INSTABILITY OF EQUATORIAL WATER WAVES IN THE $f-$PLANE

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ABSTRACT. This papers addresses the hydrodynamical stability of nonlinear geophysical equatorial waves in the $f-$plane approximation. By implementing the short-wavelength perturbation approach, we show that certain westward-propagating equatorial waves are linearly unstable when the wave steepness exceeds a specific threshold.

1. Introduction

This paper is concerned with the hydrodynamical stability of a recently derived [28] solution for equatorial water waves. This three-dimensional exact Gerstner-type solution of the governing equations is explicit in the Lagrangian formulation, and it prescribes zonal waves which propagate westwards along the equator. An important aspect of the solution is that there is no variation in the meridional direction. This is in stark contrast to the recently-derived equatorially trapped solutions which were derived in the $\beta-$plane in [7, 26], and it is precisely this aspect which enables westward propagating waves in the setting of [28], but not in the setting of [7, 26].

Allowing for westward propagating waves is interesting from the viewpoint of the complex geophysical dynamics in the equatorial region [7, 17]. For instance, the famous Equatorial Undercurrent (EUC) [31] underlies wind-generated westward-propagating surface waves, and it has been proposed that the interplay between equatorial currents in the ocean and atmosphere is one of the major generating mechanisms for El Niño and La Niña phenomena, cf. [31] and the discussions in [8, 10, 27]. Recent mathematical work [10, 27] has rigorously established the existence of westward-propagating equatorial waves which admit an underlying current of general vorticity distribution, and the derivation of an explicit solution in [28] is therefore highly interesting from both a mathematical and physical perspective.

Once an exact solution is available, the stability issue becomes important. Hydrodynamic stability is important for understanding the factors that might trigger the transition from the large-scale coherent structure represented by the exact solution to a more chaotic fluid motion [15, 16]. Due to the intractability of the governing equations, establishing the hydrodynamical stability or instability of a flow is difficult. However, it was first observed that for the celebrated Gerstner's solution, which is explicit in the Lagrangian formulation [3, 23, 25], the short wavelength instability analysis is remarkably elegant [32]. The

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short-wavelength instability method, which was independently developed by the authors of [1, 18, 33], examines how a localised and rapidly-varying infinitesimal perturbation to a flow will evolve by way of a system of ODEs.

Recently, Gerstner-type solutions have been derived and adapted to model a number of different physical and geophysical scenarios [4, 7, 9, 12, 26, 28–30, 34, 35] with explicit exact solutions in a Lagrangian framework [2]. The short-wavelength instability method was then successfully employed in the geophysical context, first in [11] and then in [22], obtaining a criterion for instability which has a remarkably explicit and elegant formulation in terms of the wave steepness. In this paper we employ the short-wavelength instability method to prove that if the wave steepness exceeds a certain value, then the equatorial water waves presented in [28] are unstable under short wavelength perturbations. The wave-steepness instability criterion which applies to the westward-travelling equatorial wave derived in [28] is presented in Proposition 4.1 below.

2. Governing equations

In a reference frame with the origin located at a point on earth's surface and rotating with the earth, we take the $x-$axis to be the longitudinal direction (horizontally due east), the $y-$axis to be the latitudinal direction (horizontally due north) and the $z-$axis to be vertically upwards. The earth is taken to be a perfect sphere of radius $R = 6378km$ with constant rotational speed of $\Omega = 73 \cdot 10^{-6} rad/s$, and $g = 9.8ms^{-2}$ is the gravitational acceleration at the surface of the earth. The governing equations for geophysical ocean waves [14, 21] are given by

$$u_t + uu_x + vu_y + wu_z + 2\Omega w \cos\phi - 2\Omega v \sin\phi = -\frac{1}{\rho} P_x, \tag{2.1a}$$

$$v_t + uv_x + vv_y + wv_z + 2\Omega u \sin\phi = -\frac{1}{\rho} P_y, \tag{2.1b}$$

$$w_t + uw_x + vw_y + ww_z - 2\Omega u \cos\phi = -\frac{1}{\rho} P_z - g, \tag{2.1c}$$

