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PREDICTION OF THE FREE-SURFACE ELEVATION FOR ROTATIONAL WATER WAVES USING THE RECOVERY OF PRESSURE AT THE BED

DAVID HENRY AND GARETH P. THOMAS

Abstract. This paper considers the pressure–streamfunction relationship for a train of regular water waves propagating on a steady current, which may possess an arbitrary distribution of vorticity, in two dimensions. The application of such work is to both nearshore and offshore environments, and in particular for linear waves we provide a description of the role which the pressure function on the sea-bed plays in determining the free-surface profile elevation. Our approach is shown to provide a good approximation for a range of current conditions.

1. Introduction

Determining the relationship between the pressure distribution and the wave surface profile is a topic which is both fascinating from a theoretical viewpoint, while being of the utmost importance in practical considerations. Measuring the surface of water waves directly is extremely difficult and costly, particularly in the ocean. A commonly employed alternative is to calculate the free-surface profile of water waves by way of the so-called pressure transfer function [1, 17, 26, 38], which recovers the free-surface elevation using measurements from submerged pressure transducers, which are most conveniently located on the sea-bed. The key to the success of this approach is in the derivation of a suitable candidate for the pressure transfer function, an issue which is the subject of a large body of experimental and theoretical research; this is outlined and reviewed below. Most theoretical studies, to this point, have focussed exclusively on irrotational travelling water waves, with much of this work primarily in the linear setting.

The dearth of literature, detailing the role that the pressure function plays in flows with vorticity, is not surprising. From a theoretical perspective, chief among the reasons for this deficit are the severe mathematical complications inherent in rotational flows [8, 13]. As an illustration of this, we note that while it has been rigorously proven that the profile-recovery problem is well-posed for solitary water waves with general vorticity distributions [20], no such result currently exists for periodic water waves. Additionally, the paucity of mathematical research analysing the pressure function in its own right no doubt stems from the fact that, for a perfect (incompressible and inviscid) and homogeneous fluid, the pressure distribution function serves primarily as a Lagrange multiplier maintaining the divergence-free

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constraint on the velocity field of the flow. Nevertheless, although it plays an apparently indirect mathematical role in prescribing the fluid kinematics, it is noteworthy that direct, rigorous mathematical analyses of the pressure distribution function itself have recently gleaned rich and detailed qualitative information concerning the fluid motion underlying various irrotational water waves prescribed by the fully nonlinear, exact governing equations [14, 19, 27]. Physically, this is not surprising since the pressure appears as the diagonal term in the stress tensor and is the driving force in the equation of motion.

However, converse to these mathematical difficulties, it is well known that flows with vorticity are relevant in a physical context, for instance being vital in the modelling of wave-current interactions [31, 35–37]. This is particularly pertinent when considering that pressure sensors are frequently located on the sea-bed, a region where currents are ubiquitous (accounting for sediment transport, for instance). Furthermore, a detailed knowledge of the pressure function is important in understanding the wave-induced force-loading experienced by various offshore maritime structures; admitting vorticity in the fluid enables us to incorporate the effects of currents, and wave-current interactions [37].

Compelled by these physical considerations, the aim of this paper is to present a systematic analysis of the role that the pressure distribution, and in particular the dynamic pressure, plays in the kinematics of rotational flows, with the primary focus being how the pressure at the bed prescribes the surface profile. We derive a new pressure-streamfunction reformulation of the governing equations, which is particularly amenable to determining the connection between the pressure function and the fluid kinematics, and obtain explicit relations by way of series solutions for the pressure, streamfunction and the vorticity distribution. In this context, we remark that interesting experimental and numerical work examining various aspects of the relationship between the dynamic pressure and fluid kinematics can be found in [2, 32, 34]. However, it is noteworthy that all theoretical and modelling considerations contained therein apply solely to irrotational flows, a shortcoming that this study aims to go some way to addressing.

The layout of the paper is as follows: in Section 2 the governing equations are presented, while in Section 3 we review previous approaches for addressing the surface-profile recovery problem, both for linear and nonlinear waves, which have almost universally focussed on irrotational flows. Section 4 offers a novel pressure-streamfunction reformulation of the fully-nonlinear, exact governing equations for flows with general vorticity distributions. Regular solutions expressed as a series of harmonic expansions are sought, and the appropriate relations for the series hierarchy are derived here (and in Appendix A.1). In Section 5 an analysis of first-order wavelike solutions will be presented for flows with arbitrary vorticity, leading to the derivation of expressions for the pressure transfer function (5.8) and the pressure amplification factor (5.9). Further implementation of a moderate current approximation renders these expressions more tractable, in the process leading to the elegant and explicit formulae (5.22) and (5.23).

2. Governing equations

For a perfect fluid, that is, inviscid and incompressible, the equations of motion take the form of the Euler equation

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\, \vec{u} = -\frac{\nabla p}{\rho} - g\hat{\vec{z}},$$

and the incompressiblity equation

$$\nabla \cdot \vec{u} = 0. \tag{2.1a}$$

Here $\vec{u}$ denotes the velocity field, p represents the pressure distribution, ρ is the fluid density (hereupon assumed to be constant) and g is the standard gravitational constant of acceleration. In this paper our focus is restricted to two-dimensional fluid motion, defining a local Cartesian coordinate system in which the $x-$axis is horizontal and the $z-$axis points vertically upwards (with $\hat{\vec{z}}$ the unit vector in this direction), and furthermore spatial periodicity is assumed with respect to the $x-$variable. The origin O is chosen to lie in the mean water level: if $z = \eta(x,t)$ represents the unknown free-surface, and $\lambda > 0$ is the characteristic wavelength, we must have

$$\int_0^\lambda \eta(x,t)dx = 0.$$

The choice of reference frame implies that $z = -h$ denotes the location of the impermeable flat bed, which is assumed to be locally horizontal and where h is the mean water depth. The Euler equation expressed component-wise in terms of $\vec{u} = (u(x,z,t), 0, w(x,z,t))$ takes the form

$$\begin{aligned} u_t + uu_x + wu_z &= -\frac{p_x}{\rho}, \\ w_t + uw_x + ww_z &= -\frac{p_z}{\rho} - g. \end{aligned} \tag{2.1b}$$

On the flat bed the kinematic boundary condition gives

$$w = 0 \qquad \text{on } z = -h, \tag{2.1c}$$

and at the free-surface the kinematic and dynamic conditions take the form

$$w = \eta_t + u\eta_x, \tag{2.1d}$$

$$p = \text{constant} \qquad \text{on } z = \eta(x,t). \tag{2.1e}$$

Typically, the constant atmospheric pressure, denoted p_a, is taken as the free-surface value in (2.1e) and the pressure can be written as

$$p = p_a - \rho g z + p_d(x,z,t), \tag{2.2}$$

where p_d denotes the dynamic pressure, measuring the deviation from the hydrostatic pressure. The nonlinear free-boundary value problem specified by the system of equations (2.1) represents the full governing equations for water waves in two dimensions. The vorticity prescribed by fluid motion is defined as the curl of the fluid velocity field, $\mathbf{\Omega} = \nabla \times \vec{u}$, which reduces in two-dimensions to $\mathbf{\Omega} = (0, \Omega, 0)$, leading to the scalar vorticity equation

$$\Omega(x, z, t) = u_z - w_x. \tag{2.3}$$

3. Profile recovery from pressure measurements: a brief review

3.1. Irrotational flow.

3.1.1. *Linear waves.* The most elementary approach to estimating the free-surface from pressure measurements is to invoke the hydrostatic approximation

