2}v^{-1}v^{-1}v'$, $\sin\chi_1 = -\frac{1}{2}v^{-1}X_1^{\frac{1}{2}}X_2^{\frac{1}{2}}$, $e^{\mathbb{i}\chi_2} = X_1^{\frac{1}{2}}X_2^{-\frac{1}{2}}$ and thus

$$2\mathbb{i}vP^{-1}e_0P = \mathbb{i}v^{-1}v'e_0 - X_2\epsilon_1 + X_1\epsilon_2 = -2(\Omega_1 - \Omega_2) = 2\mathbb{i}\Omega_x,$$

proving (3.9a).

Now

$$2P^{-1}P' = (\chi_0'\cos\chi_1 + \chi_2')e_0 + e^{-\mathbb{i}\chi_2}(\mathbb{i}\chi_0'\sin\chi_1 + \chi_1')\epsilon_1 - e^{\mathbb{i}\chi_2}(\mathbb{i}\chi_0'\sin\chi_1 - \chi_1')\epsilon_2.$$

With $C_5 := a_1b_2 - a_2b_1$ we have

$$\chi_0' = -\frac{2\mathbb{i}\nu\, C_5}{X_1X_2}, \quad \chi_1' = \frac{a_1a_2v^2 - b_1b_2v^{-2}}{\sqrt{X_1}\sqrt{X_2}}, \quad \chi_2' = -\frac{\mathbb{i}\, C_5\, v'}{vX_1X_2}. \tag{3.10}$$

Hence $2P^{-1}P' = (a_2v - b_2v^{-1})\epsilon_1 + (a_1v - b_1v^{-1})\epsilon_2 = 2\Omega_y$, proving (3.9b).

Thus

$$\begin{aligned} F^{-1}dF &= F^{-1}F_x\,dx + F^{-1}F_y\,dy \\ &= vP^{-1}e_0P\,dx + P^{-1}P'\,dy \\ &= \Omega_x\,dx + \Omega_y\,dy = \Omega. \end{aligned}$$

□

Example 3.3. The vacuum is the case in which the function v in (3.7) is constant. By the second equation in (3.7), $v \equiv v_0$ with $v_0^4 := (b_1b_2)/(a_1a_2)$. The potential for the vacuum is $\Omega_V := \Omega_z\,dz + \Omega_w\,dw$ where $2\mathbb{i}\Omega_z := a_2v_0\epsilon_1 - b_1v_0^{-1}\epsilon_2$ and $2\mathbb{i}\Omega_w := b_2v_0^{-1}\epsilon_1 - a_1v_0\epsilon_2$. Since $[\Omega_z, \Omega_w] = 0$, the extended frame of the vacuum is $F = \exp(\Omega_z z + \Omega_w w)$ with eigenvalues $\exp(\pm\frac{\mathbb{i}}{2}(rz + sw))$ where $r, s \in \mathbb{C}$ are determined by

$$r^2 = a_2b_1, \quad s^2 = a_1b_2, \quad rs = a_1a_2v_0^2.$$

This differs from the frame in Theorem 3.1 by left multiplication by a z- and w-independent element of $\mathrm{SL}_2\mathbb{C}$.

To describe equivariant cmc immersions into $\mathbb{S}^3 = \mathrm{SU}_2$ we specialize the above formulas. We first make the reduction in (3.4) that $w = \bar{z}$, which is equivalent to $x, y \in \mathbb{R}$. Given an extended frame $F_\lambda : \mathbb{R}^2 \to \mathrm{SU}_2$ and two distinct sym points $\lambda_1, \lambda_2 \in \mathbb{S}^1$, then

$$f := F_{\lambda_1}F_{\lambda_2}^{-1} \tag{3.11}$$

is a conformal immersion $f : \mathbb{R}^2 \to \mathbb{S}^3$ with constant mean curvature H given in (1.1). For the translation $\tau_\gamma : \mathbb{C} \to \mathbb{C}$, $p \mapsto p + \gamma$ we write $\tau_\gamma^* f = f \circ \tau_\gamma$. If $F_\lambda^{-1}dF_\lambda$ is periodic with period γ, then we define the monodromy M_λ of F_λ with respect to γ as

$$M_\lambda(\gamma) = \tau_\gamma^* F_\lambda\, F_\lambda^{-1}. \tag{3.12}$$

The closing condition $\tau_\gamma^* f = f$ with respect to a translation is equivalent to

$$M_{\lambda_1}(\gamma) = M_{\lambda_2}(\gamma) = \mathbb{1} \quad \text{or} \quad M_{\lambda_1}(\gamma) = M_{\lambda_2}(\gamma) = -\mathbb{1}\,. \tag{3.13}$$

If μ_1 and $\mu_2 = \mu_1^{-1}$ denote the eigenvalues of the monodromy, then (3.13) reads $\mu_j(\lambda_k) = \pm 1$, or equivalently that there exist four integers $p_{jk} \in \mathbb{Z}$ such that for $j, k = 1, 2$ we have

$$\ln\mu_j(\lambda_k) = \mathbb{i}\pi\, p_{jk} \quad \text{and} \quad p_{j1} - p_{j2} \in 2\mathbb{Z}. \tag{3.14}$$

The torus is embedded if and only if the *winding numbers* p_{jk} all have absolute value equal to one.

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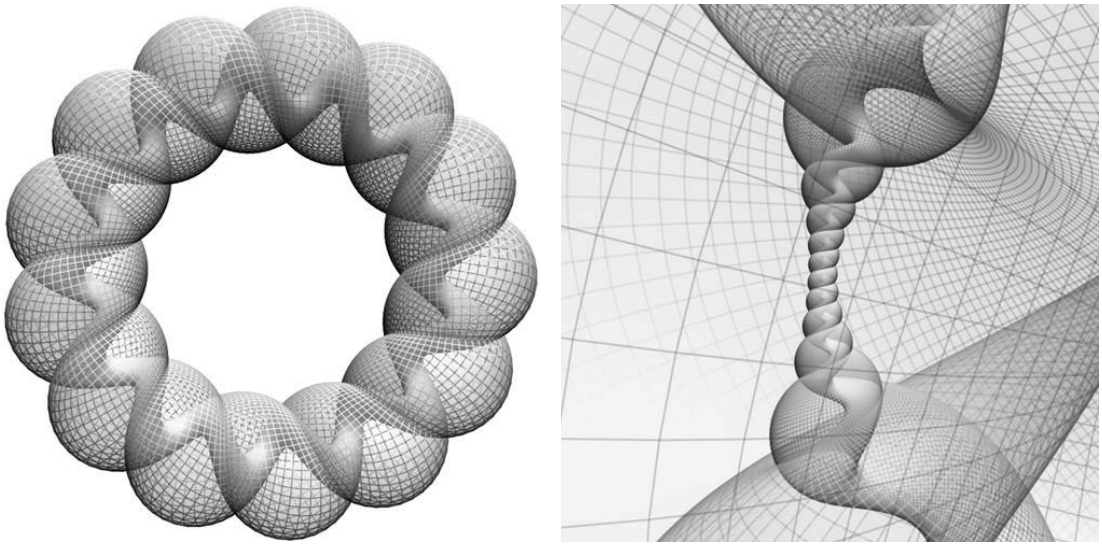

Figure 2. Two views of an equivariant cmc $(2, 1, 13)$ torus in $\mathbb{S}^3$.

3.2. Homogeneous tori in $\mathbb{S}^3$. Integrating Ω (see (3.6)) with

$$(a_1, b_1, a_2, b_2) := (\lambda, 1, \lambda^{-1}, 1), \quad v \equiv 1$$

yields an extended frame F_λ of a homogeneous surface in $\mathbb{S}^3$. By Theorem 3.1 and Example 3.3, the extended frame of any homogeneous surface (up to isometry and conformal change of coordinates) is

$$F_\lambda = \exp\big(\pi\, \mathbb{i}\big((z\lambda^{-1} + \overline{z})\epsilon_1 - (z + \overline{z}\lambda)\epsilon_2\big)\big). \tag{3.15}$$

Then also (up to isometry and conformal change of coordinates) any homogeneous immersion $f : \mathbb{R}^2 \to \mathbb{S}^3$ is of the form $f = F_{\lambda_1} F_{\lambda_2}^{-1}$ with extended frame F_λ as in (3.15) and two distinct sym points $\lambda_1, \lambda_2 \in \mathbb{S}^1$. The eigenvalues of F_λ are $\mu^{\pm 1}$ with

$$\mu(z, \lambda) = \exp\big(\pi\, \mathbb{i}(z\lambda^{-\frac{1}{2}} + \overline{z}\lambda^{\frac{1}{2}})\big). \tag{3.16}$$

Define

$$\langle\!\langle x, y \rangle\!\rangle := \frac{1}{2}\,(x\,\overline{y} + y\,\overline{x}) = \Re(x\overline{y}). \tag{3.17}$$

Then by equation (3.14) the immersion f factors through the lattice $\Gamma = \gamma_1\,\mathbb{Z} + \gamma_2\,\mathbb{Z}$ if and only if

$$\langle\!\langle \gamma_j, \lambda_k^{1/2} \rangle\!\rangle \in \mathbb{Z} \quad \text{and} \quad \langle\!\langle \gamma_j, \lambda_1^{1/2} - \lambda_2^{1/2} \rangle\!\rangle \in 2\mathbb{Z}. \tag{3.18}$$

The dual of a lattice Γ in $\mathbb{C}$ is the lattice $\Gamma^* = \{\kappa \in \mathbb{C} \mid \langle\!\langle \kappa, \gamma \rangle\!\rangle \in \mathbb{Z} \text{ for all } \gamma \in \Gamma\}$.

Proposition 3.4. (i) *A homogeneous immersion $f = F_{\lambda_1} F_{\lambda_2}^{-1}$ with extended frame F_λ (see* (3.15)*) is closed with respect to a lattice $\Gamma \subset \mathbb{C}$ if and only if $\Gamma \subseteq \Lambda^*$, where*

$$\Lambda := \kappa_1\mathbb{Z} + \kappa_2\mathbb{Z}, \quad \kappa_1 := \frac{1}{2}(\lambda_1^{1/2} + \lambda_2^{1/2}) \quad \textit{and} \quad \kappa_2 := \frac{1}{2}(\lambda_1^{1/2} - \lambda_2^{1/2}).$$

(ii) *The torus is rectangular and embedded if and only if $\Gamma = \Lambda^*$.*

(iii) *Every homogeneous torus is isogenic to a rectangular embedded homogeneous torus.*

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Proof. (i) Since $\lambda_k^{1/2} = \kappa_1 \pm \kappa_2$ the condition (3.18) is equivalent to $\Gamma \subseteq \Lambda^*$.

(ii) The torus is embedded if and only if $\langle\!\langle \gamma_j, \lambda_k^{1/2} \rangle\!\rangle \in \{\pm 1\}$, and rectangular if and only if $\gamma_1/\gamma_2 \in \mathbb{i}\,\mathbb{R}$. The corresponding periods γ_1 and γ_2 are dual to κ_1 and κ_2 and generate Λ^*. Clearly Λ, and consequently Λ^* are rectangular, since $\kappa_1/\kappa_2 \in \mathbb{i}\,\mathbb{R}$.

(iii) From (i) we know that a lattice Γ of any torus is a sublattice of Λ^*, which by (ii) is the lattice of the embedded rectangular torus. Hence there is an isogeny taking Λ^* to Γ. □

Proposition 3.5. *The lattice of an embedded homogeneous torus is square if and only if the mean curvature is zero. Swapping the sym points does not affect the period lattice of a homogeneous torus.*

Proof. Solving the four equations (3.14) for the periods gives

$$\begin{aligned} \gamma_1 &= \frac{\lambda_1^{1/2}\lambda_2\, p_{11} - \lambda_1\lambda_2^{1/2}\, p_{12}}{\lambda_2 - \lambda_1}, \\ \gamma_2 &= \frac{\lambda_1^{1/2}\lambda_2\, p_{21} - \lambda_1\lambda_2^{1/2}\, p_{22}}{\lambda_2 - \lambda_1}. \end{aligned} \tag{3.19}$$

In particular, setting $p_{11} = p_{12} = p_{21} = -p_{22} = 1$, a computation shows that the generators of a lattice of an embedded homogeneous torus satisfy

$$\frac{|\gamma_1|}{|\gamma_2|} = |\pm\sqrt{1+H^2} + H|.$$

Thus $|\gamma_1| = |\gamma_2|$ if and only if the mean curvature is zero. Setting $\lambda_1 = \mathbb{i}, \lambda_2 = -\mathbb{i}$ we obtain the generators $\gamma_1 = 1/\sqrt{2}$ and $\gamma_2 = \mathbb{i}/\sqrt{2}$ of the square lattice of the Clifford torus.

Swapping the sym points $\lambda_1 \leftrightarrow \lambda_2$ results in the integers p_{jk} swapping second indices: $p_{11} \leftrightarrow p_{12}$, $p_{21} \leftrightarrow p_{22}$. This does not change the periods γ_1, γ_2 in (3.19). □

3.3. Spectral genus one. Consider the 1-form $\Omega = \Omega_1\, dz + \Omega_2\, d\bar{z}$ where Ω_j are given in equation (3.6) with ν as in (2.6) for some $0 < |\,\mathrm{k}\,| \leq 1$ and

$$(a_1, b_1, a_2, b_2) := (\lambda, -\mathrm{k}, \lambda^{-1}, -\mathrm{k}).$$

Integrating the 1-form Ω with these choices then yields an $\mathbb{R}$-equivariant extended frame

$$F_\lambda(x, y) = \exp(x\, A_\lambda)\, P_\lambda(y)$$

for smooth maps $\lambda \mapsto A_\lambda, \mathbb{S}^1 \to \mathrm{su}_2$ and $(\lambda, y) \mapsto P_\lambda(y), \mathbb{S}^1 \times \mathbb{R} \to \mathrm{SU}_2$. Now let $\lambda_1, \lambda_2 \in \mathbb{S}^1$ be two distinct sym points. The immersion $f : \mathbb{C} \to \mathrm{SU}_2$ in (3.11) is then a conformal equivariant cmc immersion with mean curvature (1.1).