together with the equation of incompressibility

$$\nabla \cdot \mathbf{U} = 0. \tag{2.2a}$$

Here $\mathbf{U} = (u, v, w)$ is the velocity field of the fluid, the variable ϕ represents the latitude, ρ is the density of the fluid (which we take to be constant), and P is the pressure of the fluid. In the equatorial region, where the latitude ϕ is relatively small, the full governing equations for geophysical water waves (2.1) may be rendered more tractable by approximating the Coriolis terms. When ϕ is small, but not constant, the $\beta-$plane approximation [14],

$$\sin\phi \approx \phi, \cos\phi \approx 1,$$

reduces the governing equations to the following form:

$$u_t + uu_x + vu_y + wu_z + 2\Omega w - \beta y v = -\frac{1}{\rho}P_x,$$
$$v_t + uv_x + vv_y + wv_z + \beta y u = -\frac{1}{\rho}P_y,$$
$$w_t + uw_x + vw_y + ww_z - 2\Omega u = -\frac{1}{\rho}P_z - g,$$

where $\beta = 2\Omega/R = 2.28 \cdot 10^{-11} m^{-1} s^{-1}$. Effectively, the Coriolis terms for the curved earth's surface which appear in (2.1) are approximated by a planar model. If we further restrict our focus to purely Equatorial waves (and so we work at a constant latitude) we get the $f-$plane approximation

$$u_t + uu_x + vu_y + wu_z + 2\Omega w = -\frac{1}{\rho}P_x, \tag{2.2ba}$$
$$v_t + uv_x + vv_y + wv_z = -\frac{1}{\rho}P_y, \tag{2.2bb}$$
$$w_t + uw_x + vw_y + ww_z - 2\Omega u = -\frac{1}{\rho}P_z - g, \tag{2.2bc}$$

The boundary conditions for the fluid on the free-surface η are given by

$$w = \eta_t + u\eta_x + v\eta_y, \tag{2.2c}$$
$$P = P_0 \qquad \text{on } y = \eta(x, y, t), \tag{2.2d}$$

where P_0 is the constant atmospheric pressure. The kinematic boundary condition on the surface simply states that all surface particles remain confined to the surface. We assume that the fluid domain is infinitely deep, with the fluid motion vanishing rapidly with increasing depth,

$$(u, v) \to (0, 0) \quad \text{as} \quad y \to -\infty. \tag{2.2e}$$

3. Exact solution of (2.2b)

Recently, in [28], an exact solution to the $f-$plane equations (2.2b) representing steady waves travelling in the longitudinal direction due west was presented. Allowing for westward propagating waves is interesting from the viewpoint of modelling the dynamics of the equatorial region [10,17]. We remark that the equatorially-trapped exact explicit solutions in the $\beta-$plane, which were recently-derived [7,26], are inherently eastward propagating, and it is an artefact of the $f-$plane approximation that enables us to precribe westward propagating waves below.

The Eulerian coordinates of fluid particles (x, y, z) are expressed as functions of the time t and the Lagrangian labelling variables (q, r, s), for $q, s \in \mathbb{R}$, $r \in (-\infty, r_0]$ where $r_0 < 0$, as follows:

$$x = q - \frac{1}{k} e^{kr} \sin[k(q - ct)], \tag{3.1a}$$

$$y = s, \tag{3.1b}$$

$$z = r + \frac{1}{k} e^{kr} \cos[k(q - ct)]. \tag{3.1c}$$

Here k is the wavenumber and $c < 0$ is the constant speed of propagation of the waves. We can see from (3.1) that the particle trajectories for the underlying flow are closed circles in a fixed latitudinal plane. The existence of closed particle paths is typical of Gerstner-type waves, and it is a phenomenon which does not apply to most irrotational water waves. In the setting of both finite [5,13] and infinite [24] depth Stokes waves the particle trajectories are in fact not closed. Defining