$$\eta(x, t) = \frac{p_d(x, -h, t)}{\rho g}.$$

This provides a simplistic shallow water wave approximation which nevertheless is employed in the field in certain instances, an example being open-ocean buoys for tsunami detection. In order to incorporate the effects of wave motion, the following approach may be taken. In the particular setting of an irrotational fluid ($\Omega \equiv 0$) it follows from (2.3) that a potential $\phi(x, z, t)$ may be defined for the velocity field and the governing equations (2.1) recast in terms of this, thereby reducing the number of unknowns in the problem. Physically, irrotationality corresponds either to waves entering a fluid region, which is initially motionless (and hence irrotationality persists as a result of Kelvin's Circulation Theorem), or to a region where waves propagate on the surface of a constant, uniform current. Initially it is assumed that there is no underlying current. Following a standard linearisation procedure [8, 37] and choosing the ansatz

$$\eta(x, t) = a \cos(kx - \omega t) \tag{3.1}$$

for a regular linear wave solution, the linearised form of (2.1) may be solved for waves satisfying the irrotationality condition leading to

$$u = a\omega \frac{\cosh k(z + h)}{\sinh kh} \cos(kx - \omega t), \qquad w = a\omega \frac{\sinh k(z + h)}{\sinh kh} \sin(kx - \omega t), \tag{3.2}$$

and

$$p_d(x, z, t) = \rho a g \frac{\cosh k(z + h)}{\cosh kh} \cos(kx - \omega t). \tag{3.3}$$

Here a is the wave amplitude, $k = 2\pi/\lambda$ is the wavenumber and ω is the phase frequency; these quantities are not independent but are related by $c = \omega/k$, where c is the wave phasespeed. In the present setting of linear waves over a flat bed it may be shown that the dispersion relation $\omega^2 = gk \tanh kh$ holds, leading to the prescription of the wave phase-speed as

$$c = c(k) = \sqrt{g \tanh kh/k}. \tag{3.4}$$

Comparing (3.1) and (3.3) leads to the so-called transfer function formula

$$\eta(x, t) = \frac{p_d(x, z, t)}{\rho g K_p(z)} \tag{3.5}$$

where $K_p(z) = \cosh k(z+h)/\cosh kh$ is the so-called pressure response factor. The effectiveness of the transfer function formula (3.5) has been widely tested in the engineering literature, through field data and experiments: cf. [4] for an interesting comparison and contrast between a number of different data-sets, for both regular and irregular waves. At this point we remark that while some interesting experimental and numerical analyses of the pressure distribution have been undertaken for irregular waves in the literature, for example in [2, 4, 29], this paper will be focussed primarily on regular waves.

Among the issues considered in [4] is the need to offset any potential inaccuracies between theory and observations by multiplying the right-hand side of (3.5) with an empirical correction factor N, a constant which may vary depending on the local environment to which the particular data set relates. The authors outline the wide-differences of opinion which prevail regarding the necessity for, and possible behaviour of, this empirical correction factor. They conclude that such a factor is probably unnecessary — that is, it is reasonable to choose $N = 1$ — with any perceived discrepancies between theory and observation being accounted for by issues such as inaccurate measurements, instrument limitations, and analysis methods. Furthermore, they assert that "linear theory is adequate to compensate pressure data and give reliable estimates of surface wave heights" for most of the wave amplitudes considered.

We note that further experimental studies of interest regarding the transfer function are found in [26], where the authors consider a purely empirical expression for the transfer function which is derived through dimensional-analysis considerations; it is shown in [1] that this formula may be regarded as a version of the theoretically derived formula (3.5). Further analysis of these issues was undertaken in the more recent paper [38].

Two particular topics addressed throughout these studies are of relevance to the considerations of the current paper. Firstly, one source of speculation for possible discrepancies between theory and observation in the pressure transfer function (3.5) is the presence of depth-varying currents, for which the assumption of irrotationality is invalid and vorticity much be included in the fluid model. For instance, following their review of various field data and laboratory experiments, Bishop & Donelan [4] conclude that "when measuring waves with pressure transducers in shallow water the linear theory pressure response factor may require modification to account for currents". This aspiration is achieved below in the formulae (5.8) and (5.22). Secondly, much experimental work has been undertaken (cf. [4]) to establish if the transfer function is sensitive to the relative-depth at which the pressure transducers are located: we can see that the formula (3.5) for irrotational waves applies regardless of the depth at which the pressure is measured, and in particular pressure sensors need not be located on the sea-bed. The formulae that we derive below for waves with vorticity — (5.8), (5.9) and (5.22), (5.23) respectively — are similarly unrestricted with regard to the exact location of the pressure measurements, and it is a reference pressure level that is required.

Additional literature of relevance in the context of this paper are the experimental and numerical investigations found in [2, 32, 34], which analyse the relationship between the dynamic pressure and aspects of the fluid kinematics. These papers suffer from the common restriction that all theoretical and modelling considerations contained therein apply solely to irrotational flows. In this paper we go some way towards addressing this shortcoming, whereby we derive new relations between the dynamic pressure and the fluid kinematics which allow for the effects of vorticity, at the level of both linear and nonlinear waves.

We conclude by noting that a physical quantity which is of practical significance is an amplification factor Q, relating the dynamic pressure at the free surface to the dynamic pressure at the bed: expressing $p_d(x,z,t) = P_d(z)\cos(kx-\omega t)$, we have $Q = P_d(0)/P_d(-h)$. This ratio is an important quantity since the magnitude of Q may influence the acceptability of the predictive process; in the current setting of irrotational linear waves, with pressure prescribed by (2.2), we find

$$Q = \cosh kh, \tag{3.6}$$

and setting $z = -h$ in (3.5) leads to

$$\eta(x,t) = \frac{Q}{\rho g} p_d(x,-h,t).$$

The notion of an amplification factor may be extended in order to relate the dynamic pressure at the surface to the dynamic pressure of the fluid at any arbitrary depth $-h \leq z < 0$, through defining

$$Q(z) = \frac{P_d(0)}{P_d(-z)}. \tag{3.7}$$

We deduce immediately from (3.5) that, for linear irrotational water waves, we have

$$\eta(x,t) = \frac{Q(z)}{\rho g} p_d(x,z,t), \tag{3.8}$$

and so $Q(z) = 1/K_p(z)$. In our investigation of the surface–profile recovery problem for waves with vorticity we will find it more expedient to work directly with the amplification factor $Q(z)$ rather than the pressure-transfer function $K_p(z)$, although it is clear that these two entities are closely related and, as alluded to previously, the formulae derived below for the amplification factor for waves with vorticity will exhibit freedom with regard to the exact location of the pressure measurements.

The considerations involved in the derivation of (3.1)–(3.4), and the subsequent pressure recovery formulae, may be readily adapted to accommodate an irrotational velocity field of the form $(u+U,0,w)$, where $U(z) \equiv U$ is a constant underlying current, cf. [31, 34, 37]. In so doing, the form of the trigonometric terms in the wave velocity field $(u,0,w)$ matches those given by formula (3.2), however wherever the 'absolute' wave frequency ω appears as a coefficient to the trigonometric functions in (3.2) it must be directly replaced by the 'relative' wave frequency $\sigma := \omega - kU$: the presence of a constant underlying current amounts to a form of Doppler shift in the wave motion. These modifications have a number of interesting ramifications, not least that it may be discerned directly from the modified dispersion relation $\sigma^2 = gk \tanh kh$ as to which wave motions, depending on the magnitude and direction (adverse or following) of the current, are admissible, cf. Peregrine [31].

