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## A proof, a consequence and an application of Boole’s combinatorial identity

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ABSTRACT. Boole’s combinatorial identity is proved, and a consequence of it for analytic functions is derived that is used to evaluate a sequence of integrals in terms of Euler’s secant sequence of integers.

### 1. BOOLE’S IDENTITY

This features early on in [2], (cf. equation (6) on page 20) and states that if  $n$  is a nonnegative integer, then

$$\sum_{k=0}^n \binom{n}{k} (-1)^{n-k} k^n = n!. \quad (1)$$

In addition, if  $n \geq 1$ , and  $m$  is any nonnegative integer less than  $n$ , then

$$\sum_{k=0}^n \binom{n}{k} (-1)^{n-k} k^m = 0. \quad (2)$$

Both of these statements have many proofs; consult [1], and the references cited therein.

Here’s an outline of a combined proof of (1) and (2):

*Proof.* Write

$$\sigma_n(m) = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} k^m = n! \sum_{k=0}^n \frac{(-1)^k (n-k)^m}{k! (n-k)!}, \quad m, n = 0, 1, 2, \dots$$

Fix  $m$ , and observe that the sequence  $\{\sigma_n(m)/n!, n = 0, 1, \dots\}$  is the convolution of the sequences  $\{(-1)^n/n!, n = 0, 1, \dots\}$ , and  $\{n^m/n!, n = 0, 1, \dots\}$ . Hence

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\sigma_n(m)}{n!} z^n &= \sum_{n=0}^{\infty} z^n \sum_{k=0}^n \frac{(-1)^k (n-k)^m}{k! (n-k)!} \\ &= \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^n \right) \left( \sum_{n=0}^{\infty} \frac{n^m}{n!} z^n \right) \\ &= e^{-z} W_m(z), \end{aligned}$$

where

$$W_m(z) = \sum_{n=0}^{\infty} \frac{n^m z^n}{n!} = \Theta^m e^z,$$

$\Theta$  standing for the differential operator  $z \frac{d}{dz}$ , much used by Boole in his treatment of linear differential equations with variable coefficients.

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Clearly,  $W_0(z) = e^z$ ,  $W_1(z) = ze^z$ , and the following recurrence relation holds:

$$W_{m+1}(z) = zW'_m(z) + W_m(z), \quad m = 0, 1, \dots,$$

where the prime denotes differentiation. So,  $W_m(z)$  is a monic polynomial  $p_m(z)$  times  $e^z$ , and  $\deg p_m = m$ , which is easy to see by induction. Hence,

$$\sum_{n=0}^{\infty} \frac{\sigma_n(m)}{n!} z^n = p_m(z),$$

from which it follows immediately that  $\sigma_n(m) = 0, \forall n > m$  and  $\sigma_n(n) = n!$ . Thus (1) and (2) are true.  $\square$

## 2. A SIMPLE CONSEQUENCE

Suppose  $f$  is analytic on a disc  $D$  centred at 0 in the complex plane. Then, for any nonnegative integer  $n$ ,

$$\lim_{x \rightarrow 0} \frac{1}{x^n} \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} f(kx) = f^{(n)}(0). \quad (3)$$

*Proof.* Let

$$F(x) = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} f(kx), \quad \forall x \in \frac{1}{n}D.$$

Clearly,  $F$  is analytic on a subdisc of  $D$  centred at 0, on which

$$F^{(m)}(x) = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} k^m f^{(m)}(kx).$$

In particular, it follows from (2) that

$$F^{(m)}(0) = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} k^m f^{(m)}(0) = 0, \quad m = 0, 1, \dots, n-1, \quad (4)$$

and from (1) that

$$F^{(n)}(0) = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} k^n f^{(n)}(0) = n! f^{(n)}(0). \quad (5)$$

Therefore, by integrating by parts multiple times, and applying (4) repeatedly,

$$F(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} F^{(n)}(t) dt = \frac{x^n}{(n-1)!} \int_0^1 (1-s)^{n-1} F^{(n)}(xs) ds.$$

Hence

$$F(x) - x^n \frac{F^{(n)}(0)}{n!} = \frac{x^n}{(n-1)!} \int_0^1 (1-s)^{n-1} [F^{(n)}(xs) - F^{(n)}(0)] ds.$$

Let  $\epsilon > 0$ . By hypothesis, there exists  $\delta > 0$  such that  $|F^{(n)}(z) - F^{(n)}(0)| < \epsilon$  whenever  $|z| < \delta$ , and so  $|F^{(n)}(xs) - F^{(n)}(0)| < \epsilon$  whenever  $|x| < \delta$  and  $0 \leq s \leq 1$ . Consequently, if  $0 < |x| < \delta$ ,

$$\begin{aligned} \left| \frac{F(x)}{x^n} - \frac{F^{(n)}(0)}{n!} \right| &\leq \frac{1}{(n-1)!} \int_0^1 (1-s)^{n-1} |F^{(n)}(xs) - F^{(n)}(0)| ds \\ &\leq \frac{\epsilon}{(n-1)!} \int_0^1 (1-s)^{n-1} ds \\ &= \frac{\epsilon}{n!}. \end{aligned}$$