$$\chi = kr, \;\; \theta = k(q - ct),$$

the determinant of the Jacobian of the transformation (3.1) is $1 - e^{2\chi}$, which is time independent. Thus it follows that the flow defined by (3.1) must be volume preserving, ensuring that (2.2a) holds in the Eulerian setting [2]. In order for the transformation (3.1) to be well-defined it is necessary that

$$r \leq r_0 < 0.$$

As we are modelling equatorial waves, we take $v \equiv 0$ throughout the fluid, and we can express (2.2b) as

$$\frac{Du}{Dt} + 2\Omega w = -\frac{1}{\rho} P_x, \tag{3.2a}$$

$$\frac{Dv}{Dt} = -\frac{1}{\rho} P_y, \tag{3.2b}$$

$$\frac{Dw}{Dt} - 2\Omega u = -\frac{1}{\rho} P_z - g, \tag{3.2c}$$

where D/Dt is the material derivative. It can be shown by direct calculation that the motion prescribed by (3.1) satisfies (3.2) when the pressure function is given by

$$P = \rho \frac{kc^2 + 2\Omega c}{2k} e^{2\chi} - \rho g r + \rho \frac{kc^2 + 2\Omega c - g}{k} e^{\chi} \cos\theta + P_0,$$

and when the dispersion relation $kc^2 + 2\Omega c - g = 0$ holds. For westwards propagating waves, $c < 0$, this is equivalent to

$$c = \frac{-\Omega - \sqrt{\Omega^2 + kg}}{k}. \tag{3.3}$$

The flow determined by (3.1) satisfies the governing equations (2.2b), and the free-surface $z = \eta(x - ct, y)$ is defined parametrically at fixed latitudes $y = s$ by setting $r = r_0$.

Also, the steepness of the wave-profile, defined to be half the amplitude multiplied by the wavenumber, is given by

$$\tau(s) = e^{\chi}.$$

The velocity gradient tensor

$$\nabla \mathbf{U} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{pmatrix} = \frac{cke^{\chi}}{1-e^{2\chi}} \begin{pmatrix} -\sin\theta & 0 & \cos\theta + e^{\chi} \\ 0 & 0 & 0 \\ -e^{\chi} + \cos\theta & 0 & \sin\theta \end{pmatrix}, \tag{3.4}$$

and so the vorticity of the flow prescribed by (3.1) is $\omega = (w_y - v_z, u_z - w_x, v_x - u_y)$

$$= \left(0, -\frac{2kce^{2\chi}}{1-e^{2\chi}}, 0\right).$$

4. Instability analysis

In this Section we apply the short-wavelength instability method to the analysis of the flow determined by the solution (3.1) of the $f-$plane governing equations (2.2). The short-wavelength instability method examines the evolution of a localised and rapidly-varying infinitesimal perturbation represented at time t by the wave packet

$$\mathbf{u}(\mathbf{X}, t) = \varepsilon \mathbf{b}(\mathbf{X}, \xi_{\mathbf{0}}, \mathbf{b_0}, t) e^{i\Phi(\mathbf{X}, \xi_{\mathbf{0}}, \mathbf{b_0}, t)/\delta}. \tag{4.1}$$

Here $\mathbf{X} = (x, y, z)$, Φ is a scalar function, and at $t = 0$ we have

$$\Phi(\mathbf{X}, \xi_{\mathbf{0}}, \mathbf{b_0}, 0) = \mathbf{X} \cdot \xi_{\mathbf{0}}, \quad \mathbf{b}(\mathbf{X}, \xi_{\mathbf{0}}, \mathbf{b_0}, 0) = \mathbf{b_0}(\mathbf{X}, \xi_{\mathbf{0}}).$$