3.1.2. *Nonlinear travelling waves.* While the extant literature, particularly in an engineering context, focusses predominantly on the linear wave regime, it has been observed [4] that for very large amplitude, or steep, waves it is important to take into account the effects of nonlinearity. In the setting of nonlinear waves of permanent form travelling with constant speed c, a number of recent theoretical investigations have been undertaken which have successfully extracted exact surface profile recovery formulae from the pressure measurements on the flat bed for exact nonlinear water waves. Subsequent to a theoretical and experimental study of the pressure function for finite amplitude solitary waves [12], Constantin obtained an explicit parametric recovery formula for solitary waves in [9] which may be expressed (in the notation of the current paper) as

$$\begin{aligned} x(q) &= q + \int_{-\infty}^{q} \mathcal{F}^{-1}\left\{ \cosh rh \mathcal{F}\left[\frac{c}{\sqrt{c^2 - 2\mathfrak{p}}} - 1 \right](r) \right\}(s) ds, \\ \eta(q) &= \mathcal{F}^{-1}\left\{ \frac{\sinh rh}{r} \mathcal{F}\left[\frac{c}{\sqrt{c^2 - 2\mathfrak{p}}} - 1 \right](r) \right\}(q), \end{aligned} \tag{3.9}$$

where q is a parameter, $\mathfrak{p} = p_d(x, -h, t)$ is the dynamic pressure evaluated at the flat bed, all variables are in a reference frame moving with wavespeed c and $\mathcal{F}, \mathcal{F}^{-1}$ denotes the Fourier transform and its inverse, respectively. A key step in the derivation of this formula involves transforming the governing equations for the free-bounary problem (2.1) to a fixed domain by way of a conformal hodograph change of variables $(x, z) \mapsto (q, p) := \left(-\frac{1}{c}\phi(x, z), -\frac{1}{c}\Psi(x, z)\right)$, where ϕ is the velocity potential and Ψ is the streamfunction (defined below in (4.1)).

For periodic waves, it was shown in [25] that a similar approach, employing the same conformal hodograph transformation, may be adapted to yield an explicit parametric recovery formula analogous to (3.9). An interesting alternative to the conformal mapping approach was provided by Clamond & Constantin [7] where the authors, in this instance working directly with the physical variables, establish that standard iterative procedures determine the wave amplitude η_0 as the unique fixed point of the mapping

$$s \mapsto (\mathrm{Re}\mathcal{P})(0, s)/g - d,$$

where $\mathcal{P}$ is a holomorphic function whose restriction to the flat bed has a zero imaginary part, and a real part equal to the normalised pressure relative to the atmosphere. Subsequently, the wave surface is prescribed as the unique solution of the ordinary differential equation

$$\eta_x(x) = \frac{\mathrm{Im}\mathcal{P}(x, \eta)}{B - g\eta - (\mathrm{Re}\mathcal{P})(x, \eta) + gd}$$

with the initial data $\eta(0) = \eta_0$ and where $B > 0$ is the Bernoulli constant. This approach is particularly useful for numerical considerations, as is illustrated by the authors with various examples. Numerical considerations motivate an alternative, implicit reconstruction approach provided in [6] by Clamond. Finally, we note that an approach providing reconstruction formulae for exact nonlinear waves by way of a nonlocal reformulation of the governing equations is presented in [30]. While these formulae have the drawback of being implicit, and somewhat more entangled than (3.9), this paper features some interesting experimental data for solitary waves. Estimates of bounds on the wave height, which depend on pressure measurements from the flat bed, are provided in [3, 10] for travelling gravity water waves without limitation on the wave amplitudes. We note that the theoretical considerations outlined above are all strongly contingent on the flow being irrotational.

3.2. **Rotational flows.** As outlined in the introduction, there is a striking paucity of results concerning recovery formulae for flows with vorticity; to the best of our knowledge, even in the linear setting a formula analogous to (3.5) has not been derived for rotational water waves until now (cf. Section 5). This is primarily due to the technical complications that waves with vorticity pose towards mathematical analysis. Some theoretical considerations which have been established are worthy of mention however. For a current possessing constant vorticity, whereby the wavefield is irrotational, an approach similar to [30] has been implemented in [40] in order to derive a nonlocal reformulation of the governing equations. In doing so, the authors establish certain relations between the surface profile and the pressure in the fluid. A recent paper [5] has attempted to include currents with arbitrary vorticity but does not progress beyond the stage of formulation. For general, analytic vorticity distributions it was proven in [20] that the profile-recovery problem is well-posed for solitary waves of any magnitude, in the sense that the wave surface profile of a solitary wave with vorticity, described by the fully nonlinear exact governing equations (2.1), is uniquely determined by the pressure function on the flat bed. We note that, at this point, no similar result has been rigorously proven for periodic waves.

4. General pressure-streamfunction formulation

In this section we present a new pressure–streamfunction formulation of the governing equations (2.1), which is particularly amenable to our analysis of the relationship between the dynamic pressure and the wave-field kinematics. For waves interacting with non-uniform currents the fluid possesses non-zero vorticity [31, 37] and, in general, a velocity potential does not exist. Alternatively, and as a consequence of (2.1a), define the stream function $\Psi(x, z, t)$ by

$$u = \frac{\partial \Psi}{\partial z}, \quad w = -\frac{\partial \Psi}{\partial x}, \tag{4.1}$$

and it follows immediately from (2.3) and (4.1) that

$$\nabla^2 \Psi = \Omega, \tag{4.2}$$

where ∇^2 represents the (two-dimensional) Laplacian operator. The streamfunction has a natural representation in terms of the (path-independent) line integral

$$\Psi(x,z,t) = \int_{(0,-h)}^{(x,z)} \left[u dz - w dx \right], \tag{4.3}$$

and by choosing first the path from $(0,-h)$ to $(x,-h)$ (where (2.1c) applies and $dz \equiv 0$), followed by the path from $(x,-h)$ to (x,z) (where $dx \equiv 0$), (4.3) reduces to the volume flux measure

$$\Psi(x,z,t) = \int_{-h}^{z} u(x,z,t) dz.$$

It follows immediately that Ψ is itself a periodic function with respect to the $x-$variable. Eliminating p in (2.1b) and using (4.1), we obtain the equation

$$\nabla^2 \Psi_t + \Psi_z \nabla^2 \Psi_x - \Psi_x \nabla^2 \Psi_z = 0. \tag{4.4}$$

This equation is commonly referred to as the Helmholtz Vorticity Equation, particularly when Ω is deployed via (4.2). In this paper attention is focussed on regular wave solutions, by which we mean that the only x,t dependence can occur via a phase function $\theta(x,t) = kx - \omega t$, giving the $x-$ and $t-$derivatives by

$$\frac{\partial}{\partial x} = k \frac{\partial}{\partial \theta}, \qquad \frac{\partial}{\partial t} = -\,\omega \frac{\partial}{\partial \theta}. \tag{4.5}$$

For regular waves the streamfunction has a particularly elegant form, with rich structural properties being easily established upon transforming to a reference frame moving with the constant wave phasespeed, cf. [8,37]. With ω and k assumed to be locally constant, (4.4) can be written as

$$\left(\omega - k\Psi_z\right) \nabla^2 \Psi_\theta + k \Psi_\theta \nabla^2 \Psi_z = 0. \tag{4.6}$$

With respect to applications in water waves, this equation provides the basis for the method introduced by Dean [16] for computing the kinematics of nonlinear waves, albeit specifically for irrotational motion ($\Omega \equiv 0$) when (4.6) reduces to $\nabla^2 \Psi = 0$, matching (4.2). This was extended by Dalrymple [15] for nonlinear waves interacting with a rotational current. Although it may be possible to formulate the governing equations in terms of p or Ψ alone, as in (4.6) for Ψ, it should be noted that the two quantities are in fact related via the dynamic surface boundary condition (2.1e); we remark that although (2.1e) appears quite innocuous, it transforms to a highly complex, nonlinear Bernoulli relation when expressed in terms of the streamfunction.