Equivariance reduces the Gauss equation (3.7) to

$$(v')^2 + (v^2 - 1)(v^2 - \mathrm{k}^2) = 0.$$

A solution to this equation is the *square root of the conformal factor* $v : \mathbb{R} \to \mathbb{R}_+$ given by $v(y) = \mathrm{dn}(y, 1 - \mathrm{k}^2)$ where $\mathrm{dn}(y, m)$ is the Jacobi elliptic function. All other solutions are of the form $v(y + y_0)$, $y_0 \in \mathbb{R}$. The function v is even and has no zeros on $\mathbb{R}$. The range

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of v is $[\min(1,|\mathrm{k}|),\max(1,|\mathrm{k}|)]$. The function satisfies $v(0)=1$ and $v'(0)=0$, and limits to $\lim_{|\mathrm{k}|\to 1} v(y)=1$ and $\lim_{\mathrm{k}\to 0} v(y)=\operatorname{sech} y$. The period of v depends on the parameter $0<|\mathrm{k}|\leq 1$ and is equal to $2\mathbf{K}'(\mathrm{k})$. Now a straightforward calculation shows that the fundamental forms of such an equivariant conformal cmc immersion are

$$-\frac{(\lambda_2-\lambda_1)^2}{4\lambda_1\lambda_2}\,v^2\,(dx^2+dy^2), \tag{3.20a}$$

$$(v^2H+\operatorname{Re}(Q))\,dx^2-\operatorname{Im}(Q)\,dx\,dy+(v^2H-\operatorname{Re}(Q))\,dy^2, \tag{3.20b}$$

with H as in (1.1) and Hopf differential $Q\,dz^2$ with $Q:=\frac{\mathrm{i}}{4}\,\mathrm{k}(\lambda_2^{-1}-\lambda_1^{-1})$.

Proposition 3.6. *A period of an equivariant extended frame is of the form*

$$\gamma = x\,\pi + 2\mathrm{i}\,p\,\mathbf{K}' \tag{3.21}$$

where $x\in\mathbb{R}$, $p\in\mathbb{Z}$, and $2\mathbf{K}'$ is the period of v. The monodromy (3.12) *with respect to such a period is*

$$M_\lambda(\gamma)=\exp(\pi(x\,\nu+p\,\omega)e_0). \tag{3.22}$$

Proof. The imaginary part of a frame period has to be a period of the square root of the conformal factor $v:\mathbb{R}\to\mathbb{R}_+$, $y\mapsto \mathrm{dn}(y,1-\mathrm{k}^2)$. Since v has period $2\mathbf{K}'(\mathrm{k})$, the imaginary part of γ in (3.21) has to be of the form $2\,p\,\mathbf{K}'$ for some $p\in\mathbb{Z}$. From (3.8) the extended frame of an equivariant cmc torus is of the form

$$F(x,y):=\exp\left(\left(x\,\nu+\frac{1}{2}\,\chi_0\right)e_0\right)\exp\left(\frac{1}{2}\chi_1 e_1\right)\exp\left(\frac{1}{2}\chi_2 e_0\right).$$

The middle and right factor are periodic in y and do not depend on x. Hence both these factors have trivial monodromy if $y\in 2\,\mathbf{K}'\mathbb{Z}$, and the monodromy with respect to a translation by $\gamma=x\,\pi+2\mathrm{i}\,p\,\mathbf{K}'$ is

$$F(x\pi,2n\mathbf{K}')=\exp\left(\left(\pi\,x\,\nu+\frac{1}{2}\,\chi_0(2p\mathbf{K}')\right)e_0\right).$$

Clearly $\chi_0(2p\mathbf{K}')=p\chi_0(2\mathbf{K}')$ so it suffices to show that $\chi_0(2\mathbf{K}',\mathrm{k},\lambda)=2\pi\omega(\mathrm{k},\lambda)$ to conclude the proof. Let $\lambda=e^{i\theta}$; then a calculation yields

$$d(\nu J_1)=-\frac{\frac{d}{d\theta}(-\mathrm{k}(\lambda+\lambda^{-1}))}{4\nu\,\mathrm{k}(\lambda^{-1}-\lambda)}\left(\frac{-\mathrm{k}\,\lambda^{-1}}{J_1}-\frac{1}{2}\frac{d^2}{dt^2}\log J_1(t)\right)d\theta, \tag{3.23}$$

$$d(\nu J_2)=\frac{\frac{d}{d\theta}(-\mathrm{k}(\lambda+\lambda^{-1}))}{4\nu\,\mathrm{k}(\lambda^{-1}-\lambda)}\left(\frac{-\mathrm{k}\,\lambda}{J_2}-\frac{1}{2}\frac{d^2}{dt^2}\log J_2(t)\right)d\theta. \tag{3.24}$$

Since J_1 and J_2 are functions of v, they are also periodic with period $2\mathbf{K}'$, so

$$\int_0^{2\mathbf{K}'}\frac{d^2}{dt^2}\log J_k\,dt=\left.\frac{\frac{d}{dt}J_k}{J_k}\right|_0^{2\mathbf{K}'}=0.$$

Subtracting (3.24) from (3.23) and integrating over the interval $[0,2\mathbf{K}']$ gives

$$d\,\chi_0=\frac{1}{4\nu}\big(4\mathbf{E}'-2\mathbf{K}'\,\mathrm{k}\,(\lambda+\lambda^{-1})\big)\,d\theta=2\pi\,d\omega.$$

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Further, since

$$\chi_0(2\mathbf{K}', \mathrm{k}, \lambda) = 2\mathbb{i}\nu \int_0^{2\mathbf{K}'} \frac{\mathrm{k}}{v(\lambda^{-1}v - \mathrm{k}\, v^{-1})} - \frac{\mathrm{k}}{v(\lambda v - \mathrm{k}\, v^{-1})}\, dt$$

and $\nu(\lambda^{-1}) = \nu(\lambda)$ it follows that $\chi_0(2\mathbf{K}', \mathrm{k}, \lambda^{-1}) = -\chi_0(2\mathbf{K}', \mathrm{k}, \lambda)$. Thus χ_0 shares the properties of ω which determine it uniquely. Hence $\chi_0(2\mathbf{K}', \mathrm{k}, \lambda) = 2\pi\omega(\mathrm{k}, \lambda)$. □

By (3.22) we have

$$\begin{aligned}\tau_\gamma f &= M_{\lambda_1}(\gamma) f M_{\lambda_2}^{-1}(\gamma)\\ &= \exp(\pi(x\,\nu_1 + n\,\omega_1)\, e_0)\, f\, \exp(-\pi(x\,\nu_2 + n\,\omega_2)\, e_0),\end{aligned}$$

so translation by a period γ (see (3.21)) induces an ambient isometry. The *equivariant action* is the action of the 1-parameter group K of isometries of $\mathbb{S}^3$ defined by

$$K = \{g_x \in \mathrm{Iso}(\mathbb{S}^3) \mid g_x(p) = \exp(x\,\nu_1\, e_0)\, p\, \exp(-x\,\nu_2\, e_0),\ x \in \mathbb{R}\}. \tag{3.25}$$

Since $\nu_1 \neq 0 \neq \nu_2$, the commutator of the equivariant action (3.25)

$$\hat{K} = \{g \in \mathrm{Iso}(\mathbb{S}^3) \mid gk = kg \text{ for all } k \in K\} \tag{3.26}$$

in the group $\mathrm{Iso}(\mathbb{S}^3)$ of orientation preserving isometries of $\mathbb{S}^3$ is a 2-dimensional torus. With the exception of two geodesics the orbits of $\hat{K}$ are 2-dimensional embedded tori. These geodesics, which we call the *axes* of the surface, are linked, and are situated so that every geodesic 2-sphere through one is orthogonal to the other. Every orbit of the equivariant action (3.25), with the exception of the two axes, is an (m, n)-torus knot in the corresponding orbit of $\hat{K}$, with

$$\frac{m}{n} = \frac{\nu_1 - \nu_2}{\nu_1 + \nu_2}. \tag{3.27}$$

If we identify

$$\mathbb{S}^3 = \mathrm{SU}_2 = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \,\middle|\, |a|^2 + |b|^2 = 1 \right\} \quad \text{and choose} \quad e_0 = \begin{pmatrix} \mathbb{i} & 0 \\ 0 & -\mathbb{i} \end{pmatrix},$$

the equivariant action extends to an action on $(a, b) \in \mathbb{C}^2$ given by $R(s, t)(a, b) = (e^{\mathbb{i}s}a, e^{\mathbb{i}t}b)$, called the *extended action*. In particular, the translation τ_γ by a period γ (see (3.21)) induces the ambient isometry $R(s, t)$ with

$$s = x\,(\nu_1 - \nu_2) + p\,(\omega_1 - \omega_2) \quad \text{and} \quad t = x\,(\nu_1 + \nu_2) + p\,(\omega_1 + \omega_2).$$

Proof of Proposition 2.5. For a homogeneous torus this can always be achieved since we have not used the freedom of the Möbius transformation. A spectral genus one torus is a surface of revolution if and only if the equivariant action is the rotation around a geodesic, or equivalently if (3.25) fixes point wise one geodesic of $\mathbb{S}^3$. The generator of the extended action has eigenvalues $\mathbb{i}(\nu_1 \pm \nu_2)$. Thus there exists a zero eigenvalue if and only if $\nu_2 = \pm\nu_1$, which is equivalent to $\lambda_2 = \lambda_1^{-1}$. □

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Proposition 3.7. *A spectral genus one cmc surface in $\mathbb{S}^3$ is closed along two independent periods if and only if there exists a vector $T \in \mathbb{Z}^3 \setminus \{0\}$ with $T \cdot X = 0 = T \cdot Y$ for $X := (0, \nu_1, \nu_2)$ and $Y := (1, \omega_1, \omega_2)$.*

Proof. Suppose we have two $\mathbb{R}$-independent periods $\gamma_j = x_j \pi + 2\mathbb{i}\, p_{j0}\mathbf{K}' \in \mathbb{C}^\times$ for some $x_j \in \mathbb{R}$ and $p_{j0} \in \mathbb{Z}$, $j = 1, 2$. Let $M_\lambda(\gamma_j)$ be the respective frame monodromies with eigenvalues $\mu_j^{\pm 1}$. Then there exist four further integers $p_{jk} \in \mathbb{Z}$ as in (3.14) for $j, k = 1, 2$. Hence $p_{jk} = x_j\, \nu_k + p_{j0}\, \omega_k$, and we write this system as

$$\begin{pmatrix} x_1 & p_{10} \\ x_2 & p_{20} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{pmatrix} := \begin{pmatrix} P \\ Q \end{pmatrix}. \tag{3.28}$$

Hence $T := P \times Q$ is in $\mathbb{Z}^3 \setminus \{0\}$ and satisfies $T \cdot X = 0$ and $T \cdot Y = 0$.

Conversely suppose that there exists a $T \in \mathbb{Z}^3 \setminus \{0\}$ satisfying $T \cdot X = 0$ and $T \cdot Y = 0$. Let $\{P, Q\}$ be a basis for the lattice $\Lambda = \{P \in \mathbb{Z}^3 \mid T \cdot P = 0\}$. Then there exist $x_1, x_2 \in \mathbb{R}$ and $p_{10}, p_{20} \in \mathbb{Z}$ such that (3.28) holds. Then $\gamma_j = x_j \pi + 2\mathbb{i}\, p_{j0}\mathbf{K}' \in \mathbb{C}^\times$ generate a lattice with respect to which the surface is doubly periodic. □

Remark 3.8. The closing conditions in Proposition 3.7 can be used to describe the intersection of the zero-sets of two functions on the parameter space $(\mathrm{k}, \mathrm{q}, \mathrm{h})$. The curve forming the intersection of two level sets then integrates to the vector field (2.12). For X, Y as in Proposition 3.7, there exists an $s = (s_0, s_1, s_2) \in \mathbb{Z}^3$ such that $s \cdot X = 0$ and $s \cdot Y = 0$. The closing conditions are thus $F = 0, G = 0$, where $F := s \cdot X$ and $G := s \cdot Y$. If we set $\lambda_k = e^{2\mathbb{i}\theta_k}$, then the system of implicit flow equations is

$$\begin{pmatrix} \frac{\partial F}{\partial \mathrm{k}} & \frac{\partial F}{\partial \theta_1} & \frac{\partial F}{\partial \theta_2} \\ \frac{\partial G}{\partial \mathrm{k}} & \frac{\partial G}{\partial \theta_1} & \frac{\partial G}{\partial \theta_2} \end{pmatrix} \begin{pmatrix} \dot{\mathrm{k}} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = 0,$$

of which we next compute the matrix on the right hand side. Up to scale, $(\dot{\mathrm{k}}, \dot{\theta}_1, \dot{\theta}_2)$ is a cross product of its rows. Hence

$$\begin{pmatrix} \frac{\partial F}{\partial \mathrm{k}} & \frac{\partial F}{\partial \theta_1} & \frac{\partial F}{\partial \theta_2} \\ \frac{\partial G}{\partial \mathrm{k}} & \frac{\partial G}{\partial \theta_1} & \frac{\partial G}{\partial \theta_2} \end{pmatrix} = \begin{pmatrix} s_1 \frac{\partial \nu_1}{\partial \mathrm{k}} + s_2 \frac{\partial \nu_2}{\partial \mathrm{k}} & s_1 \frac{\partial \nu_1}{\partial \theta_1} & s_2 \frac{\partial \nu_2}{\partial \theta_2} \\ s_1 \frac{\partial \omega_1}{\partial \mathrm{k}} + s_2 \frac{\partial \omega_2}{\partial \mathrm{k}} & s_1 \frac{\partial \omega_1}{\partial \theta_1} & s_2 \frac{\partial \omega_2}{\partial \theta_2} \end{pmatrix}.$$