The normalised wave vector $\xi_{\mathbf{0}}$ is subject to the transversality condition $\xi_{\mathbf{0}} \cdot \mathbf{b_0} = 0$, and $\mathbf{b_0}$ is the normalised amplitude of the short-wavelength perturbation of the flow which has the velocity field $\mathbf{U}(\mathbf{X}) \equiv (u\ v\ w)^T(x, y, z)$. Then the evolution in time of $\mathbf{X}$, of the perturbation amplitude $\mathbf{b}$, and of the wave vector $\xi = \nabla\Phi$, is governed at the leading order in the small parameters ϵ and δ by the system of ODEs

$$\begin{cases} \dot{\mathbf{X}} = \mathbf{U}(\mathbf{X}, t), \\ \dot{\xi} = -(\nabla \mathbf{U})^T \xi, \\ \dot{\mathbf{b}} = -L\mathbf{b} - (\nabla \mathbf{U})\mathbf{b} + ([L\mathbf{b} + 2(\nabla \mathbf{U})\mathbf{b}] \cdot \xi) \frac{\xi}{|\xi|^2}, \end{cases} \tag{4.2}$$

with initial conditions

$$\mathbf{X}(0) = \mathbf{X_0},\ \xi(0) = \xi_{\mathbf{0}},\ \mathbf{b}(0) = \mathbf{b_0}.$$

Here $(\nabla\mathbf{U})^T$ is the transpose of the velocity gradient tensor (3.4) and, for the system defined by (3.1), $L = L(\mathbf{X})$ is given by

$$L = \begin{pmatrix} 0 & 0 & 2\Omega \\ 0 & 0 & 0 \\ -2\Omega & 0 & 0 \end{pmatrix},$$

cf. [11, 22] for details. The system of ODEs (4.2) describing the evolution of a rapidly-varying perturbation were independently derived in [1, 18, 33]. While in (4.2) the second and third equations are linear, the first equation is usually nonlinear but decouples from the other two. The first equation in (4.2) provides the particle trajectory of the basic (undisturbed) flow, while the second and third equation govern to leading order the evolution along this trajectory of the local wave vector and of the amplitude of the perturbation, respectively. The associated instability criterion for Lagrangian flows, for which $\mathbf{X}(0)=\mathbf{X_0}$, is determined by the exponent

$$\Lambda(\mathbf{X_0})=\limsup_{t\to\infty}\frac{1}{t}\ln\left(\sup_{|\xi_\mathbf{0}|=|\mathbf{b_0}|=1,\xi_\mathbf{0}\cdot\mathbf{b_0}=0}\{|\mathbf{b}(\mathbf{X},\xi_\mathbf{0},\mathbf{b_0},t)|\}\right). \tag{4.3}$$

If $\Lambda(\mathbf{X_0})>0$ for a given fluid trajectory then particles become separated at an exponential rate, and accordingly the flow is unstable [19]. Therefore, determining (4.3) provides us with a criterion for instability of a flow. For certain solutions [11, 22, 32] which have an explicit Lagrangian formulation, it transpires that the short wavelength instability analysis is remarkably elegant, and the criterion for instability (4.3) takes on a tangible and explicit formulation in terms of the wave steepness. We now derive this instability criterion for the flow determined by (3.1).

To prove the instability of the geophysical fluid flow (3.1), it is not necessary to investigate the associated system (4.2) for all initial data. It suffices to exhibit a choice for the initial disturbance that results in an exponentially growing amplitude $\mathbf{b}$. This is our aim. We choose the latitudinal wave vector $\xi_\mathbf{0}=(0\ 1\ 0)^T$, and from (3.4) we then have $\xi(t)=(0\ 1\ 0)^T$ for all $t\geq 0$. It follows that the evolution of $\mathbf{b}=(b_1,b_2,b_3)$ is governed by

$$\begin{cases}\dot{b_1}=-2\Omega b_3+\dfrac{kce^{\chi}\sin\theta}{1-e^{2\chi}}b_1+\dfrac{kce^{\chi}(e^{\chi}-\cos\theta)}{1-e^{2\chi}}b_3,\\ \dot{b_2}=0,\\ \dot{b_3}=2\Omega b_1-\dfrac{kce^{\chi}(e^{\chi}+\cos\theta)}{1-e^{2\chi}}b_1-\dfrac{kce^{\chi}\sin\theta}{1-e^{2\chi}}b_3.\end{cases} \tag{4.4}$$