In the present approach, attention is directed at the pressure as specified by (2.1b), and it is assumed that (4.6) can be solved, as required, even for rotational flow. Returning to (2.1b), and employing (4.1), enables the first equation to be expressed as

$$\frac{1}{\rho} p_x = -\frac{\partial}{\partial t} \Psi_z - \frac{\partial}{\partial x} \left[\frac{1}{2} \left(\Psi_x^2 + \Psi_z^2 \right) - \Psi \nabla^2 \Psi \right] - \Psi \nabla^2 \Psi_x,$$

and the second equation, following some rearrangement, becomes

$$\frac{1}{\rho}p_z = -g + \frac{\partial}{\partial t}\Psi_x - \frac{\partial}{\partial z}\left[\frac{1}{2}\left(\Psi_x^2 + \Psi_z^2\right) - \Psi\nabla^2\Psi\right] - \Psi\nabla^2\Psi_z.$$

Imposing the quest for wavelike solutions onto these two equations by way of (4.5) finally gives the form

$$\begin{aligned}\frac{1}{\rho}\frac{\partial p}{\partial \theta} &= \frac{\partial H}{\partial \theta} - \Psi\,\nabla^2\Psi_\theta,\\ \frac{1}{\rho}\frac{\partial p}{\partial z} &= \frac{\partial H}{\partial z} - \frac{\omega}{k}\nabla^2\Psi - \Psi\,\nabla^2\Psi_z,\\ H(\theta,z) &= -gz + \frac{\omega}{k}\Psi_z - \frac{1}{2}\left\{\Psi_z^2 + k^2\Psi_\theta^2\right\} + \Psi\,\nabla^2\Psi.\end{aligned} \tag{4.7}$$

The function $H(\theta, z)$ here represents a form of integrability in the equations and also contains a contribution from the vorticity $\nabla^2\Psi$. Those terms on the right-hand side of (4.7) not included in $H(\theta, z)$ are strictly rotational terms. Note that (4.7) reduces to (4.6) on the elimination of p, and furthermore that the system can be integrated to a form equivalent to a Bernoulli condition for irrotational flow. Although no link has been established, it may well be that the form of (4.7) plays a role in the variational formulation [22] of the equations of water waves with vorticity.

Writing the terms not included in $H(\theta, z)$ in the two pressure equations of (4.7) as $-F(\theta, z)$ and $-G(\theta, z)$, respectively, enables an integration of the form

$$\begin{aligned}\frac{1}{\rho}p(\theta,z) &= H(\theta,z) - \int^{\theta} F(\theta,z)d\theta - A(z)\\ &= H(\theta,z) - \int^{z} G(\theta,z)dz - B(\theta).\end{aligned} \tag{4.8}$$

Evaluation of each equation should, in principle, enable $A(z)$ and $B(\theta)$ to be determined. In the absence of analytic solutions this will still prove a difficult task. It may appear that the $A(z)$ and $B(\theta)$ terms in (4.8) are unimportant. This is not the case, as together with the $p_0(z)$ term they will describe mean flow contributions from currents and waves.

4.1. **Series solutions.** When analytic or semi-analytic solutions are sought in a pressure-streamfunction formulation, it is usual to represent the streamfunction and the pressure-type terms as

$$\begin{aligned}\Psi(\theta,z) &= \sum_{n=0}^{\infty}\psi_n(z)\cos n\theta,\\ p(\theta,z) &= -\rho g z + \sum_{n=0}^{\infty}p_n(z)\cos n\theta,\end{aligned} \tag{4.9}$$

where p is now the pressure relative to the atmospheric pressure. For convenience, we also write the vorticity $\Omega(=\nabla^2\Psi)$ in the same way,

$$\Omega(\theta,z)=\sum_{n=0}^{\infty}\left[\psi_n''(z)-n^2k^2\psi_n(z)\right]\cos n\theta=\sum_{n=0}^{\infty}\Omega_n(z)\cos n\theta, \tag{4.10}$$

where $'$ indicates differentiation with respect to z. It is known that analytic solutions have been obtained only for the special cases of $\Omega_0=0$ or $\Omega_0=$constant, both corresponding to irrotational wave motions. Also, from (4.7) and (4.9), the hydrostatic component of the pressure is identified explicitly and need not be included in the forthcoming analysis.

With the forms for Ψ, p and Ω from (4.9) and (4.10) utilised, the resulting expressions for $H(\theta,z)$, $F(\theta,z)$ and $G(\theta,z)$ are given in Appendix 1. In particular, from (4.8) we have

$$\int^{\theta}F(\theta,z)d\theta+A(z)=\int^{z}G(\theta,z)dz+B(\theta)$$

and examining (A.7) and (A.8) suggests that

$$A(z)=A_0+\frac{\omega}{k}\int^{z}\Omega_0(z)dz+\int^{z}\psi_0(z)\Omega_0'(z)dz+\frac{1}{2}\sum_{n=1}^{\infty}\left\{\int^{z}\psi_n(z)\Omega_n'(z)dz\right\}, \tag{4.11}$$

where A_0 is some constant that can only be determined by the imposition of appropriate physical conditions. The importance of $A(z)$ is that it is associated with mean-flow quantities and particularly those associated with the vorticity.

4.2. **Hierarchy of series harmonics.** Collecting the terms on the right-hand side of the first equation in (4.8), from (A.6), (A.7) and (4.11), gives the full pressure-streamfunction relationship. With the hydrostatic component omitted, following some manipulation, the final form of the dynamic pressure is

$$\begin{aligned}
\frac{1}{\rho}\sum_{n=0}^{\infty}p_n(z)\cos n\theta &= -A_0-\frac{1}{2}\left(\psi_0'(z)\right)^2+\int^{z}\psi_0'(z)\Omega_0(z)dz\\
&-\frac{1}{4}\sum_{m=1}^{\infty}\left[\left(\psi_m'(z)\right)^2+\left(mk\psi_m(z)\right)^2-2\int^{z}\psi_m'(z)\Omega_m(z)dz\right]\\
&+\sum_{n=1}^{\infty}\left\{\frac{\omega}{k}\psi_n'(z)-\frac{1}{2}\left[\psi_n'(z)\psi_0'(z)-2\psi_n(z)\Omega_0(z)\right]\right\}\cos n\theta \qquad (4.12)\\
&-\frac{1}{4}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left[\psi_n'(z)\psi_m'(z)-nmk^2\psi_n(z)\psi_m(z)-\frac{2n}{m+n}\psi_n(z)\Omega_m(z)\right]\cos(n+m)\theta\\
&-\frac{1}{4}\sum_{n=0}^{\infty}\sum_{\substack{m=1\\ m\neq n}}^{\infty}\left[\psi_n'(z)\psi_m'(z)+nmk^2\psi_n(z)\psi_m(z)+\frac{2n}{m-n}\psi_n(z)\Omega_m(z)\right]\cos(m-n)\theta,
\end{aligned}$$

where the identity $\Omega_0 = \psi_0''$ has been employed. This relation is valid up to constants of integration, which vanish identically since (4.3) implies that $\psi_n(-h) = 0$ for all $n \geq 0$.

The next step is to match the harmonic components on each side. Taking just the $n = 0$ term initially from (4.12) we have

$$\frac{1}{\rho}p_0(z) = -A_0 - \frac{1}{2}\left(\psi_0'(z)\right)^2 + \int^z \psi_0'(z)\Omega_0(z)dz$$
$$-\frac{1}{4}\sum_{m=1}^{\infty}\left[\left(\psi_m'(z)\right)^2 + \left(mk\psi_m(z)\right)^2 - 2\int^z \psi_m'(z)\Omega_m(z)dz\right], \quad (4.13)$$

in which the presence of the vorticity is immediately identifiable.