Since $G = s_1 \nu_1 + s_2 \nu_2 = 0$, this matrix is a scalar multiple of

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} := \begin{pmatrix} \nu_2 \frac{\partial \nu_1}{\partial \mathrm{k}} - \nu_1 \frac{\partial \nu_2}{\partial \mathrm{k}} & \nu_2 \frac{\partial \nu_1}{\partial \theta_1} & -\nu_1 \frac{\partial \nu_2}{\partial \theta_2} \\ \nu_2 \frac{\partial \omega_1}{\partial \mathrm{k}} - \nu_1 \frac{\partial \omega_2}{\partial \mathrm{k}} & \nu_2 \frac{\partial \omega_1}{\partial \theta_1} & -\nu_1 \frac{\partial \omega_2}{\partial \theta_2} \end{pmatrix}.$$

Hence $(\dot{\mathrm{k}}, \dot{\theta}_1, \dot{\theta}_2)$ is a scalar multiple of $V_1 \times V_2$. The derivatives of ν and ω with respect to k and θ, where $2\mathbb{i}\,\theta := \ln \lambda$ were computed in (2.11), (2.13) and (2.14). A calculation yields that $V_1 \times V_2$ is a scalar multiple of

$$\begin{pmatrix} 2\,\mathrm{k}(\mathbf{E}' \cos(\theta_1 + \theta_2) - \mathrm{k}\,\mathbf{K}' \cos(\theta_1 - \theta_2)) \\ -\frac{(1+\mathrm{k}^2)\mathbf{E}' - 2\mathbf{K}'}{1-\mathrm{k}^2} \sin\big(\theta_1 + \theta_2\big) + \frac{2\mathbf{E}' - (1+\mathrm{k}^2)\mathbf{K}'}{1-\mathrm{k}^2} \sin\big(\theta_1 - \theta_2\big) \\ -\frac{(1+\mathrm{k}^2)\mathbf{E}' - 2\mathbf{K}'}{1-\mathrm{k}^2} \sin\big(\theta_1 + \theta_2\big) - \frac{2\mathbf{E}' - (1+\mathrm{k}^2)\mathbf{K}'}{1-\mathrm{k}^2} \sin\big(\theta_1 - \theta_2\big) \end{pmatrix}.$$

Changing variables from $(\mathrm{k}, \theta_1, \theta_2)$ to $(\mathrm{k}, \mathrm{q}, \mathrm{h})$, and rescaling, yields (2.12).

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Figure 3. A sampling of $(k, 13)$ cmc tori of revolution in $\mathbb{S}^3$.

4. The moduli space of equivariant cmc tori in $\mathbb{S}^3$

We next determine those homogeneous tori which allow a bifurcation into genus $g = 1$ cmc tori. These are precisely those homogeneous tori whose spectral curves have a double point on the unit circle. We will show in Theorem 4.5 that the spectral genus one cmc tori lie in flow families which begin at a homogeneous torus with a double point on $\mathbb{S}^1$. These homogeneous tori hence serve as initial conditions for the spectral genus $g = 1$ flow (2.12). The purpose of this section is to classify the homogeneous tori in $\mathbb{S}^3$ with a double point on $\mathbb{S}^1$.

We will show that three integers determine a primitive homogeneous torus with a double point on $\mathbb{S}^1$ up to the $\mathbb{S}^1$ action and complex conjugation on the spectral curve. This is done by parameterizing the configuration space of three marked points on $\mathbb{S}^1$ as follows: Let $\Delta \subset \mathbb{T}^3 := \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$ denote the union of all triples of unimodular numbers in which two of the three entries coincide. Let $\mathcal{T} = (\mathbb{T}^3 \setminus \Delta)/\!\sim$, where we identify triples obtained by the three transformations:

(A) Rotation $(\lambda_0, \lambda_1, \lambda_2) \mapsto \delta(\lambda_0, \lambda_1, \lambda_2)$ with $\delta \in \mathbb{S}^1$.

(B) Inversion $(\lambda_0, \lambda_1, \lambda_2) \mapsto (\lambda_0^{-1}, \lambda_1^{-1}, \lambda_2^{-1})$.

(C) Swapping of the sym points $(\lambda_0, \lambda_1, \lambda_2) \mapsto (\lambda_0, \lambda_2, \lambda_1)$.

Now we define a mapping

$$\phi : (\lambda_0, \lambda_1, \lambda_2) \mapsto \left(\frac{|s_0|}{\max\{|s_1 + s_2|, |s_1 - s_2|\}}, \frac{\min\{|s_1 + s_2|, |s_1 - s_2|\}}{\max\{|s_1 + s_2|, |s_1 - s_2|\}} \right) \tag{4.1}$$

where

$$(s_0, s_1, s_2) = \mathbb{i}(\lambda_0^{1/2}, \lambda_1^{1/2}, \lambda_2^{1/2}) \times (\lambda_0^{-1/2}, \lambda_1^{-1/2}, \lambda_2^{-1/2}).$$

Proposition 4.1. *The map* (4.1) *is bijective from* $\mathcal{T}$ *onto*

$$\mathcal{R} = \{(t_0, t_1) \in [0, 1)^2 \mid t_1 < t_0\}.$$

Proof. For $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{T}^3$ choose square roots and set

$$m := (\lambda_0^{1/2}, \lambda_1^{1/2}, \lambda_2^{1/2}).$$

Then $s = \mathbb{i}m \times \overline{m}$ and $\overline{s} = -\mathbb{i}\overline{m} \times m = \mathbb{i}m \times \overline{m} = s$, so s is real. If $s = 0$, then $\overline{m}$ is contained in the $\mathbb{S}^1$ orbit of m, and $\lambda_0 = \lambda_1 = \lambda_2$ so $(\lambda_0, \lambda_1, \lambda_2) \in \Delta$. This shows $s \neq 0$, and $s \in \mathbb{RP}^2$.

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For the map $\mathbf{s} : \mathbb{RP}^3 \to \mathbb{R}$ defined by

$$\begin{aligned}\mathbf{s}(s_0, s_1, s_2) &= -(s_0 + s_1 + s_2)(-s_0 + s_1 + s_2)(s_0 - s_1 + s_2)(s_0 + s_1 - s_2)\\ &= \big(\lambda_0^{1/2}\lambda_1^{-1/2} - \lambda_1^{1/2}\lambda_0^{-1/2}\big)^2\big(\lambda_0^{1/2}\lambda_2^{-1/2} - \lambda_2^{1/2}\lambda_0^{-1/2}\big)^2\\ &\quad\times\big(\lambda_1^{1/2}\lambda_2^{-1/2} - \lambda_2^{1/2}\lambda_1^{-1/2}\big)^2\end{aligned}$$

we have $\mathbf{s}(s) \le 0$ since each of the three squared factors is non-positive. Furthermore, $\mathbf{s}(s) = 0$ is equivalent to $(\lambda_0, \lambda_1, \lambda_2) \in \Delta$. Hence we have $\mathbf{s}(s) < 0$. Define

(A') Rotation $(\lambda_0^{1/2}, \lambda_1^{1/2}, \lambda_2^{1/2}) \mapsto \delta(\lambda_0^{1/2}, \lambda_1^{1/2}, \lambda_2^{1/2})$ with $\delta \in \mathbb{S}^1$.

(B') Inversion $(\lambda_0^{1/2}, \lambda_1^{1/2}, \lambda_2^{1/2}) \mapsto (\lambda_0^{-1/2}, \lambda_1^{-1/2}, \lambda_2^{-1/2})$.

(C') Swapping of the sym points $(\lambda_0^{1/2}, \lambda_1^{1/2}, \lambda_2^{1/2}) \mapsto (\lambda_0^{1/2}, \lambda_2^{1/2}, \lambda_1^{1/2})$.

(D') Changing the signs of the entries of $(\lambda_0^{1/2}, \lambda_1^{1/2}, \lambda_2^{1/2})$ independently.

The transformations (A')–(B') do not change $s \in \mathbb{RP}^2$, while (C') negates signs and swaps s_1 and s_2, while (D') negates two signs in s for each sign change. If $s_j = 0$ for at least one index $j \in \{0, 1, 2\}$, then $\mathbf{s}(s_0, s_1, s_2) \ge 0$. Hence if $s \in \mathbb{RP}^2$, then $\mathbf{s}(s) < 0$ implies $s_j \ne 0$ for all $j \in \{0, 1, 2\}$. Then $s = \mathbb{i}m \times \overline{m}$ if and only if $s \cdot m = 0$ and $s \cdot \overline{m} = 0$. These are equivalent to

$$\begin{aligned}\lambda_1 - 2A_1\lambda_1^{1/2}\lambda_0^{1/2} + \lambda_0 &= 0,\\ \lambda_2 - 2A_2\lambda_2^{1/2}\lambda_0^{1/2} + \lambda_0 &= 0,\\ s_0\lambda_0^{1/2} + s_1\lambda_1^{1/2} + s_2\lambda_2^{1/2} &= 0,\end{aligned}$$

where

$$\begin{aligned}A_1 &= \frac{-s_0^2 - s_1^2 + s_2^2}{2s_0s_1} = 1 + \frac{\mathbf{s}(s)}{(2s_0s_1)^2},\\ A_2 &= \frac{-s_0^2 + s_1^2 - s_2^2}{2s_0s_1} = 1 + \frac{\mathbf{s}(s)}{(2s_0s_2)^2}.\end{aligned}$$

Thus $s \in \mathbb{RP}^2$ with $\mathbf{s}(s) < 0$ determines the following elements of $\mathbb{T}^3$ uniquely up to (A') and (B'):

$$(\lambda_0^{1/2}, \lambda_1^{1/2}, \lambda_2^{1/2}) = \left(1, \frac{-s_0^2 - s_1^2 + s_2^2 \pm \sqrt{\mathbf{s}(s)}}{2s_0s_1}, \frac{-s_0^2 + s_1^2 - s_2^2 \mp \sqrt{\mathbf{s}(s)}}{2s_0s_2}\right). \tag{4.2}$$

Furthermore the numbers

$$(\ell_0, \ell_1, \ell_2) = \frac{1}{2}\left(|s_0|, \min\{|s_1 + s_2|, |s_1 - s_2|\}, \max\{|s_1 + s_2|, |s_2 - s_2|\}\right) \tag{4.3}$$

determine (s_0, s_1, s_2) up to the transformations (C') and (D'). Due to

$$\mathbf{s}(s) = (s_0^2 - (s_1 - s_2)^2)(s_0^2 - (s_1 + s_2)^2) = 4(\ell_0^2 - \ell_1^2)(\ell_0^2 - \ell_2^2)$$

the condition $\mathbf{s}(s) < 0$ is equivalent to $0 \le \ell_1 < \ell_0 < \ell_2$. This shows that ϕ is surjective from $\mathbb{T}^3 \setminus \Delta$ onto $\mathcal{R}$, whose pre-images are uniquely determined up to (A)–(C). □

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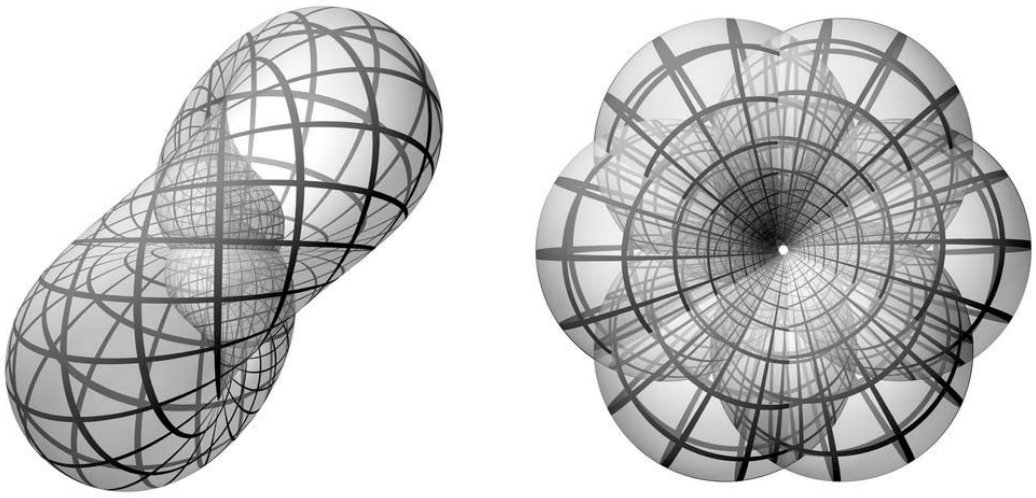

Figure 4. The minimal $(2,1,3)$ and $(3,1,6)$ tori. Stereographically projected from $\mathbb{S}^3$ to $\mathbb{E}^3$.

The *spectral data* of a homogeneous torus with a double point is a triple $(\lambda_0,\lambda_1,\lambda_2)\in\mathcal{T}$ of values of the spectral parameter at the double point and the two sym points. We identify triples obtained by the transformations (A)–(C). Hence the set of spectral data is isomorphic to $\mathcal{R}\cap\mathbb{Q}^2$.

The *turning number* of a plane curve is the degree of its Gauss map; we take the turning number to be unsigned. The *total turning number* of a collection of immersed curves is the sum of their turning numbers. For twizzled surfaces, a *profile curve set* of the surface with respect to one of its axes A is constructed as follows: Let $\mathbb{S}^2_A$ be a geodesic 2-sphere containing A. The axis A divides $\mathbb{S}^2_A$ into two hemispheres. Then a profile curve set of the torus with respect to A is the intersection of the surface with $\mathbb{S}^2_A$ or with one of the hemispheres of $\mathbb{S}^2_A$. By the equivariant action, all profile curve sets associated with an axis are isometric. For surfaces of revolution, there is a profile curve set with respect to the axis which is not the revolution axis.