Noting that the choice $b_2(0)=0$ implies $b_2(t)=0$ for all $t\geq 0$, and accordingly $\xi(t)\cdot\mathbf{b}(t)=0$, the system (4.4) reduces to

$$\dot{B}=\begin{pmatrix}\frac{kce^{\chi}\sin\theta}{1-e^{2\chi}} & -2\Omega+\frac{kce^{\chi}(e^{\chi}-\cos\theta)}{1-e^{2\chi}}\\ 2\Omega-\frac{kce^{\chi}(e^{\chi}+\cos\theta)}{1-e^{2\chi}} & -\frac{kce^{\chi}\sin\theta}{1-e^{2\chi}}\end{pmatrix}B, \tag{4.5}$$

where $B=\begin{pmatrix}b_1\\ b_3\end{pmatrix}$. This system is nonautonomous, however the change of variables induced by the matrix

$$P=\begin{pmatrix}\cos(kct/2) & \sin(kct/2)\\ -\sin(kct/2) & \cos(kct/2)\end{pmatrix}$$

transforms the planar system (4.5) to an autonomous system for $Q=P^{-1}B$ which is of the form

$$\frac{d}{dt}Q(t)=DQ(t),$$

where

$$D = \begin{pmatrix} \frac{kce^{\chi}}{1-e^{2\chi}}\sin(kq) & -\frac{kce^{\chi}}{1-e^{2\chi}}\cos(kq) - 2\Omega + \frac{kce^{2\chi}}{1-e^{2\chi}} - \frac{kc}{2} \\ -\frac{kce^{\chi}}{1-e^{2\chi}}\cos(kq) + 2\Omega - \frac{kce^{2\chi}}{1-e^{2\chi}} + \frac{kc}{2} & -\frac{kce^{\chi}}{1-e^{2\chi}}\sin(kq) \end{pmatrix}.$$

Interestingly, the form of the matrix D matches that for the $\beta-$plane formulation which was analysed in [11,22], with differences arising through the form of the dispersion relation for the wavespeed c in (3.3). Since $B = PQ$, and P is periodic with time, we deduce that the short-wavelength rapidly-varying perturbation $\mathbf{u}$, defined in (4.1), grows exponentially with time if the matrix D has a positive eigenvalue. The eigenvalues of D are given by the expression

$$\lambda^2 = \frac{-(4\Omega + 3kc)^2 e^{4\chi} + (10k^2c^2 + 32\Omega^2 + 32\Omega kc)e^{2\chi} - (4\Omega + kc)^2}{4(1 - e^{2\chi})^2}.$$

Therefore, instability of the system occurs if

$$e^{\chi} > \frac{4\Omega + kc}{4\Omega + 3kc},$$

with the exponential growth rate of the perturbation then given by the positive value of the root λ. Together with the dispersion relation (3.3), this allows us to state our main result as follows.

Proposition 4.1. *The westward propagating equatorial waves prescribed by* (3.1) *are unstable under short wavelength perturbations if the steepness of the wave profile is such that*

$$e^{kr_0} > \frac{\sqrt{\Omega^2 + kg} - 3\Omega}{3\sqrt{\Omega^2 + kg} - \Omega}. \tag{4.6}$$

Therefore, if the steepness of the wave is sufficiently large at the equator, the localised small perturbation (4.1) of grows at an exponential rate and the flow is consequently unstable. We can re-express (4.6) as

$$\frac{1 - 3\tilde{\epsilon}}{3 - \tilde{\epsilon}} \lessapprox \frac{1}{3},$$

where, since $\Omega \ll 1$, we have

$$\tilde{\epsilon} = \frac{\Omega}{\sqrt{\Omega^2 + kg}} \ll 1.$$

Remark 4.2. We note that if we ignore the Coriolis effects of the earth's rotation, by setting $\Omega = 0$, we recover the instability criterion for Gerstner's wave which was presented in [32]. Furthermore, the steepness criterion (4.6) differs from those derived in the $\beta-$plane context in the sense that, in [11, 22], the Coriolis parameter Ω served to increase the steepness threshold for the wave by rendering the term on the right-hand side of (4.6) slightly larger than 1/3. Here, the Coriolis effects decrease the threshold slightly.

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