Three contexts arise in the flow of waves over a horizontal bed and a free-surface, depending on the base vorticity $\Omega_0(z)$. Firstly, if the base vorticity is zero, then $\Omega_0 = 0$ and the motion will be irrotational. This corresponds to $\Omega_n = 0$ for all $n \geq 1$ throughout (4.12) and (4.13), and applies to a constant current and in the absence of an imposed current. The pressure p_0 from (4.13) becomes

$$\frac{1}{\rho}p_0(z) = -A_0 - \frac{1}{2}\left(\psi_0'(z)\right)^2 - \frac{1}{4}\sum_{m=1}^{\infty}\left[\left(\psi_m'(z)\right)^2 + \left(mk\psi_m(z)\right)^2\right].$$

In this setting it is more usual to employ a velocity potential than a streamfunction, via Bernoulli's equation, and the historical context is discussed in [23] by Jonsson & Kofoed-Hansen.

Secondly, if Ω_0 is constant, then the current possesses a linear profile. The wave-field is still irrotational, as shown by Tsao [39], and a velocity potential may be employed as $\Omega_n = 0$ for $n \geq 1$. Analytic solutions are not easy to obtain beyond the first three terms in a Stokes-type approach but there is a unexpected cancellation in (4.10) to give

$$\frac{1}{\rho}p_0(z) = -A_0 - \frac{1}{4}\sum_{m=1}^{\infty}\left[\left(\psi_m'(z)\right)^2 + \left(mk\psi_m(z)\right)^2\right].$$

A numerical method for this case based upon the streamfunction has been given by Dalrymple (1974) in [15].

The third case is when $\Omega_0(z)$ is considered to be arbitrary and is of interest here. Note that from (4.10)

$$\psi_m'(z)\Omega_m(z) = \psi_m'(z)\left[\psi_m''(z) - (mk)^2\psi_m(z)\right] = \frac{1}{2}\left[\left(\psi_m'(z)\right)^2 - \left(mk\psi_m(z)\right)^2\right]',$$

enabling integration in (4.13). If $\Omega_m(z)$ does not vanish, then

$$\frac{1}{\rho}p_0(z) = -A_0 - \frac{1}{2}\sum_{m=1}^{\infty}\left(mk\psi_m(z)\right)^2, \quad (4.14)$$

which is a rather surprising result and can only be interpreted in a discussion concerning A_0. In all three cases, A_0 must be chosen to ensure that $p_0(0) = 0$, bearing in mind the boundary condition (2.1e) and the decompostion (2.2).

The harmonic expansions for the pressure, streamfunction and vorticity in (4.9) and (4.10) do not assume any ordering on the relative magnitudes of the functions of z. With reference to (4.12) for the matching of $\cos q\theta$, for a strictly positive integer q, to enable the determination of $p_q(z)$, an assessment of the individual terms shows a single term ($n = q$), a finite series ($m + n = q$) and an infinite series ($|m - n| = q$). For the practical implementation of these methods for irrotational waves, the series (4.9) is curtailed to be finite and, as discussed in [33] by Rienecker & Fenton (1981), this characterises a numerical method capable of describing nonlinear waves. Nonetheless, it is shown numerically that the magnitudes of the unknown functions decrease in magnitude with increasing q. In the present context, the waves are considered to be weakly nonlinear and it is appropriate to introduce the usual scaling parameter $\epsilon = ak$ (the wave slope) so that $p_q(z)$ is of $\mathcal{O}(\epsilon^q)$. Equivalence with Stokes' waves can be obtained by expanding $\psi_q(z)$ as a series in ϵ,

$$\psi_q(z) = \epsilon^q \left[\psi_{q0}(z) + \epsilon\psi_{q1}(z) + \epsilon^2\psi_{q2}(z) + \ldots\right], \quad q = 1, 2, \ldots,$$

as in equation (B1) of [36].

Before proceeding with arbitrary $\Omega_0(z)$, it is salient to make important comments about the first two cases identified above. It was shown by Ismail (1984) in [21] that, for regular wave-current interactions in two-dimensions, the current can be modelled by a constant and linear profile for following and opposing currents respectively. This was subsequently confirmed by Groeneweg & Klopman (1998) in [18]. A consequence is that for two-dimensional applications, and provided that the current is not generated by an external mechanism, then it may be sufficient in the first instance to employ irrotational wave models. Such an approach has obvious advantages if higher order solutions are required.

5. First order solutions

The first order terms in (4.9) capture linear wave motion and wave-current interactions. If the current takes the form $(U(z), 0, 0)$, then from (4.1) and (4.9), $\psi_0(z)$ can be easily determined to be

$$\psi_0(z) = \int_{-h}^{z} U(z)dz. \tag{5.1}$$

At first order, with the assumption that $\psi_q(z)$ is of $\mathcal{O}(\epsilon^q)$, the combination of (4.6) and (4.9) yield the following equation for $\psi_1(z)$:

$$\psi_1''(z) - \left(k^2 - \frac{kU''(z)}{\omega - kU(z)}\right)\psi_1(z) = 0. \tag{5.2}$$

This is the Rayleigh equation of hydrodynamic stability theory, or the inviscid form of the Orr-Sommerfeld equation. An alternative formulation to (5.2), expressed in terms of $p_1(z)$ rather than $\psi_1(z)$, may be employed for linear solutions and this is described in Appendix 2. For regular waves over a horizontal bed with mean water depth h, the appropriate boundary conditions at the bed $z=-h$ and surface $z=0$ are

$$\begin{aligned} &(c-U(0))\psi_1''(0)+[(c-U(0))\,U'(0)-g]\,\psi_1(0)=0, && \\ &\psi_1(0)=a\,(c-U(0)) && \text{on } z=0, \\ &\psi_1(-h)=0 && \text{on } z=-h, \end{aligned} \tag{5.3}$$

where a is the wave amplitude, defined to be the magnitude of the first harmonic of the surface elevation series, and $c=\omega/k$ is the phase velocity. In the derivation of the surface conditions the usual assumptions of linearity apply.

The first harmonic of dynamic pressure $p_1(z)$ is obtained from (4.12), with the appropriate ordering implemented and (5.1) utilised, resulting in

$$\frac{p_1(z)}{\rho}=\left(\frac{\omega}{k}-U(z)\right)\psi_1'(z)+U'(z)\psi_1(z). \tag{5.4}$$

Bearing in mind (4.9), the decomposition (2.2) and the standard linear wave profile (3.1), $p_1(z)$ must also satisfy

$$p_1(0)=\rho g a \tag{5.5}$$

to ensure that condition (2.1e) holds. A noteworthy consequence of (5.5) is that $p_n(0)=0$ for the higher order ($n\geq 2$) dynamic pressure terms. Thus, if $\psi_1(z)$ can be determined from (5.2) and (5.3), then $p_1(z)$ can be obtained from (5.4).