Theorem 4.2. (1) *The set of spectral data of homogeneous tori with a double point on* $\mathbb{S}^1$ *up to transformations* (A)–(C) *is in one-to-one correspondence with integer triples*

$$(\ell_0,\ell_1,\ell_2)\in\mathbb{Z}^3 \text{ with } \gcd(\ell_0,\ell_1,\ell_2)=1 \text{ satisfying } 0\le\ell_1<\ell_0<\ell_2. \tag{4.4}$$

(2) *Let* $\mathbb{T}^2$ *be a* (ℓ_0,ℓ_1,ℓ_2) *homogeneous torus. Then* $\mathbb{T}^2$ *covers its underlying homogeneous embedded rectangular cmc torus* ℓ_0 *times. Each profile curve set* $\mathcal{C}_k$ $(k=1,2)$ *of* $\mathbb{T}^2$ *has total turning number* ℓ_0. *The set is a union of* $\gcd(\ell_0,\ell_k)$ *coinciding circles, and each circle is wrapped* $\ell_0/\gcd(\ell_0,\ell_k)$ *times.*

(3) *The case* $\ell_1=0$ *occurs if and only if the torus is rotational.*

Proof. Let $m=(\lambda_0^{1/2},\lambda_1^{1/2},\lambda_2^{1/2})$ be the square roots of spectral data of a homogeneous torus, and assume the torus has period lattice $\Gamma=\gamma_1\mathbb{Z}\times\gamma_2\mathbb{Z}$. We can frame the torus by an extended frame (3.15), and then the logarithmic eigenvalues of the monodromy with respect to these periods are $\ln\mu(\gamma_j,\lambda_k)$ as in (3.16). Then there exist six integers $p_{jk}\in\mathbb{Z}$ such that $\ln\mu(\gamma_j,\lambda_k)=\pi\,\mathbb{i}\,p_{jk}$ for $j=1,2$ and $k=0,1,2$. As in Proposition 3.4 let $\Lambda=\kappa_1\mathbb{Z}\times\kappa_2\mathbb{Z}$,

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and $\Lambda^* = \gamma_1^*\mathbb{Z} \times \gamma_2^*\mathbb{Z}$ the lattice of the underlying embedded rectangular torus with orthogonal basis $\langle\!\langle \gamma_j^*, \kappa_k \rangle\!\rangle = \delta_{jk}$. Since $\Gamma \subset \Lambda^*$, we can expand

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} \langle\!\langle \gamma_1, \kappa_1 \rangle\!\rangle & \langle\!\langle \gamma_1, \kappa_2 \rangle\!\rangle \\ \langle\!\langle \gamma_2, \kappa_1 \rangle\!\rangle & \langle\!\langle \gamma_2, \kappa_2 \rangle\!\rangle \end{pmatrix} \begin{pmatrix} \gamma_1^* \\ \gamma_2^* \end{pmatrix}. \tag{4.5}$$

Since $p_{jk} = \langle\!\langle \gamma_j, \lambda_k^{1/2} \rangle\!\rangle$, the determinant of the above change of basis is

$$|\det(\langle\!\langle \gamma_j, \kappa_k \rangle\!\rangle)| = \frac{1}{2}|p_{12}p_{21} - p_{11}p_{22}|. \tag{4.6}$$

(1) Define $(\tilde{s}_0, \tilde{s}_1, \tilde{s}_2) := (p_{10}, p_{11}, p_{12}) \times (p_{20}, p_{21}, p_{22})$. By construction we have $(\tilde{s}_0, \tilde{s}_1, \tilde{s}_2) \in \mathbb{Z}^3$. A computation gives

$$(\tilde{s}_0, \tilde{s}_1, \tilde{s}_2) = (\overline{\gamma}_1\gamma_2 - \gamma_1\overline{\gamma}_2)\, m \times \overline{m}.$$

Note that $\overline{\gamma}_1\gamma_2 - \gamma_1\overline{\gamma}_2 \in \mathbb{i}\mathbb{R}$, and is never zero, since γ_1, γ_2 are not collinear. From equation (3.18) we additionally have that $p_{11} - p_{12} \in 2\,\mathbb{Z}$ and $p_{21} - p_{22} \in 2\,\mathbb{Z}$. Hence it follows that $\tilde{s}_0, \tilde{s}_1 + \tilde{s}_2, \tilde{s}_1 - \tilde{s}_2 \in 2\,\mathbb{Z}$. There exists a unique representative $(s_0, s_1, s_2) \in \mathbb{Z}^3$ of $(\tilde{s}_0, \tilde{s}_1, \tilde{s}_2) \in \mathbb{RP}^2$ with $\gcd(s_0, s_1 + s_2, s_1 - s_2) = 2$. The corresponding numbers (ℓ_0, ℓ_1, ℓ_2) defi+ned in (4.3) then obey (4.4). We thus have a map from spectral data to integer triples obeying (4.4), and by Proposition 4.1 this map is bijective.

(2) The vector $s = (s_0, s_1, s_2)$ determines a lattice $\Gamma_s \subset \Lambda^*$ defined by

$$\Gamma_s = \left\{ \frac{p_1 + p_2}{2}\gamma_1^* + \frac{p_1 - p_2}{2}\gamma_2^* \;\middle|\; \left(p_0, \frac{p_1 + p_2}{2}, \frac{p_1 - p_2}{2}\right) \in \mathbb{Z}^3 \text{ with } s_0p_0 + s_1p_1 + s_2p_2 = 0 \right\}. \tag{4.7}$$

The lattice Γ_s contains all the periods of the torus with respect to which the logarithmic eigenvalues of the monodromy do not change their values at the double point. The lattice Γ_s does not change if s is multiplied by some integer, or the sign of s_0 is changed. Due to (4.3) the integers s are determined by (ℓ_0, ℓ_1, ℓ_2) up to transformations (C') and (D'). Two co-linear s correspond to the same lattice Γ_s. Switching the signs of s_0 does not change Γ_s. Therefore the lattices Γ_s of the homogeneous tori with triple (ℓ_0, ℓ_1, ℓ_2) satisfying (4.4) are those Γ_s with one of the following vectors s given by

$$\begin{aligned} &(2\ell_0, \ell_1 + \ell_2, \ell_1 - \ell_2), \qquad (2\ell_0, \ell_1 + \ell_2, \ell_2 - \ell_1), \\ &(2\ell_0, \ell_1 - \ell_2, \ell_1 + \ell_2), \qquad (2\ell_0, \ell_2 - \ell_1, \ell_1 + \ell_2). \end{aligned} \tag{4.8}$$

The transformations (C') and (D') act on sublattices $\Upsilon \subset \Lambda^*$ as follows:

(C") $C"\Upsilon = \{n_1\gamma_1^* + n_2\gamma_2^* \mid n_1\gamma_1^* - n_2\gamma_2^* \in \Upsilon\}$,

(D") $D"\Upsilon = \{n_1\gamma_1^* + n_2\gamma_2^* \mid n_2\gamma_1^* + n_1\gamma_2^* \in \Upsilon\}$.

Since $p_0 2\ell_0 + p_1(\ell_1 + \ell_2) + p_2(\ell_1 - \ell_2) = 2(p_0\ell_0 + \frac{p_1+p_2}{2}\ell_1 + \frac{p_1+p_2}{2}\ell_2)$, the lattices corresponding to the four vectors s in (4.8) are respectively equal to

$$\begin{aligned} &\Gamma_{[\ell_0,\ell_1,\ell_2]} = \{n_1\gamma_1^* + n_2\gamma_2^* \mid n_1\ell_1 + n_2\ell_2 \in \ell_0\,\mathbb{Z}\}, \\ &C"\Gamma_{[\ell_0,\ell_1,\ell_2]}, \quad D"\Gamma_{[\ell_0,\ell_1,\ell_2]} \quad \text{and} \quad D'C"\Gamma_{[\ell_0,\ell_1,\ell_2]}. \end{aligned} \tag{4.9}$$

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For homogeneous tori the 2-dimensional group $\hat{K}$ (see (3.26)) acts on the torus. Since the axes do not depend on the subgroup $K \subset \hat{K}$ (see (3.25)), the two subgroups corresponding to the rotations of the embedded torus act on a geodesic sphere containing one of the axes. Hence the smallest periods in $\Gamma_s \cap \gamma_1^* \mathbb{Z}$ and $\Gamma_s \cap \gamma_2^* \mathbb{Z}$ represent components of the profile curve sets. These wrapping numbers are $\ell_0/\gcd(\ell_0, \ell_1)$ and $\ell_0/\gcd(\ell_0, \ell_2)$. The number of components times these wrapping numbers is equal to $|\Lambda^*/\Gamma_s|$. But $\ell_0 = |\Lambda^*/\Gamma_s|$ by (4.6). Hence the total turning number is equal to ℓ_0. Moreover, the corresponding embedded torus is covered ℓ_0 times.

(3) Clearly $\ell_1 = 0$ holds if and only if $s_1 = \pm s_2$. We may assume that $\lambda_0 = 1$ and then by (4.2) this holds if and only if $\lambda_2 = \lambda_1^{-1}$. By Proposition 2.5 we conclude that $\ell_1 = 0$ holds if and only if the torus is rotational. □

Proposition 4.3. *The family of non-rotational spectral genus one cmc tori starting at the* (ℓ_0, ℓ_1, ℓ_2) *homogeneous torus ends at the* $(\ell_1 + \ell_2 - \ell_0, \ell_1, \ell_2)$ *homogeneous torus.*

Proof. Due to Proposition 2.6 (2) the difference

$$(\cos(2\theta_1) - \cos(2\theta_2))^2 = 4(1 - \mathrm{q}^2)(1 - \mathrm{h}^2)$$

is uniformly bounded away from zero. After possibly swapping the sym points we may assume $\operatorname{sign}(\mathrm{k}) \cos(2\theta_1) < \operatorname{sign}(\mathrm{k}) \cos(2\theta_2)$ during the flow. We remark that sign(k) is constant throughout the flow. In particular, λ_1 cannot pass through $\lambda_1 = \operatorname{sign}(\mathrm{k})$ and λ_2 cannot pass through $\lambda_2 = -\operatorname{sign}(\mathrm{k})$. Due to Proposition 2.7 the function $\mathrm{q} - \operatorname{sign}(\mathrm{k})\,\mathrm{h}$ changes the sign, and the flow passes through a root of this function. This implies that λ_2 has to pass through $\lambda_2 = \operatorname{sign}(\mathrm{k})$. We want to describe how the final six integers in (3.28) are related to the corresponding initial six integers. Since ω is multi-valued on the fixed point set $\mathbb{S}^1$ of ϱ, we cut this circle at $\lambda = \operatorname{sign}(\mathrm{k})$. At the end points of the flow with $\mathrm{k} = \pm 1$ this point is a double point. This is a good choice for the cut, since at the end points of the flow the sym points cannot sit there. The multi-valued function ω is single-valued on $\mathbb{S}^1 \setminus \{\operatorname{sign}(\mathrm{k})\}$. The difference between the two boundary values of ω on this interval is equal to ± 2 due to Legendre's relation [1, equation 17.3.13]. With $p_{jk} = x_j\, \nu_k + p_{j0}\, \omega_k$ with respect to the periods $\gamma_j = x_j \pi + 2\mathbb{i}\, p_{j0} \mathbf{K}'$ as in (3.28), the final and initial integers are related by

$$\begin{pmatrix} p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{pmatrix}\Bigg|_{t=t_{\max}} = \begin{pmatrix} p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{pmatrix}\Bigg|_{t=t_{\min}} \pm 2 \begin{pmatrix} 0 & 0 & p_{10} \\ 0 & 0 & p_{20} \end{pmatrix}.$$

We remark that due to our choice these integers change their values only when λ_2 passes through $\lambda_2 = \operatorname{sign}(\mathrm{k})$. Now let (ν_1, ω_1) and (ν_2, ω_2) denote the values of (ν, ω) at the sym points. After possibly independent hyperelliptic involutions, we have $0 < \nu_2 < \nu_1$ in agreement with our choice of the sym points.

Note that

$$\begin{aligned}(s_0, s_1, s_2) &= (p_{10}, p_{11}, p_{12}) \times (p_{20}, p_{21}, p_{22}) \\ &= (x_1 p_{20} - x_2 p_{10})\, (0, \nu_1, \nu_2) \times (1, \omega_1, \omega_2).\end{aligned}$$

Since the integers change only when λ_2 passes through $\lambda_2 = \operatorname{sign}(\mathrm{k})$, we can calculate the change of the s_j in terms of the change of the values of ν and ω at $\lambda_2 = \operatorname{sign}(\mathrm{k})$.