In other words,

$$\lim_{x \rightarrow 0} \frac{F(x)}{x^n} = f^{(n)}(0),$$

by (5), as claimed.  $\square$

In particular, if  $f$  has a power series expansion about 0 so that, for some  $r > 0$ ,

$$f(x) = \sum_{m=0}^{\infty} a_m x^m, \quad \forall |x| < r,$$

then

$$\lim_{x \rightarrow 0} \frac{1}{x^n} \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} f(kx) = n! a_n$$

by (3).

### 3. AN APPLICATION

Consider the sequence of integrals

$$I_n = \int_0^{\infty} \frac{(\ln(x))^n}{1+x^2} dx, \quad n = 0, 1, 2, \dots$$

It's familiar that  $I_0 = \pi/2$ , and clear that

$$\begin{aligned} I_n &= \int_0^1 \frac{(\ln(x))^n}{1+x^2} dx + \int_1^{\infty} \frac{(\ln(x))^n}{1+x^2} dx \\ &= \int_0^1 \frac{(\ln(x))^n}{1+x^2} dx + \int_0^1 \frac{(\ln(\frac{1}{x}))^n}{1+x^2} dx \\ &= (1 + (-1)^n) \int_0^1 \frac{(\ln(x))^n}{1+x^2} dx. \end{aligned}$$

Hence,  $I_{2n+1} = 0$ ,  $n = 0, 1, 2, \dots$ . It's an exercise on page 134 in [3] (Titchmarsh's Theory of Functions) that  $I_2 = \pi^3/8$ , while the computer package MAPLE spews out values of  $I_{2n}$  for  $n = 2, 3, 4, 5, 6$ , according to which

$$I_4 = \frac{5\pi^5}{2^5}, I_6 = \frac{61\pi^7}{2^7}, I_8 = \frac{1385\pi^9}{2^9}, I_{10} = \frac{50521\pi^{11}}{2^{11}}, I_{12} = \frac{13936098\pi^{13}}{2^{13}}.$$

The numbers 1, 5, 61, 1385, 50521, 139360981 are the first six terms of the integer sequence named Euler's secant sequence, and numbered A000364 in [4] (Sloane's online encyclopedia of integer sequences). If  $a(n)$  denotes the  $n$ th term of this sequence, it's tempting to conjecture that

$$I_{2n} = \frac{a(n)\pi^{2n+1}}{2^{2n+1}}, \quad n = 0, 1, 2, \dots$$

One way to confirm this is as follows.

*Proof.* Recall that, for  $x > 0$ ,  $\ln x$  is the limit of the decreasing sequence,  $m(\sqrt[m]{x} - 1)$ ,  $m = 1, 2, \dots$ . Hence

$$\begin{aligned} I_n &= \lim_{m \rightarrow \infty} m^n \int_0^{\infty} \frac{(x^{1/m} - 1)^n}{1+x^2} dx \\ &= \lim_{m \rightarrow \infty} m^n \int_0^{\infty} \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \frac{x^{k/m}}{1+x^2} dx \\ &= \lim_{m \rightarrow \infty} m^n \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} J(k/m), \end{aligned}$$

where, for  $|\Re\alpha| < 1$ ,

$$J(\alpha) = \int_0^\infty \frac{x^\alpha}{1+x^2} dx = \frac{\pi}{2} \sec\left(\frac{\pi\alpha}{2}\right).$$

Since  $\sec$  admits of a power series expansion about 0 of the form

$$\sec x = \sum_{n=0}^{\infty} \frac{a(n)}{(2n)!} x^{2n},$$

that is valid for all  $|x| < \pi/2$ , it follows that

$$\begin{aligned} I_n &= \frac{\pi}{2} \lim_{m \rightarrow \infty} m^n \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \sec\left(\frac{k\pi m}{2n}\right) \\ &= \frac{\pi^{2n+1}}{2^{2n+1}} \sec^{(n)}(0), \end{aligned}$$

by (3), and so, in particular,  $I_{2n+1} = 0$ ,  $n = 0, 1, \dots$ , as we noted above, and

$$I_{2n} = \frac{a(n)\pi^{2n+1}}{2^{2n+1}},$$

as desired. □

**Remark 3.1.** The connection between the values of the sequence  $I_n$  of integrals, and terms of the sequence A000364, doesn't appear to have been noticed before.

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