For the pressure recovery problem, the aim is to measure the dynamic bed-pressure $p_b=p_1(-h)$ and use this to determine the surface amplitude a. In this formulation, the opposite approach is taken, namely that a is used to determine $\psi_1(z)$ and hence the pressure via (5.4). However, $\psi_1(z)$ is linearly proportional to a, as can be seen from (5.2) and (5.3), so we can write

$$\psi_1(z)=a\chi_1(z), \tag{5.6}$$

and from (5.4),

$$p_b=p_1(-h)=\rho\,[c-U(-h)]\,\psi_1'(-h)=\rho a\,[c-U(-h)]\,\chi_1'(-h), \tag{5.7}$$

as $\psi_0(-h)=\psi_1(-h)=0$ from (5.1) and (5.3). Thus if p_b is the known quantity, then

$$a=\frac{p_b}{\rho\,[c-U(-h)]\,\chi_1'(-h)}. \tag{5.8}$$

The amplification factor $Q(-h)$, defined in (3.7), relating the pressure at the free surface to the pressure at the bed now follows directly from (5.5) and (5.7):

$$Q(-h)=\frac{g}{[c-U(-h)]\,\chi_1'(-h)}. \tag{5.9}$$

This formula is the generalisation of (3.6) to the setting of water waves with general vorticity distributions; formula (3.6) for irrotational waves is recovered immediately upon setting $U(z)\equiv U$, where U is a constant underlying current. This is easily verified by taking $U=0$, using the dispersion relation (3.4) for the wave phase speed c and (3.2) for determining $\chi_1'(z)$. Furthermore, it is clear from our approach that we are not restricted to working at the flat bed ($z = -h$) in taking pressure measurements, although matters are slightly more complicated otherwise; indeed, we may choose instead to derive $Q(z) = p_1(0)/p_1(z)$, working again with (5.5) and inputting an arbitrary depth $-h \leq z < 0$ in formula (5.4), leading to a generalisation of (5.7) and (5.9).

The drawback of the described approach of using (5.4) in analysing the relationship between the dynamic pressure and the wave-field kinematics is that, for arbitrary $U(z)$, (5.2) and (5.3) can only be solved numerically as described by Thomas in [35], and for this reason approximate solutions are required in general. Nevertheless, in the special cases of a uniform underlying current $U(z) = U_c$ (zero vorticity), and also for constant vorticity distributions (representing a linearly–sheared current profile $U(z) = U_s + \Omega z$), solutions analogous to (3.2) and (3.4) can be derived, cf. Thomas & Klopman [37], in which case the recovery formula (5.9) is applicable. Accordingly, it is worthy of mention that many current profiles can be approximated by a number of linear components and with appropriate matching conditions applied at the interfaces.

5.1. **The Moderate Current Approximation.** Although numerical solutions are not difficult to obtain at this order, the lack of an analytic solution prevents simple insights to be gained. For this reason a perturbation approach is developed for weakly nonlinear waves, consistent with Stokes waves for irrotational wave motion and with the pressure–streamfunction formulation developed in §4. In contrast to most procedures in water waves, two non-dimensional perturbation parameters are employed — the wave slope ε, already utilised above, and δ as a measure of the current strength relative to the phase speed of the waves: typically $\delta = \hat{U}/c$, with $\hat{U}$ a characteristic current measure. The presence of δ is formally recognised by writing

$$U(z) = \delta V(z), \tag{5.10}$$

so that $V(z)$ has the same dimensions as the current but is of comparable magnitude to the wave phase speed c.

Perturbation solutions for the streamfunction Ψ, surface elevation η and pressure p are sought of the form

$$\begin{aligned}
\Psi &= \Psi_{00} + \delta\,\Psi_{01} + \delta^2\,\Psi_{02} + \cdots + \varepsilon\,(\Psi_{10} + \delta\,\Psi_{11} + \cdots) + \varepsilon^2\,(\Psi_{20} + \delta\,\Psi_{21} + \cdots) + \ldots \\
\eta &= \eta_{00} + \delta\,\eta_{01} + \delta^2\,\eta_{02} + \cdots + \varepsilon\,(\eta_{10} + \delta\,\eta_{11} + \cdots) + \varepsilon^2\,(\eta_{20} + \delta\,\eta_{21} + \cdots) + \ldots \\
p &= -\rho g z + p_{00} + \delta\,p_{01} + \delta^2 p_{02} + \cdots + \varepsilon\,(p_{10} + \delta\,p_{11} + \cdots) + \varepsilon^2\,(p_{20} + \delta\,p_{21} + \cdots) + \ldots
\end{aligned} \tag{5.11}$$

An intuitive interpretation of the representation defined by (5.11) is that the formal setting $\delta = 0$ removes the imposed current but permits mean flows associated with the waves, depending upon the choice of reference frame. Similarly the setting $\varepsilon = 0$ corresponds to the case of a current alone in the absence of waves and mixed terms of $\mathrm{O}(\varepsilon^i\delta^j)$ describe the interaction terms. To maintain consistency with established practice in Stokes wave theory, it is also necessary to expand the frequency ω of the waves (or phase speed c) in a similar manner,

$$\omega = \omega_{00} + \delta\omega_{01} + \delta^2\omega_{02} + \cdots + \varepsilon\,(\omega_{10} + \delta\omega_{11} + \cdots) + \ldots, \tag{5.12}$$

and this is only meaningful when ε is non-zero.

The scheme in (5.11) and (5.12) requires the imposition of a relative ordering upon ε and δ. Thomas & Klopman [37] proposed a classification scheme of three regimes based upon the relative magnitude of the parameters, together with a discussion of the salient issues involved. In the terminology of that paper, the Moderate Current Approximation (MCA) is defined by the regime $\varepsilon << \delta << 1$ and is the one of interest here. It is noted that the Strong Current Approximation defined by $\varepsilon << 1$, $\delta \sim O(1)$ is the same as the one employed linearly earlier in §5.

Implementation involves substituting the series for Ψ, η and ω into (4.6) and formulating a hierarchy in the usual manner. It is hoped that these can be solved when appropriate boundary conditions are applied and p can then be obtained from (4.7). As mentioned previously (cf. Appendix B), the problem can be formulated in terms of the pressure at $O(\varepsilon)$ but this cannot be readily extended beyond $O(\varepsilon)$ and thus Ψ is employed here. The basic current term is at $O(\delta)$ and is given by

$$\Psi_{01}(z) = \int_{-h}^{z} V(z)\, dz \,. \tag{5.13}$$

The first wavelike term at $O(\varepsilon)$ is the incident wave in the absence of a current. As the streamfunction component $\Psi_{10}(z,\theta)$ satisfies $\nabla^2\Psi_{10} = 0$ and is the known linear wave solution we have (cf. (3.1), (3.2) and the analogous dispersion relation)

$$\eta_{10} = \frac{1}{k}\cos\theta(x,t), \quad \Psi_{10} = \frac{\omega_{00}}{k^2}\,\frac{\sinh k(z+h)}{\sinh kh}\,\cos\theta(x,t), \quad \omega_{00}^2(k) = gk\tanh kh \,. \tag{5.14}$$

If kinematic evaluation is required, then the wave slope $\varepsilon = ak$ must be included as in (5.11).