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At the beginning and end of the flow we have

$$(s_0, s_1, s_2)|_{t=t_{\min}} = (0, \nu_1, \nu_2) \times (1, \omega_1, \omega_2)$$

and

$$(s_0, s_1, s_2)|_{t=t_{\max}} = (0, \nu_1, \nu_2) \times (1, \omega_1, \omega_2 \pm 2).$$

Due to $0 < \nu_2 < \nu_1$ we have

$$(\ell_0, \ell_1, \ell_2) = |x_1 p_{20} - x_2 p_{10}|(|\nu_1\omega_2 - \nu_2\omega_1|, \nu_1 - \nu_2, \nu_1 + \nu_2). \tag{4.10}$$

Therefore the final values are $(|\ell_0 \pm (\ell_1 + \ell_2)|, \ell_1, \ell_2)$ in terms of the initial values. The inequality $\ell_0 < \ell_2$ excludes the plus sign, and concludes the proof. □

4.1. Formulas. In a few places we will need formulas for equivariant cmc tori and their profile curves in terms of the extended frame (3.8). Identifying the unit quaternions with $\mathbb{i} := \exp((\pi/2)e_0)$, $\mathbb{j} := \exp(\frac{\pi}{2}e_1)$, $\mathbb{k} := \exp(\frac{\pi}{2}e_2)$, and identifying $\mathbb{i} = \sqrt{-1}$, a computation gives

$$\begin{aligned} f = {} & \alpha_1\alpha_2^{-1}\beta_1\beta_2^{-1}(\gamma_1\gamma_2^{-1}c_1c_2 + \gamma_1^{-1}\gamma_2 s_1 s_2) \\ & + \alpha_1\alpha_2\beta_1\beta_2(\gamma_1^{-1}\gamma_2 s_1 c_2 - \gamma_1\gamma_2^{-1}c_1 s_2)\mathbb{j}, \end{aligned} \tag{4.11}$$

$$\alpha = \exp(\mathbb{i}x\nu), \qquad \beta = \exp\left(\frac{\mathbb{i}}{2}\chi_0\right), \quad \gamma = \exp\left(\frac{\mathbb{i}}{2}\chi_2\right), \tag{4.12}$$

$$c = \cos\left(\frac{1}{2}\chi_1\right), \quad s = \sin\left(\frac{1}{2}\chi_1\right), \tag{4.13}$$

$$\begin{aligned} \alpha_k &= \alpha(\lambda_k), & \beta_k &= \beta(\lambda_k), & \gamma_k &= \gamma(\lambda_k), \\ c_k &= c(\lambda_k), & s_k &= s(\lambda_k), & k &= 1, 2. \end{aligned} \tag{4.14}$$

With τ (see (2.8)), we have $\tau^*\nu = \nu$, $\tau^*\chi_0 = -\chi_0$, $\tau^*\chi_1 = \chi_1$, $\tau^*\chi_2 = -\chi_2$. Applying these symmetries to the formula (4.11) for the immersion at $y = 0$ shows the profile curve of an equivariant cmc surface of revolution in $\mathbb{S}^3$ is

$$f_0 = \exp(\mathbb{i}\chi_0)(\cos\chi_2 + \mathbb{i}\cos\chi_1 \sin\chi_2) + \mathbb{i}\sin\chi_1 \sin\chi_2 \mathbb{j}.$$

More explicitly,

$$f_0 = \exp(\mathbb{i}\chi_0)(g_1 + \mathbb{i}g_2) + g_0\mathbb{k}, \tag{4.15}$$

$$g_0 := \frac{1}{2}\nu^{-1}v\sin(2\theta_1), \quad g_1 := \mathrm{C}^{-1}(v\cos(2\theta_1) - v^{-1}\,\mathrm{k}), \tag{4.16}$$

$$g_2 := \frac{1}{2}\,\mathrm{C}^{-1}\,\nu^{-1}v'\sin(2\theta_1),$$

$$\mathrm{C}^2 := v^{-1}v' - 4\nu^2 = v^2 - 2\,\mathrm{k}\cos(2\theta_1) + \mathrm{k}^2\,v^{-2}. \tag{4.17}$$

4.2. Sphere bouquets. For $m, n \in \mathbb{Z}$ with $1 \le m < n$ and $\gcd(m, n) = 1$, the (m, n) *sphere bouquet* is constructed as follows. With $\alpha = \exp(2\pi\,\mathbb{i}\,\frac{m}{n}) \in \mathbb{C}^\times$, let C be the union of the n circles through α^k and α^{k+1} perpendicular to $\mathbb{S}^1$ ($k = 0, \ldots, n-1$). Inverse stereographic projection of C to a geodesic 2-sphere and revolution about the image of $\mathbb{S}^1$ as axis produces the (m, n) sphere bouquet, consisting of n pairwise tangent spheres forming a necklace. Since the radius of each circle is $r = \pi\frac{m}{n}$, the mean curvature of the (m, n) sphere bou-

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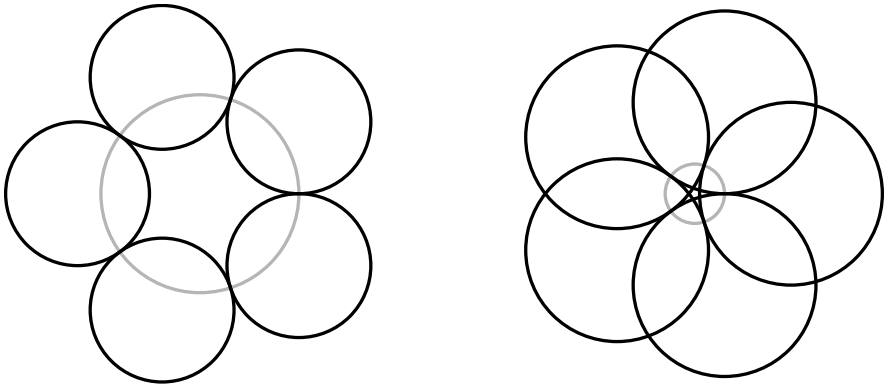

Figure 5. The profile curves of the two five-lobed sphere bouquets $(1,5)$ and $(2,5)$, stereographically projected to $\mathbb{E}^2$. The gray central circles are the axes of revolution

quet is $\cot(\pi\frac{m}{n})$. Since the (m,n) sphere bouquet is the same as the $(n-m,m)$ sphere bouquet, it follows that for $n>2$, the number of distinct sphere bouquets with n spheres is half the number of generators of $\mathbb{Z}_n$. The $(1,2)$ sphere bouquet is a special case because its two spheres coincide. In the rotational case we write $(\ell_0,0,\ell_2)=(\ell_0,\ell_2)$ for brevity.

Lemma 4.4. *The (ℓ_0,ℓ_2) family of rotational cmc tori converges to the (ℓ_0,ℓ_2) sphere bouquet as* $\mathrm{k}\to 0$.

Proof. For rotational cmc tori we have $(\lambda_2,\nu_2,\omega_2)=(\lambda_1^{-1},-\nu_1,-\omega_1)$. By Proposition 3.7 there exists an $(s_0,s_1,s_2)\in\mathbb{Z}^3$ that is perpendicular to $(0,\nu_1,\nu_2)$ and $(1,\omega_1,\omega_2)$. Then $s_2=-s_1$ and $s_0+2s_1\omega_1=0$. Hence $\ell_1=0$ and $\ell_0/\ell_2=\frac{1}{2}|s_0/s_1|=|\omega_1|$.

Define $\theta_1:\mathbb{R}\to\mathbb{R}$ by $2\mathbb{i}\,\theta_1:=\ln\lambda_1$. The limiting profile curve as $\mathrm{k}\to 0$ can be computed from the profile curve for cmc tori of revolution f_0 in (4.15). Let $\mathrm{k}(t)$, $\theta_1(t)$ vary with the flow parameter $t\in[t_{\min},t_{\max}]$, and assume without loss of generality that $\lim_{t\to t_{\min}}\mathrm{k}(t)=0$.

From $\lim_{\mathrm{k}\to 0}\omega=1-2\theta/\pi$ we conclude that

$$\theta_0=\lim_{t\to t_{\min}}\theta_1(t)=\frac{\pi}{2}\left(1-\frac{\ell_0}{\ell_2}\right).$$

With $\lim_{\mathrm{k}\to 0}v=\operatorname{sech}x$ and $\lim_{\mathrm{k}\to 0}2\nu=1$ we have

$$\lim_{\mathrm{k}\to 0}g_0=\sin(2\theta_0)\operatorname{sech}x,\quad \lim_{\mathrm{k}\to 0}g_1=\cos(2\theta_0),\quad \lim_{\mathrm{k}\to 0}g_2=-\sin(2\theta_0)\tanh x.$$

Since the integrand in χ_0 goes to 0 as $\mathrm{k}\to 0$, we get $\lim_{\mathrm{k}\to 0}\chi_0=0$. The limiting profile curve as $\mathrm{k}\to 0$ is thus

$$\gamma_0(x):=\lim_{\mathrm{k}\to 0}f_0=(\cos(2\theta_0)-\mathbb{i}\sin(2\theta_0)\tanh x)+(\sin(2\theta_0)\operatorname{sech}x)\,\mathbb{j}.$$

Since $\cos(2\theta_0)-\mathbb{i}\sin(2\theta_0)\tanh x$ traces out a straight line segment in $\mathbb{C}$, it follows that $\gamma_0(x)$ traces out a circle. The discrete rotational symmetry of γ implies that the limiting set as $\mathrm{k}\to 0$ is a sphere bouquet.

The limiting points are $\gamma_0(\pm\infty)=(e^{2\mp\mathbb{i}\theta_0},0)$ on $\mathbb{S}^1$. The angle between the radii is $4\theta_0=2\pi(1-\ell_0/\ell_2)$. Hence the limiting curve is the $(\ell_2-\ell_0,\ell_2)$ sphere bouquet, which is the same as the (ℓ_0,ℓ_2) sphere bouquet. Note that in the case $(\ell_0,\ell_2)=(1,2)$ the circle is a geodesic. □

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We bring together Proposition 2.7, Proposition 2.8 and Proposition 4.3 in the following theorem.

Theorem 4.5. *Spectral genus one cmc tori lie in* 1*-parameter families with monotonic mean curvature. The family starting at the* (ℓ_0, ℓ_1, ℓ_2) *homogeneous torus ends at the* $(\ell_1 + \ell_2 - \ell_0, \ell_1, \ell_2)$ *homogeneous torus.*

Proof. By Propositions 2.7 and 2.8 the mean curvature is monotonic, and by Proposition 2.7, every flow starts and ends at a homogeneous torus with a double point on $\mathbb{S}^1$. The integers associated to these two homogeneous tori endpoints are computed next. As shown in Lemma 4.4, the two flows ending at the $(\ell_0, 0, \ell_2)$ and $(\ell_2 - \ell_0, 0, \ell_2)$ homogeneous tori at $\mathrm{k} = 1$ start at the same sphere bouquet at $\mathrm{k} = 0$. Because only cmc tori of revolution flow to sphere bouquets, we conclude that every sphere bouquet is the limit of these two flows and no others. While the flow is singular at $\mathrm{k} = 0$, Proposition 4.3 nevertheless holds for the family constructed by gluing these two families together along the sphere bouquet. □

5. Geometry

5.1. The torus knot and symmetry group. Every orbit of the equivariant action (3.25) with the exception of the two axes is a (p, q)-torus knot in the corresponding orbit of $\hat{K}$. Due to (3.27) and (4.10) this implies that the orbit of a point on (ℓ_0, ℓ_1, ℓ_2) torus is generically a torus knot in the corresponding orbit of $\hat{K}$. If a (ℓ_0, ℓ_1, ℓ_2) torus does not meet the axes, then the linking numbers of the K orbit of a point on the torus and the two axes are $(\frac{\ell_1}{\gcd(\ell_1,\ell_2)}, \frac{\ell_2}{\gcd(\ell_1,\ell_2)})$.

Proposition 5.1. *With* $n := \gcd(\ell_1, \ell_2)$*, the symmetry group of an* (ℓ_0, ℓ_1, ℓ_2) *cohomogeneity one cmc torus is a semidirect product of* $\mathbb{S}^1 \times \mathbb{Z}_n$ *and* $\mathbb{Z}_2$ *if it is twizzled, and a semidirect product of* $\mathbb{S}^1 \times \mathbb{Z}_n$ *and* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *if it is a torus of revolution.*

Proof. Let $T := (t_0, t_1, t_2) \in \mathbb{Z}^3$ be a point with $\gcd(t_0, t_1, t_2) = 1$ and $t_0 \neq 0$, and let $n := \gcd(t_1, t_2)$. We first show that

$$\mathcal{Z} := \{n_0 \in \mathbb{Z} \mid T \cdot (n_0, n_1, n_2) = 0 \text{ for some } n_1, n_2 \in \mathbb{Z}\} = n\mathbb{Z}.$$

If $T \cdot (n_0, n_1, n_2) = 0$, then since $n \mid t_1$ and $n \mid t_2$, we get $n \mid (t_0 n_0)$. Since $\gcd(t_0, t_1, t_2) = 1$, it follows that $\gcd(t_0, n) = 1$. Hence we have $n \mid n_0$. Thus n divides every element of the set $\mathcal{Z}$. Since $\gcd(t_1/n, t_2/n) = 1$, by the Euclidean algorithm, there exist $x_1, x_2 \in \mathbb{Z}$ such that $n + t_1 x_1 + t_2 x_2 = 0$. Hence with $N = t_0(1, x_1, x_2)$, then $T \cdot N = 0$. This shows that $n \in \mathcal{Z}$. Hence $\mathcal{Z} = n\mathbb{Z}$.

There exists a basis $\gamma_1, \gamma_2 \in \mathbb{C}^\times$ for the torus lattice so that

$$\langle\!\langle \gamma_1, \lambda_0^{1/2} \rangle\!\rangle = 0 \quad \text{and} \quad p_{20} := \langle\!\langle \gamma_2, \lambda_0^{1/2} \rangle\!\rangle = \gcd(\ell_1, \ell_2),$$

where $\lambda_0^{1/2} = \mathbb{i}$. Then $\gamma_1 \in \mathbb{R}$ and $\gamma_2 = \pi x_2 + 2\mathbb{i}\, p_{20} \mathbf{K}'$ for some $x_2 \in \mathbb{R}$.

By formula (3.20), the first fundamental form is preserved if and only if v is preserved. The symmetry group thus contains the three conformal automorphisms $z \mapsto z + t\gamma_1$, $t \in \mathbb{R}$, $z \mapsto z + \gamma_2/n_2$ and $z \mapsto -z$. For cmc tori of revolution, the sym points satisfy $\lambda_2 = \lambda_1^{-1}$ by

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Proposition 2.5. Hence the coefficient $\frac{1}{2}Q$ of the Hopf differential in (3.20) is real. Since the mean curvature H is real, the second fundamental form is preserved under complex conjugation. Hence in this case there is a further anti-conformal automorphism $z \mapsto \overline{z}$. □

5.2. Lobe counts. The two lobe counts are the orders of the two orientation-preserving cyclic subgroups of the symmetry group which fix one or the other axis point wise.

Proposition 5.2. *The lobe counts of a twizzled (ℓ_0, ℓ_1, ℓ_2) torus are ℓ_1 and ℓ_2, and for a rotational torus it is ℓ_2.*

Proof. Let $\hat{K}$ be the 2-dimensional torus (3.26). Let G be a subgroup of the orientation preserving subgroup of the isometry group of the torus which fixes one axis of the equivariant action point wise. Such groups are homeomorphic to $\mathbb{S}^1$, and thus closed. Closed subgroups of $\mathbb{S}^1$ are either finite or all of $\mathbb{S}^1$, and since we are not considering surfaces of revolution, the group G is finite, and thus cyclic. Let $H \subset G$ be a subgroup which fixes every orbit of the equivariant action set wise. We compute $\operatorname{ord}(G/H)$ and $\operatorname{ord}(H)$. From the proof of Proposition 5.1 we conclude that

$$n_2 = \operatorname{ord}(G/H) = \gcd(\ell_1, \ell_2),$$

since $H = G \cap K$. Now we have $G \subset \{(1, e^{is}) \mid s \in \mathbb{R}\}$ and $K = \{(e^{\mathbb{i}\ell_1 t}, e^{\mathbb{i}\ell_2 t}) \mid t \in \mathbb{R}\}$, and $\operatorname{ord}(\{(1, e^{is}) \mid s \in \mathbb{R}\} \cap \{(e^{\mathbb{i}\ell_1 t}, e^{\mathbb{i}\ell_2 t}) \mid t \in \mathbb{R}\})$ is equal to ℓ_1/n, and generated by the element $(1, e^{2\pi\mathbb{i}n/\ell_1}) \in G$, the order coincides with the order of $K \cap G$. Hence

$$\operatorname{ord}(G) = \operatorname{ord}(G/H)\operatorname{ord}(G) = \ell_1.$$

Similarly for the other axis. This proves the claim for the twizzled case. For the rectangular case a similar argument holds, and concludes the proof. □

In view of Proposition 5.2 we call ℓ_1 and ℓ_2 respectively the *minor and major lobe counts*. The cmc tori shown in Figure 1 have major and minor lobe counts n and 1 respectively. By Theorem 4.5 and Theorem 4.2 we have

Proposition 5.3. *The major lobe count of a non-rotational spectral genus one cmc torus is at least* 3*, and that of a rotational spectral genus one cmc torus of revolution is at least* 2*.*

5.3. Profile curve sets. The discrepancy between the two endpoints of the $g = 1$ flow in Proposition 4.3 is associated to the fact that at two points during the flow, the corresponding torus intersects one and then the other of its axes. At each of these two cmc tori, one of the torus knots degenerates to a circle. The combinatorics of the profile curve sets of equivariant cmc tori are almost invariant during the flow: they are invariant on two disjoint intervals. When the torus intersects its axis, the connectivity and turning numbers of the profile curve set jumps as described in Lemma 5.4. This phenomenon is depicted in Figure 6.

Lemma 5.4. *If a profile curve set is immersed, then*

$$(\mathrm{h}\,\mathrm{k} - \mathrm{q})(\mathrm{h} - \mathrm{k}\,\mathrm{q}) \neq 0.$$

The total turning number of each profile curve set of a non-homogeneous twizzled (ℓ_0, ℓ_1, ℓ_2) cmc torus is ℓ_0 or $\ell_1 + \ell_2 - \ell_0$.

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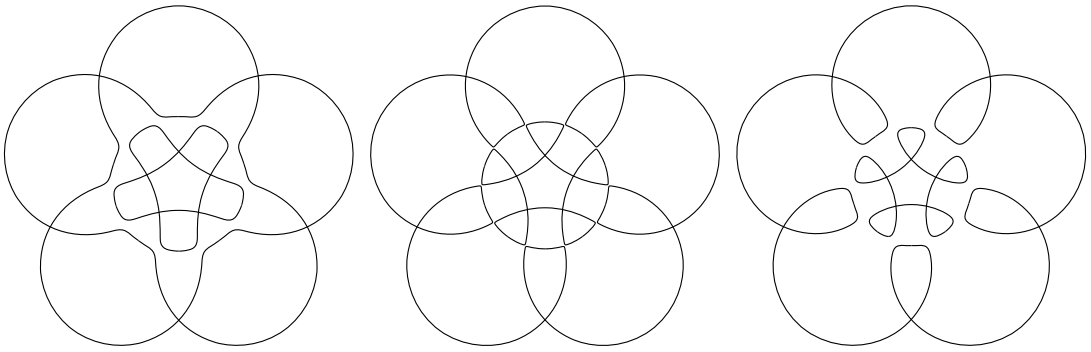

Figure 6. Profile curves of the $(\ell_0, \ell_1, \ell_2) = (2, 1, 5)$ twizzled torus family as the torus flows through its axis. The turning number of the inner profile curve jumps from $\ell_0 = 2$ to $\ell_1 + \ell_2 - \ell_0 = 4$. Figure 1 shows a 5-lobed torus in the family of which these are cross-sections.

Proof. First, we will prove the following:

Claim 1. *If a profile curve set is not immersed, then* $(\mathrm{h\,k} - \mathrm{q})(\mathrm{h} - \mathrm{k\,q}) = 0$.

To this end, let f be the immersion of the torus as in (4.11). Writing $f = f_1 + f_2\,\mathbb{j}$, the two profile curve sets are defined implicitly by $\operatorname{Re} f_1 = 0$ and $\operatorname{Re} f_2 = 0$ respectively. The profile curves are singular wherever $\operatorname{Re} f_1$, $(\operatorname{Re} f_1)_x$ and $(\operatorname{Re} f_1)_y$ all vanish or $\operatorname{Re} f_2$, $(\operatorname{Re} f_2)_x$ and $(\operatorname{Re} f_2)_y$ all vanish. The function f_k decouples into

$$f_k = \phi_k(x)\psi_k(y),$$

where $\gamma = \gamma_1\gamma_2^{-1}$ and $\phi_1 = \alpha_1\alpha_2^{-1}$ and $\psi_1 = \beta_1\beta_2^{-1}(\gamma c_1 c_2 + \gamma^{-1} s_1 s_2)$, and $\phi_2 = \alpha_1\alpha_2$ and $\psi_2 = \beta_1\beta_2(\gamma^{-1} s_1 c_2 - \gamma c_1 s_2)$. Then

$$2\operatorname{Re} f_k = \phi_k\psi_k + \overline{\phi}_k\overline{\psi}_k, \quad 2(\operatorname{Re} f_k)_x = (\phi_k)_x\psi_k + \overline{(\phi_k)}_x\overline{\psi}_k, \quad k = 1, 2.$$

For $k = 1, 2$, since $(\phi_k)_x$ never vanishes, it follows that $\operatorname{Re} f_k$ and $(\operatorname{Re} f_k)_x$ vanish if and only if ψ_k vanishes. The additional condition that $(\operatorname{Re} f_k)_y$ vanishes is ignored; it specifies for which values of x, if any, the curve fails to be immersed. Since β_1 and β_2 are unimodular and c_j and s_j are real, this occurs if $\gamma^4 = 1$ and either $c_1^2c_2^2 - s_1^2s_2^2 = 0$ or $s_1^2c_2^2 - c_1^2s_2^2 = 0$. We have

$$2(c_1^2c_2^2 - s_1^2s_2^2) = \cos(\chi_1(\lambda_2)) + \cos(\chi_1(\lambda_1)) = v^{-1}v'\left(\frac{1}{2}(\nu^{-1}(\lambda_2) + \nu^{-1}(\lambda_1))\right),$$
$$2(s_1^2c_2^2 - c_1^2s_2^2) = \cos(\chi_1(\lambda_2)) - \cos(\chi_1(\lambda_1)) = v^{-1}v'\left(\frac{1}{2}(\nu^{-1}(\lambda_2) - \nu^{-1}(\lambda_1))\right).$$

The zero-sets of each of these expressions is the zero-set of v'. At the zeros of v', $v = 1$ or $v = \mathrm{k}$. A computation shows that the zero-set of $\gamma^4 - 1$ is the zero set of $\mathrm{h}\,v^2 - \mathrm{q\,k}$. Hence if the curve is not immersed, then either $v = \mathrm{k}$ and $\mathrm{h\,k} = \mathrm{q}$, or else $v = 1$ and $\mathrm{h} = \mathrm{q\,k}$. This proves Claim 1.

Let

$$\mathcal{I}_1 = \{\mathrm{q} - \mathrm{k\,h} < 0\}, \quad \widehat{\mathcal{I}}_1 = \{\mathrm{q} - \mathrm{k\,h} > 0\}, \quad \mathcal{I}_2 = \{\mathrm{h} - \mathrm{k\,q} > 0\}, \quad \widehat{\mathcal{I}}_2 = \{\mathrm{h} - \mathrm{k\,q} < 0\},$$
$$(\tilde{\ell}_0, \tilde{\ell}_1, \tilde{\ell}_2) = (\ell_1 + \ell_2 - \ell_0, \ell_1, \ell_2), \quad c_{jk} = \gcd(\ell_j, \ell_k), \quad \hat{c}_{jk} = \gcd(\hat{\ell}_j, \hat{\ell}_k).$$

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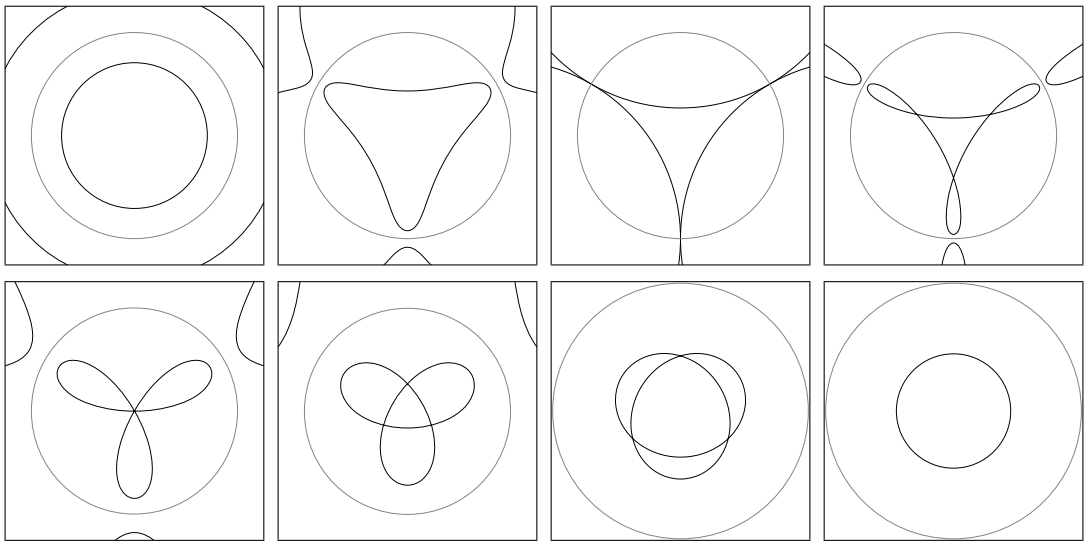

Figure 7. The flow through the 3-lobed cmc tori of revolution. Starting at a singly wrapped homogeneous torus, the inner profile curve of the embedded $(1,3)$ cmc tori flows to the $(1,3)$ sphere bouquet in the third frame. It continues through the non-embedded $(2,3)$ cmc tori, crosses itself in the fifth frame, and ends at a doubly wrapped homogeneous torus. As the inner profile curve passes through the origin, and the outer curve through infinity, their turning numbers remain fixed but their winding numbers jump. The curves are stereographically projected to $\mathbb{E}^2$; the central gray circle is the axis of revolution.

Claim 2. *On $\mathcal{I}_k$ (respectively $\widehat{\mathcal{I}}_k$), $\mathcal{C}_k$ has c_k (respectively $\hat{c}_k$) components.*

Each component of $\mathcal{C}_k$ has turning number ℓ_0/c_k (respectively $\hat{\ell}_0/c_k$). The total turning number of $\mathcal{C}_k$ is ℓ_0 on each of $\mathcal{I}_k$ and $\widehat{\mathcal{I}}_k$. By Theorem 4.2 (2), at the homogeneous torus at the beginning (respectively end) of the flow, the profile curve sets are ℓ_0 (respectively $\ell_1+\ell_2-\ell_0$) wrapped circles. Since turning numbers of the immersed profile curves are homotopy invariants, the total turning number of $\mathcal{C}_1$ is preserved in $\mathcal{I}_1$ and $\widehat{\mathcal{I}}_1$. Similarly the total turning number of $\mathcal{C}_2$ is preserved in $\mathcal{I}_2$ and $\widehat{\mathcal{I}}_2$. □

Lemma 5.4 simplifies in the case of cmc tori of revolution. By Theorem 4.2, the profile curve at the homogeneous torus is an ℓ_0 wrapped circle, with turning number ℓ_0. Since the flow $\mathrm{k} \in (0,1]$ induces a regular homotopy of the profile curve, by the Whitney–Graustein Theorem, every profile curve in the flow has turning number ℓ_0. Figure 7 illustrates the profile curves of the 3-lobed cmc tori of revolution.

5.4. Embeddedness. We show that twizzled cmc tori are never embedded, and classify embedded cmc tori of revolution. As a corollary of Lemma 5.4 we have

Corollary 5.5. *A non-rotational spectral genus one cmc torus in $\mathbb{S}^3$ is never embedded.*

Proof. By Lemma 5.4, the profile curve sets of a (ℓ_0,ℓ_1,ℓ_2) twizzled cmc torus have total turning number ℓ_0 or $\ell_1+\ell_2-\ell_0$. By Theorem 4.2, each of these turning numbers is strictly bigger than 1. Hence the profile curve sets are not embedded. To show the surface is not embedded assume first that $(\mathrm{h}\,\mathrm{k}-\mathrm{q})(\mathrm{h}-\mathrm{k}\,\mathrm{q}) \neq 0$. If f where embedded, then by Lemma 5.4

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the profile curve is immersed. By the Inverse Function Theorem the inverse image under f of the profile curve is embedded, hence if it where embedded the profile curve would be embedded, which is not true since its turning number is at least two by Lemma 5.4, giving the contradiction. Because embeddedness is an open condition, during the flow $(\mathrm{h}\,\mathrm{k}-\mathrm{q})(\mathrm{h}-\mathrm{k}\,\mathrm{q}) \neq 0$ away from isolated points, the surface is embedded also at the zeroes. □

Theorem 5.6. *An (ℓ_0, ℓ_2) torus of revolution is embedded if and only if $\ell_0 = 1$.*

Proof. A surface of revolution in $\mathbb{S}^3$ is embedded if and only if its profile curve is embedded and does not meet the revolution axis. To show the embeddedness, we show that the curvature of the orthographic projection of the profile curve $f_0 = \exp(\mathbb{i}\,\chi_0)(g_1 + \mathbb{i}\,g_2) + g_0\mathbb{k}$ in (4.15) is strictly positive. Write $\exp(\mathbb{i}\,\chi_0)(g_1 + \mathbb{i}\,g_2) = r\exp(\mathbb{i}\,\psi)$ and $s = g_0$ so the profile curve is $f_0 = re^{\mathbb{i}\psi} + s\mathbb{k}$. To compute ψ', note that

$$\psi = \chi_0 + \arg(g_1 + \mathbb{i}\,g_2).$$

The expression for ${\chi_0}'$ in (3.10) yields after a calculation

$$\begin{aligned} \psi' &= {\chi_0}' + \frac{g_1 g_2' - g_1' g_2}{g_1^2 + g_2^2} \\ &= \frac{2\nu\sin(2\theta_1)(v^2\cos(2\theta_1) - \mathrm{k})}{v^2\sin^2(2\theta_1) - 4\nu^2} \end{aligned} \tag{5.1}$$

with $2\mathbb{i}\,\theta_1 = \ln\lambda_1$. The curvature of the plane curve $re^{\mathbb{i}\psi}$ is $\kappa = 8\,\mathrm{C}^{-3}\,\nu^2\,\mathrm{k}\,v^{-1}$. Note that the plane curve $re^{\mathbb{i}\psi}$ is an orthographic projection of the hemisphere to $\mathbb{R}^2$, not stereographic; the curvature of the stereographic projection may change sign, as seen in Figure 7.