The primary interaction term occurs at $O(\varepsilon\delta)$ and the streamfunction component Ψ_{11} has previously been obtained in [37] and presented here in a slightly different form. If $S(z), C(z), S2(z)$ and $C2(z)$ denote $\sinh k(z+h)$, $\cosh k(z+h)$, $\sinh 2k(z+h)$ and $\cosh 2k(z+h)$ respectively, and the functions $I_s(z)$ and $I_c(z)$ are defined by

$$I_s(z) = \int_{-h}^{z} V(z)\sinh 2k(z+h)\, dz\,, \quad I_c(z) = \int_{-h}^{z} V(z)\cosh 2k(z+h)\, dz\,,$$

then if $\Psi_{11}(\theta,z)$ is written as $\widetilde{\Psi}_{11}(z)\cos\theta(x,t)$, $\widetilde{\Psi}_{11}(z)$ is given by

$$\widetilde{\Psi}_{11} = -\frac{S(z)}{S(0)}.\frac{V(z)}{k} + \frac{2}{S(0)}.\left[I_c(z)C(z) - \frac{C2(0)}{S2(0)}.I_c(0)S(z)\right] + 2\frac{S(z)}{S(0)}.\left[I_s(0) - I_s(z)\right] \tag{5.15}$$

and necessitates that the unknown function $\omega_{01}(k)$ satisfies

$$\omega_{01}(k) = \frac{2k^2}{\sinh 2kh}\int_{-h}^{0} V(z)\cosh 2k(z+h)dz. \tag{5.16}$$

Thus from (5.12), the dispersion relation at this order is

$$\omega = \omega_{00}(k) + \delta\,\omega_{01}(k) + ... = \sqrt{gk\tanh kh} + \delta\,\frac{2k^2}{\sinh 2kh}\int_{-h}^{0} V(z)\cosh 2k(z+h)dz + ...,$$

In terms of the physical current $U(z)$ and correct to the order of working, this can also be written as

$$(\omega - k\widetilde{U}(k))^2 = gk\tanh kh, \quad \widetilde{U}(k) = \frac{2k}{\sinh 2kh}\int_{-h}^{0} U(z)\cosh 2k(z+h)dz \qquad (5.17)$$

which enables k to be determined once ω, h and $U(z)$ are specified.

To obtain the pressure, it is necessary to take the perturbation respresentation for Ψ and p from (5.11) and (5.12) and substitute into (4.7) or (4.12). Comparison of the appropriate $\varepsilon^m\delta^n$ combination will give p_{mn} in terms of the Ψ_{mn}. For waves alone, this is

$$\frac{1}{\rho}p_{10} = \frac{\omega_{00}}{k}.\frac{\partial\Psi_{10}}{\partial z}\ . \qquad (5.18)$$

At $O(\varepsilon\delta)$, the two equations from (4.7), relating equations relating p_{11}, Ψ_{11} and Ψ_{10} are

$$\begin{aligned}
\frac{1}{\rho}\frac{\partial p_{11}}{\partial\theta} &= \frac{\partial H_{11}}{\partial\theta} - \Psi_{01}\,\nabla^2\left(\frac{\partial\Psi_{10}}{\partial\theta}\right) - \Psi_{10}\,\nabla^2\left(\frac{\partial\Psi_{01}}{\partial\theta}\right),\\
\frac{1}{\rho}\frac{\partial p_{11}}{\partial z} &= \frac{\partial H_{11}}{\partial z} - \frac{1}{k}\left[\omega_{00}\nabla^2\Psi_{11} + \omega_{01}\nabla^2\Psi_{10} + \omega_{10}\nabla^2\Psi_{01}\right] - \Psi_{01}\,\nabla^2\left(\frac{\partial\Psi_{10}}{\partial z}\right)\\
&\quad -\Psi_{10}\,\nabla^2\left(\frac{\partial\Psi_{01}}{\partial z}\right),\\
H_{11}(\theta,z) &= \frac{1}{k}\left\{\omega_{00}\frac{\partial\Psi_{11}}{\partial z} + \omega_{01}\frac{\partial\Psi_{10}}{\partial z} + \omega_{10}\frac{\partial\Psi_{01}}{\partial z}\right\} - \frac{\partial\Psi_{01}}{\partial z}.\frac{\partial\Psi_{10}}{\partial z} + k^2\,\frac{\partial\Psi_{01}}{\partial\theta}.\frac{\partial\Psi_{10}}{\partial\theta}\\
&\quad +\Psi_{01}\,\nabla^2\Psi_{10} + \Psi_{10}\,\nabla^2\Psi_{01}\,.
\end{aligned}$$

Considerable simplification is possible, as Ψ_{01} is a function of z, from (5.12), and $\Psi_{10}(\theta,z)$ satisfies Laplace's equation. It is straightforward, though tedious, to show that the two equations possess the solution

$$\begin{aligned}
\frac{1}{\rho}p_{11}(\theta,z) = \frac{1}{k}\left\{\omega_{00}\frac{\partial\Psi_{11}}{\partial z} + \omega_{01}\frac{\partial\Psi_{10}}{\partial z}\right\} - \frac{\partial\Psi_{01}}{\partial z}.\frac{\partial\Psi_{10}}{\partial z}\\
+k^2\,\frac{\partial\Psi_{01}}{\partial\theta}.\frac{\partial\Psi_{10}}{\partial\theta} + \Psi_{01}\,\nabla^2\Psi_{10} + \Psi_{10}\,\nabla^2\Psi_{01} + \gamma_{11}
\end{aligned}$$

where the arbitrary constant γ_{11} can be shown to be zero by application of the boundary conditions. Employing the predetermined properties of $\Psi_{01}(z)$, $\Psi_{10}(\theta,z)$ and $\Psi_{11}(\theta,z)$ from (5.12)-(5.14), enables this expression to be determined as

$$\frac{1}{\rho}p_{11}(\theta,z) = \frac{\omega_{00}}{k}\frac{\partial\Psi_{11}}{\partial z} + \left(\frac{\omega_{01}}{k} - V(z)\right)\frac{\partial\Psi_{10}}{\partial z} + \Psi_{10}\,\frac{dV}{dz}\ . \qquad (5.19)$$

The physical wavelike pressure component at this order is retrieved by including ε and δ to give

$$\begin{aligned} p(\theta, z) &= \varepsilon\,(p_{10} + \delta\, p_{11}) \\ &= \rho a \left[\omega_{00} . \frac{\partial \Psi_{10}}{\partial z} + \omega_{00} \frac{\partial \Psi_{11}}{\partial z} + \left(\frac{\omega_{01}}{k} - U(z)\right) \frac{\partial \Psi_{10}}{\partial z} + \Psi_{10}\, \frac{dU}{dz}\right] \\ &= \rho a \left[(\omega_{00} + \omega_{01} - kU(z)) . \frac{\partial \Psi_{10}}{\partial z} + \omega_{00} \frac{\partial \Psi_{11}}{\partial z} + \Psi_{10}\, k \frac{dU}{dz}\right] \end{aligned} \tag{5.20}$$

and is the MCA equivalent of (5.4) correct to $O(\varepsilon\delta)$. As the $\theta-$dependency is the same in all terms of (5.20), it is convenient to write (5.20) as

$$p(\theta, z) = \rho a \; \Pi(z) \cos\theta \; . \tag{5.21}$$

A similar analysis to that conducted in the earlier part of this section may now be undertaken, with the difference being that $\Pi(z)$ is known and can be evaluated. If the bed and surface pressures are written as

$$p(\theta, -h) = \rho a \; \Pi(-h) \cos\theta = p_b \cos\theta, \quad p(\theta, 0) = a \; \Pi(0) \cos\theta,$$

then the surface elevation a is

$$a = \frac{p_b}{\rho \Pi(-h)} \; , \tag{5.22}$$

and the pressure ampification factor Q is

$$Q = \frac{\Pi(0)}{\Pi(-h)} \; . \tag{5.23}$$

In contrast to the corresponding expressions for a and Q in (5.8) and (5.9), all terms in (5.22) and (5.23) are known and can be evaluated for a given current profile $U(z)$. Also, with $\Pi(-h)$ obtained from (5.20) and (5.21) with $z = -h$, the last term in the expression is zero since $\Psi_{10}(\theta, -h) = 0$ by construction. Furthermore, in common with the considerations of previous sections there is no restriction here on the depth $-h \le z < 0$ at which pressure measurements are taken, and we may choose to work with $Q(z) = \Pi(0)/\Pi(z)$ as required.