We next show that the profile curve does not meet the revolution axis for $\mathrm{k} \in (0, 1]$. Since the range of v is $[\mathrm{k}, 1]$, the range of s is $[\frac{1}{2}\nu^{-1}\,\mathrm{k}\sin(2\theta_1), \frac{1}{2}\nu^{-1}\sin(2\theta_1)]$. Hence $s > 0$, because $\nu > 0$, $\mathrm{k} > 0$ and $\sin(2\theta_1) > 0$. Hence $|r| < 1$, so the profile curve does not meet the axis of revolution.

If the profile curve is embedded, then its turning number is 1. But by the discussion after Theorem 5.4, its turning number is ℓ_0. Hence $\ell_0 = 1$.

Conversely, assume $\ell_0 = 1$, so its turning number is 1. The curvature κ of the orthographic projection $re^{\mathbb{i}\psi}$ of the profile curve to $\mathbb{C}$ computed above is strictly positive, so the profile curve is convex and hence embedded (see e.g. [11, Section 5-7, Proposition 1]). □

Theorem 5.7. *An equivariant cmc torus in $\mathbb{S}^3$ is Alexandrov embedded if and only if it is a surface of revolution and singly wrapped with respect to the rotational period.*

Proof. A rotational cmc torus is Alexandrov embedded if and only if there exists an immersion $\mathbb{S}^1 \times [0, 1] \to \mathbb{S}^2_+$ into a hemisphere $\mathbb{S}^2_+$ such that $\mathbb{S}^1 \times \{0\}$ is mapped to the equator of $\mathbb{S}^2_+$ and $\mathbb{S}^1 \times \{1\}$ is mapped onto a profile curve of the torus. The resulting 3-manifold, obtained by rotating the strip is then the Alexandrov embedding.

Every homogeneous torus is a covering of an embedded homogeneous torus. Hence the 3-manifold is always a solid torus, and thus has fundamental group $\mathbb{Z}$. The compact coverings correspond to proper subgroups of $\mathbb{Z}$. Hence Alexandrov embedded homogeneous tori have to be singly wrapped. This condition is stable under continuous deformations which stay away from bouquets of spheres. □

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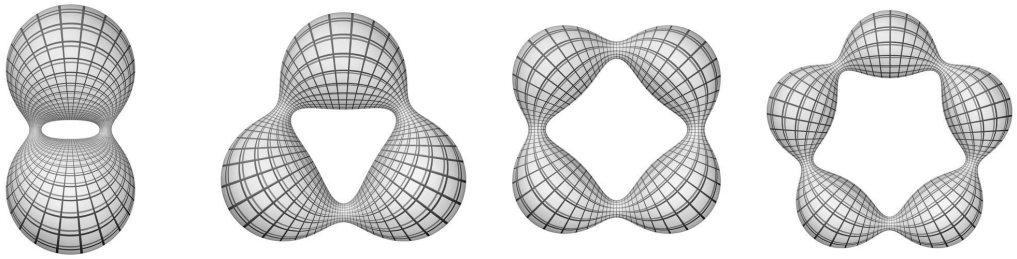

Figure 8. Embedded $(1, n)$ cmc tori of revolution in $\mathbb{S}^3$, with $n = 2, 3, 4, 5$.

5.5. Mean curvature and minimal tori. There are infinitely many minimal tori in the 3-sphere $\mathbb{S}^3$, see [10]. There are in fact already infinitely many minimal equivariant ones [18]. Theorem 5.8 shows the existence of infinitely many minimal twizzled tori. For example the $(\ell_0, \ell_1, \ell_2) = (n-k, n, n+k)$ flow family with $0 < k < n$ is a fixed point of the involution of Proposition 4.3. The flow starts and ends at homogeneous tori with opposite mean curvature, and hence it contains a minimal torus. In Figure 1, a (non-minimal) example from the $(\ell_0, \ell_1, \ell_2) = (2, 1, 3)$ family is shown.

Lemma 5.8. *A spectral genus one flow family with the two endpoints* (ℓ_0, ℓ_1, ℓ_2) *and* $(\ell_1 + \ell_2 - \ell_0, \ell_1, \ell_2)$ *contains exactly one minimal torus if*

$$(\ell_1^2 + \ell_2^2)^{1/2} \geq \sqrt{2} \max\{\ell_0, \ell_1 + \ell_2 - \ell_0\}$$

and no minimal tori otherwise.

Proof. Consider the flow from a homogeneous torus to a homogeneous torus through spectral genus one cmc tori as described in Theorem 4.5. Since the mean curvature (2.3) is monotonic, the flow contains a minimal torus if and only if the mean curvature of the homogeneous tori have opposite signs, or if one of the homogeneous tori is minimal.

By a calculation, $\ell_1 = \ell_0 \min\{X, Y\}$, $\ell_2 = \ell_0 \max\{X, Y\}$, where $X = \sqrt{(1+\mathrm{q})/(1-\mathrm{h})}$ and $Y = \sqrt{(1-\mathrm{q})/(1+\mathrm{h})}$, and then

$$H = \operatorname{sign}(\mathrm{q}+\mathrm{h}) \frac{\ell_1^2 + \ell_2^2 - 2\ell_0^2}{2\sqrt{(\ell_2^2 - \ell_0^2)(\ell_0^2 - \ell_1^2)}}. \tag{5.2}$$

Since $\mathrm{q}+\mathrm{h}$ is negative at the beginning of the flow and positive at the end (Theorem 4.5), by Proposition 4.3, mean curvatures H_0 and H_1 of the homogeneous tori at the beginning and end of the flow are respectively

$$H_0 := \frac{\ell_1^2 + \ell_2^2 - 2\ell_0^2}{2\sqrt{(\ell_2^2 - \ell_0^2)(\ell_0^2 - \ell_1^2)}}, \quad H_1 := -\frac{\ell_1^2 + \ell_2^2 - 2\hat{\ell_0}^2}{2\sqrt{(\ell_2^2 - \hat{\ell_0}^2)(\hat{\ell_0}^2 - \ell_1^2)}} \tag{5.3}$$

where $\hat{\ell_0} := \ell_1 + \ell_2 - \ell_0$. The flow contains a minimal torus if and only if $\ell_1^2 + \ell_2^2 - 2\ell_0^2$ and $\ell_1^2 + \ell_2^2 - 2(\ell_1 + \ell_2 - \ell_0)^2$ have the same sign, or either is 0. Since the sum of these two integers is equal to $4(\ell_2 - \ell_0)(\ell_0 - \ell_1) > 0$, they are not both negative. The condition that they have the same sign, or either is 0, is then equivalent to the asserted inequality. □

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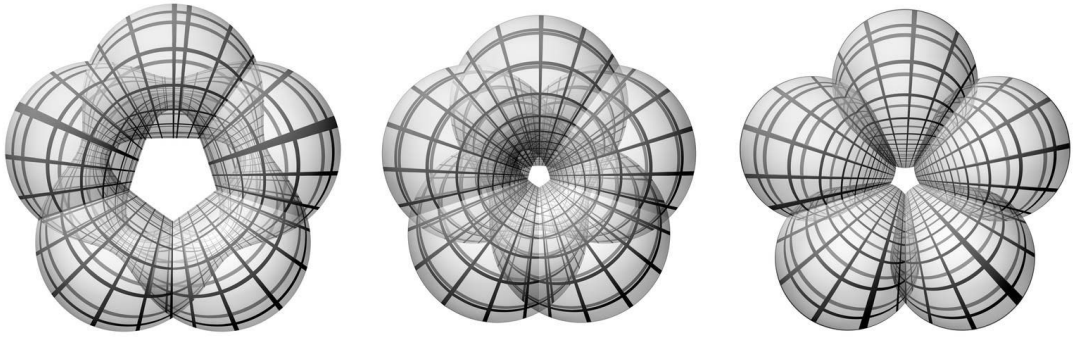

Figure 9. Alexandrov embedded five-lobed $(k, 5)$ cmc tori of revolution in $\mathbb{S}^3$. The turning number of the inner profile curve is $k = 2, 3, 4$.

Consider the (ℓ_0, ℓ_2) family of cmc tori of revolution, and let $\varrho := \ell_0/\ell_2$. The mean curvature for the homogeneous tori is chosen to be positive for $\varrho \in (0, 1/\sqrt{2})$ and negative for $\varrho \in (1/\sqrt{2}, 1)$.

Lemma 5.9. *For spectral genus one cmc tori of revolution, the mean curvature decreases monotonically from the mean curvature* $H_0 = (1 - 2\varrho^2)/(2\varrho\sqrt{1-\varrho^2})$ *at the homogeneous torus to* $H_s = \cot \pi\varrho$ *at the sphere bouquet. This family contains exactly one minimal torus if* $\varrho \in (\frac{1}{2}, 1/\sqrt{2}]$*, and no minimal tori otherwise.*

Proof. By (5.3), the mean curvature of the homogeneous torus at the end of the flow ($\mathrm{k} = \pm 1$) is H_0 as in the assertion. By Lemma 4.4, the (ℓ_0, ℓ_2) family of cmc tori of revolution converges to the (ℓ_0, ℓ_2) sphere bouquet as $\mathrm{k} \to 0$. The limiting sphere bouquet has mean curvature $H_s = \cot \pi\varrho$. The mean curvature is monotonic by Theorem 4.5 and hence has the specified range. The family contains a minimal torus if and only if the mean curvature has different signs at the endpoints of the flow. This occurs if and only if $\varrho \in (\frac{1}{2}, 1/\sqrt{2}]$. □

We thus obtain an independent proof of the following result, which is a special case of the affirmation of the Pinkall–Sterling conjecture by Andrews and Li [4].

Corollary 5.10. *The Clifford torus is the only minimal embedded rotational torus in* $\mathbb{S}^3$.

By combining the above results we have shown that among the infinitely many minimal equivariant tori in the 3-sphere, only one is embedded (see also [18]). This of course is merely a special case of the affirmation of the Lawson conjecture by Brendle [7].

Theorem 5.11. *The Clifford torus is the only embedded minimal equivariant torus in* $\mathbb{S}^3$.

6. Connectedness of the moduli space

Let M denote the set of cmc immersions from the oriented 2-torus $\mathbb{T}^2$ into the oriented 3-sphere $\mathbb{S}^3$. We define an equivalence relation by identifying two maps in M if they differ by

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an orientation preserving diffeomorphism of $\mathbb{T}^2$ and an orientation preserving isometry of $\mathbb{S}^3$, and set $\mathcal{M} = M/\!\sim$. We denote the spectral genus zero maps by $\mathcal{M}_0 \subset \mathcal{M}$, that is,

$$\mathcal{M}_0 = \{\text{equivalence classes of homogeneous tori in } \mathbb{S}^3\}.$$

Thus $\mathcal{M}_0$ consists of infinitely many $\mathbb{R}$-families of homogeneous tori. Even though each of these families by Proposition 3.4 (iii) is a finite cover of the family of the underlying embedded rectangular torus, we need the full diversity of $\mathcal{M}_0$ to bifurcate into all possible spectral genus one cmc tori. The spectral genus one cmc tori will be denoted by $\mathcal{M}_1 \subset \mathcal{M}$, that is,

$$\mathcal{M}_1 = \{\text{equivalence classes of spectral genus one cmc tori in } \mathbb{S}^3\}.$$

Since deformation families of rotational cmc tori in $\mathcal{M}_1$ flow into bouquets of spheres, we take the closure of $\mathcal{M}_1$ by supplementing it with the limiting bouquets of spheres, and set

$$\overline{\mathcal{M}}_1 = \mathcal{M}_1 \cup \{\text{equivalence classes of sphere bouquets in } \mathbb{S}^3\}.$$

The aim of this section is to show that this *completed* moduli space of equivariant cmc tori $\mathcal{M}_0 \cup \overline{\mathcal{M}}_1$ is connected (Theorem 6.5). We have already seen that the moduli space of equivariant cmc tori in the 3-sphere is a graph, which consists of:

(1) edges of families of spectral genus zero cmc tori,

(2) edges of families of spectral genus one cmc tori,

(3) 'bifurcation' vertices in $\mathcal{M}_0$ that connect with other vertices in $\mathcal{M}_0$ via spectral genus one edges by Theorem 4.5.

By Proposition 3.4 (iii) any element in $\mathcal{M}_0$ is isogenic to the unique (up to isomorphism) embedding of a rectangular lattice with the same mean curvature. Hence each edge in $\mathcal{M}_0$ contains a unique minimal torus, obtained via an isogeny from the Clifford torus. If we identify two isogenies which differ only by an isomorphism of the domain, then we have a one-to-one correspondence between isomorphy classes of isogenies and co-finite sublattices of Λ^*. Hence we can identify the connected components of $\mathcal{M}_0$ with co-finite sublattices of Λ^*. We say that two such sublattices are *connected* if the corresponding genus zero edges are connected in $\mathcal{M}_0 \cup \overline{\mathcal{M}}_1$.

We associated to the bifurcation vertices triples $(\ell_0, \ell_1, \ell_2) \in \mathbb{Z}^3$ with $\gcd(\ell_0, \ell_1, \ell_2) = 1$ and $0 \leq \ell_1 < \ell_0 < \ell_2$. In the proof of Theorem 4.2 (2) we showed that triples (4.4) correspond to the lattices (4.9). The genus one edges described in Theorem 4.5 yield four isomorphisms of each of these lattices onto one of those lattices corresponding to the triples $(\ell_1 + \ell_2 - \ell_0, \ell_1, \ell_2)$. Furthermore, the genus zero edge corresponding to a sublattice of one of the former lattices is connected by an isogenic genus one edge with the corresponding sublattice of one of the latter lattices. But we do not use these isomorphisms of the lattices (4.9) onto those lattices corresponding to $(\ell_1 + \ell_2 - \ell_0, \ell_1, \ell_2)$. We shall make use of these isomorphisms only in case of embedded cmc tori with $\ell_0 = 1$ and $\ell_1 = 0$. In this case rotation periods are preserved, and up to transformations (C') and (D') these isomorphisms are of the form

$$\begin{aligned} \Lambda^* &\to p\,\gamma_1^*\mathbb{Z} \oplus q\,\gamma_2^*\mathbb{Z}, \\ n_1\gamma_1^* + n_2\gamma_2^* &\mapsto p\,n_1\gamma_1^* \oplus q\,n_2\gamma_2^* \end{aligned} \tag{6.1}$$

with $p = 1$ or $q = 1$.

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Proposition 6.1. *The following statements hold:*

(i) *The edge of a lattice $\Gamma \subset \Lambda^*$ contains a vertex with triple (ℓ_0, ℓ_1, ℓ_2) if and only if Γ is a sublattice of one of the lattices in* (4.9)*.*

(ii) *For any ℓ_0, ℓ_1, ℓ_2 we have that $\Gamma_{[\ell_0,\ell_1,\ell_2]} = \Lambda^*$ if and only if $\ell_0 = 1$.*

(iii) *The unique edge of embedded rotational cmc tori in $\mathcal{M}_0$ contains only the vertices with triples of the form $(\ell_0, \ell_1, \ell_2) = (1, 0, \ell_2)$ with $\ell_2 \geq 2$.*

Proof. (i) In the proof of Theorem 4.2 (2) we determined the lattices that correspond to a triple (4.4). They are given in (4.9).

(ii) If $\ell_0 = 1$, then $n_1\ell_1 + n_2\ell_2 \in \mathbb{Z}$ holds for all $(n_1, n_2) \in \mathbb{Z}^2$. Hence $\Gamma_{[1,\ell_1,\ell_2]} = \Lambda^*$. Conversely, if $\Gamma_{[\ell_0,\ell_1,\ell_2]} = \Lambda^*$, then $n_1\ell_1 + n_2\ell_2 \in \ell_0\mathbb{Z}$ holds for all pairs $(n_1, n_2) \in \mathbb{Z}^2$. In particular we have $\ell_1, \ell_2 \in \ell_0\mathbb{Z}$, which implies $\ell_0 = 1$ so as not to contradict the assumption $\gcd(\ell_0, \ell_1, \ell_2) = 1$.

(iii) By Proposition 4.2 we have that $\ell_1 = 0$ in the rotational case. By Theorem 5.6 the (ℓ_0, ℓ_2) torus of revolution is embedded if and only if $\ell_0 = 1$. □

Lemma 6.2. *The moduli of rotational cmc tori in the* 3*-sphere, supplemented by bouquets of spheres is connected.*

Proof. We will show that any edge of rotational cmc tori in $\mathcal{M}_0$ is connected to the edge of embedded rotational cmc tori, or equivalently that any lattice $p\,\gamma_1^*\mathbb{Z} + q\,\gamma_2^*\mathbb{Z}$ of a rotational cmc torus is connected to the Λ^* lattice. By Proposition 6.1 (iii) the edge of embedded rotational cmc tori in $\mathcal{M}_0$ contains all the vertices $(1, 0, \ell_2)$ with $\ell_2 \geq 2$. This edge is connected with all the edges that contain the vertices $(\ell_2 - 1, 0, \ell_2)$ with $\ell_2 \geq 2$. Hence the lattice $\Gamma_{[1,0,\ell_2]} = \Lambda^*$ is connected to the lattice $\Gamma_{[\ell_2-1,0,\ell_2]} = \gamma_1^*\mathbb{Z} + (\ell_2 - 1)\gamma_2^*\mathbb{Z}$. Furthermore, the lattice $D\Gamma_{[1,0,\ell_2]} = \Lambda^*$ is connected to the lattice $D\Gamma_{[\ell_2-1,0,\ell_2]} = (\ell_2 - 1)\gamma_1^*\mathbb{Z} + \gamma_2^*\mathbb{Z}$. Sublattices $\Gamma = p\,\gamma_1^*\mathbb{Z} + \gamma_2^*\mathbb{Z} \subset \Gamma_{[1,0,\ell_2]}$ are connected along genus one edges isogenic to the former genus one edges with

$$p\,\gamma_1^*\mathbb{Z} + (\ell_2 - 1)\,\gamma_2^*\mathbb{Z} \subset \Gamma_{[\ell_2-1,0,\ell_2]}$$

by the isomorphism (6.1). □

In the following we shall combine deformations to pass between bifurcation vertices in $\mathcal{M}_0$. There are three different types of deformations:

① The deformation through spectral genus one, possibly also passing through bouquets of spheres as described in Theorem 4.5. In this case we write

$$(\ell_0, \ell_1, \ell_2) \overset{①}{\to} (\ell_2 + \ell_1 - \ell_0, \ell_1, \ell_2).$$

② The deformation along an edge of homogeneous tori, passing from one 'bifurcation' vertex to another one: Suppose $(\ell_0, \ell_1, \ell_2) \in \mathbb{Z}^3$ is such that $0 < \ell_1$ and $\ell_2 < 2\ell_0$. Then from $n_1\ell_1 + n_2\ell_2 \in \ell_0\mathbb{Z}$ get that also $n_1(\ell_1 + \ell_0) + n_2(\ell_2 - \ell_0) \in \ell_0\mathbb{Z}$. In this case the transformation $(\ell_0, \ell_1, \ell_2) \mapsto (\ell_0, \ell_2 - \ell_0, \ell_0 + \ell_1)$ acts on the corresponding lattices as the transformation (D"), which interchanges the four lattices (4.9). In such a case we write

$$(\ell_0, \ell_1, \ell_2) \overset{②}{\to} (\ell_0, \ell_2 - \ell_0, \ell_0 + \ell_1).$$

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③ If ℓ_1 is odd, then we have

$$n_1\gamma_1^* + n_2\gamma_2^* \in \Gamma_{[\ell_0,\ell_1,\ell_2]} \iff 2n_1\gamma_1^* + n_2\gamma_2^* \in \Gamma_{[2\ell_0,\ell_1,2\ell_2]}. \tag{6.2}$$

Due to Lemma 6.2 both genus zero edges corresponding to the lattices $2\mathbb{Z}\gamma_1^* + \mathbb{Z}\gamma_2^*$ and $\mathbb{Z}\gamma_1^* + 2\mathbb{Z}\gamma_2^*$ are connected with the edge corresponding to Λ^*. If we apply an isogeny to these families we obtain with (6.1) a deformation of a genus zero edge corresponding to the lattice $\Gamma \subset \Lambda^*$ to another genus zero edge corresponding to the lattices

$$\{2n_1\gamma_1^* + n_2\gamma_2^* \mid n_1\gamma_1^* + n_2\gamma_2^* \in \Gamma\} \quad \text{and} \quad \{n_1\gamma_1^* + 2n_2\gamma_2^* \mid n_1\gamma_1^* + n_2\gamma_2^* \in \Gamma\},$$

respectively. In combination with (6.2) we write

$$(\ell_0, \ell_1, \ell_2) \xrightarrow{③} (2\ell_0, \ell_1, 2\ell_2).$$

Lemma 6.3. *For co-prime integers $0 \le \ell_1 < \ell_0 < \ell_2$ the lattices* (4.9) *are connected with Λ^*.*

Proof. We shall show that it is possible to successively reduce ℓ_1 until $\ell_1 = 0$. We can add to ℓ_2 multiples of ℓ_0 without changing the lattices (4.9). We pick the smallest of all possible ℓ_2 and obtain $\ell_2 \le 2\ell_0$. In case of equality, the lattices (4.9) are of the form $p\mathbb{Z}\gamma_1^* + q\mathbb{Z}\gamma_2^*$. In this case Lemma 6.2 connects Γ with Λ^*. Therefore we may assume $\ell_2 < 2\ell_0$.

In the deformation ① we pick the smaller of the first entries ℓ_0 and $\ell_2 + \ell_1 - \ell_0$. Hence we can assume that $2\ell_0 \le \ell_1 + \ell_2$, and now we have the following sequence of deformations:

$$\begin{aligned}(\ell_0, \ell_1, \ell_2) \xrightarrow{②} (\ell_0, \ell_2 - \ell_0, \ell_0 + \ell_1) \xrightarrow{①} (\ell_2 + \ell_1 - \ell_0, \ell_2 - \ell_0, \ell_0 + \ell_1)\\ \xrightarrow{②} (\ell_2 + \ell_1 - \ell_0, 2\ell_0 - \ell_2, \ell_1 + 2\ell_2 - 2\ell_0).\end{aligned}$$

If $2\ell_0 < \ell_1 + \ell_2$, then $2\ell_0 - \ell_2 < \ell_1$, so ℓ_1 has decreased by this deformation.

If $2\ell_0 = \ell_1 + \ell_2$, then we distinguish two cases: If ℓ_1 and $\ell_0 - \ell_1$ were both even, then ℓ_2 would be even, contradicting that $\gcd(\ell_0, \ell_1, \ell_2) = 1$. Hence we just need to consider the two cases $\ell_0 - \ell_1$ is odd, and ℓ_1 is odd.

If $\ell_0 - \ell_1$ is odd, then

$$\begin{aligned}(\ell_0, \ell_1, 2\ell_0 - \ell_1) \xrightarrow{②} (\ell_0, \ell_0 - \ell_1, \ell_0 + \ell_1) \xrightarrow{③} (2\ell_0, \ell_0 - \ell_1, 2\ell_0 + 2\ell_1)\\ \xrightarrow{①} (\ell_0 + \ell_1, \ell_0 - \ell_1, 2\ell_0 + 2\ell_1).\end{aligned}$$

If ℓ_1 is odd, then

$$(\ell_0, \ell_1, 2\ell_0 - \ell_1) \xrightarrow{③} (2\ell_0, \ell_1, 4\ell_0 - 2\ell_1) \xrightarrow{①} (2\ell_0 - \ell_1, \ell_1, 4\ell_0 - 2\ell_1).$$

Obviously for $\ell_2 = 2\ell_0$ the lattices (4.9) are of the form $p\mathbb{Z}\gamma_1^* + q\mathbb{Z}\gamma_2^*$. Hence in all cases either the lattices (4.9) are connected with Λ^* or ℓ_1 is reduced. By repeating the above procedure finitely many times we eventually achieve $\ell_1 = 0$, which we have already dealt with in Lemma 6.2. □

Lemma 6.4. *For every co-finite sublattice $\Gamma \subsetneq \Lambda^*$ there exists a triple $(\ell_0, \ell_1, \ell_2) \in \mathbb{Z}^3$ with $\gcd(\ell_0, \ell_1, \ell_2) = 1$, $0 \le \ell_1 < \ell_0 < \ell_2$ and $\ell_0 > 1$, such that Γ is contained in a lattice of* (4.9).

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Proof. Since $\Gamma \neq \Lambda^*$, there exists a $\gamma_1 \in \Lambda^*$ with $\gamma_1 \notin \Gamma$, and γ_1 is not a multiple of another element in Λ^*. Let $\gamma_2 \in \Lambda^*$ such that $\Lambda^* \cong \gamma_1\mathbb{Z} \oplus \gamma_2\mathbb{Z}$. Since Λ^*/Γ is finite, there exist smallest integers $p, q \in \mathbb{Z}$ with $p \geq 2, q \geq 0$ such that $\gamma_1 p \in \Gamma$ and $\gamma_1 q \oplus \gamma_2 \in \Gamma$.

Consider the homomorphism $g : \Lambda^* \to \mathbb{Z}, \gamma_1 l \oplus \gamma_2 m \mapsto l - qm$. By definition of p, q we have that $\Gamma \subset \gamma_1 p\mathbb{Z} \oplus (\gamma_1 q + \gamma_2)\mathbb{Z}$, so that g maps Γ to a sublattice of $p\mathbb{Z}$. Every such homomorphism is of the form $a\gamma_1^* + b\gamma_2^* \in \Lambda^* \mapsto \ell_1 a + \ell_2 b$ with $\ell_1, \ell_2 \in \mathbb{Z}$. By adding appropriate multiples of $\ell_0 = p$ to ℓ_1, ℓ_2 we can achieve $0 \leq \ell_1 < \ell_0 < \ell_2$. □

Theorem 6.5. *The completed moduli space of equivariant cmc tori in the* 3*-sphere is connected.*

Proof. If $\Gamma \subsetneq \Lambda^*$ is a co-finite sublattice, then by Lemma 6.4 the corresponding genus zero edge contains a bifurcation vertex with triple (ℓ_0, ℓ_1, ℓ_2) and integers $0 \leq \ell_1 < \ell_0 < \ell_2$. By Lemma 6.3 the corresponding lattices (4.9) are connected to the lattice Λ^*. An isogeny of this path connects the edge corresponding to Γ with a genus zero edge corresponding to Γ' with $|\Lambda^*/\Gamma'| < |\Lambda^*/\Gamma|$. Repeating this argument we can successively reduce the order until $|\Lambda^*/\Gamma'| = 1$. □

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