6. Discussion

The accuracy of the adopted approach depends upon the profile of the input current velocity $U(z)$. Writing $U(z) = U_M + [U(z) - U_M]$, where U_M is the depth-averaged value of $U(z)$, enables (5.17) to be interpreted as

$$\widetilde{U}(k) = U_M + \frac{2k}{\sinh 2kh} \int_{-h}^{0} [U(z) - U_m] \cosh 2k(z+h) dz$$

and the contribution from the integral term represents the difference of $U(z)$ from its mean value over the profile. More specifically, it is the difference from the mean with a weighting towards the upper part of the profile due to the presence of the $\cosh 2k(z+h)$ term within the integral. Kirby & Chen [24] suggest that the ω_{12} term in (5.12) is required to approximate the dispersion for some profiles in water of finite depth, whereas Thomas & Klopman [37] confirm good kinematics agreement for a single corrective term alone. Both ω_{12} and Ψ_{12} are known but the increase in complexity, for Ψ_{12} in particular, does not seem to justify their inclusion at this stage. This remains an area of investigation with particular emphasis placed upon those profiles encountered in the physical environment.

The problem has been formulated in terms of the measured bed pressure being employed to predict the free surface elevation, albeit within the application given only to the linear wave regime. There are inherent reservations associated with such an approach and these are worthy of investigation. If the pressure amplification factor Q is large, corresponding to deep-water waves, then an error in p_b will result in an increased error at the surface; this decreases as the wave regime moves more towards shallow water. A better procedure, as proposed by some authors (such as Sobey & Hughes [34]), might be to take a reference pressure from a transducer placed above the bed, say at $z = -h_r$, where $0 < h_r < h$. The reference pressure will be larger in this case and the influence of possible viscous effects from the vicinity of the bed will also be less; to use another reference pressure simply requires the replacement of $\Pi\left(-h\right)$ by $\Pi\left(-h_r\right)$ in (5.22) and (5.23).

An important aspect of the practical implementation of the present method is that $U(z)$ must provide a good approximation to the physical current. This is not an issue within the laboratory environment, where measuring systems can be controlled for purposes of model validation, but it is certainly more challenging in the targeted physical field. However, the development and utilisation of Acoustic Doppler Current Profilers (ADCPs) in recent years has enabled the acquisition of good-quality current information over the vertical profile; while ADCPs may lose some accuracy close to the free surface, it is widely acknowledged that they are capable of providing important inputs to predictive models.

The linear approach described herein via the MCA may not be deemed sufficient for waves bordering upon the limits of the approximation and an extension to $O\left(\varepsilon^2\right)$ and $O\left(\varepsilon^2\delta\right)$ may be considered desirable. There is a considerable difference between the the two terms, Ψ_{20} and Ψ_{21} respectively. The term Ψ_{20} corresponds to the standard second order term of the Stokes wave expansion, represented as a streamfunction rather than as the usual velocity potential. In contrast, Ψ_{21} denotes a higher order reference term and is correspondingly more complex. The forms for Ψ_{21} and the pressure p_{21} have been determined but not utilised in any meaningful manner.

The two-parameter expansion of (5.11) and (5.12), employed intially to obtain approximate solutions to (5.2), is discussed in detail by Thomas & Klopman [37] and where three regimes were readily identified. Only the parameter balance of the MCA enables progress to be made within an analytical framework and without the recourse to numerical models. It is readily acknowledged that it provides only a first step to obtaining a prediction of the free surface but the availability of a model that avoids compution and possesses the possibility of accuracy and extension is worthy of further study.

7. Conclusion

A preliminary study on the prediction of the surface waves following the recovery of pressure at the bed, in the presence of a current, has been undertaken. The waves are regular and restricted to two dimensions, though the current may possess an arbitrary distribution of vorticity. A short review of existing methods utilising pressure recovery is presented initially, with an emphasis on work related to the presence of current; these are mainly restricted to constant currents or those with constant vorticity, for which the wavefield is irrotational. The pressure–streamfunction relations are then presented for the general case of regular waves interacting with a current possessing an arbitrary profile and the importance of the vorticity is identified. Although such an approach normally requires numerical evaluation, the Moderate Current Approximation is employed to provide a good description of the bed pressure – surface elevation relationship in water of finite depth.

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Appendix A. Appendix 1

A.1. **The function $H(\theta, z)$.** Neglecting the gravitational component, $H(\theta, z)$ is given by (4.7) as

$$H(\theta, z) = \frac{\omega}{k}\Psi_z - \frac{1}{2}\left\{\Psi_z^2 + k^2\Psi_\theta^2\right\} + \Psi\nabla^2\Psi. \tag{A.1}$$

and with the expansion given in (4.9) and (4.10) each term needs to be represented in a form analogous to the harmonic series employed. The first term in (A.1) is

$$\frac{\omega}{k}\Psi_z = \frac{\omega}{k}\sum_{n=0}^{\infty}\psi_n'(z)\cos n\theta. \tag{A.2}$$

The first term in the kinetic energy in (A.1), with the constant omitted, is

$$\begin{aligned}
\Psi_z^2 &= \left\{\sum_{n=0}^{\infty} \psi_n'(z)\cos n\theta\right\}\left\{\sum_{m=0}^{\infty} \psi_m'(z)\cos m\theta\right\} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \psi_n'(z)\psi_m'(z)\cos n\theta\cos m\theta \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \psi_n'(z)\psi_m'(z)\left[\cos(m+n)\theta + \cos(m-n)\theta\right] \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \psi_n'(z)\psi_m'(z)\cos(m+n)\theta + \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \psi_n'(z)\psi_m'(z)\cos(m-n)\theta \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\left[\psi_n'(z)\right]^2 + \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \psi_n'(z)\psi_m'(z)\cos(m+n)\theta \\
&\qquad + \frac{1}{2}\sum_{n=0}^{\infty}\psi_n'(z)\left\{\sum_{\substack{m=0\\ m\neq n}}^{\infty} \psi_m'(z)\cos(m-n)\theta\right\}.
\end{aligned} \tag{A.3}$$

The second term in the kinetic energy in (A.1), with the constant omitted, is

$$\Psi_\theta^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} nm\psi_n(z)\psi_m(z)\sin n\theta\sin m\theta$$

where the series starts at 1 because there will be a zero contribution when either n or m are zero. Using the relation $\sin n\theta \sin m\theta = \frac{1}{2}\left[\cos(m-n)\theta - \cos(m+n)\theta\right]$ gives

$$\begin{aligned}
\Psi_\theta^2 &= \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} nm\psi_n(z)\psi_m(z)\left[\cos(m-n)\theta - \cos(m+n)\theta\right] \\
&= -\frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} nm\psi_n(z)\psi_m(z)\cos(m+n)\theta + \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} nm\psi_n(z)\psi_m(z)\cos(m-n)\theta \\
&= \frac{1}{2}\sum_{n=1}^{\infty} n^2\psi_n^2(z) - \frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} nm\psi_n(z)\psi_m(z)\cos(m+n)\theta \\
&\qquad + \frac{1}{2}\sum_{n=1}^{\infty} n\psi_n(z)\left\{\sum_{\substack{m=1\\ m\neq n}}^{\infty} m\psi_m(z)\cos(m-n)\theta\right\}.
\end{aligned} \tag{A.4}$$

The last term in (A.1) is $\Psi\nabla^2\Psi$ and this can be written down immediately as

$$\Psi\nabla^2\Psi = \frac{1}{2}\sum_{n=0}^{\infty}\psi_n(z)\Omega_n(z) + \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\psi_n(z)\Omega_m(z)\cos(m+n)\theta$$
$$+ \frac{1}{2}\sum_{n=0}^{\infty}\psi_n(z)\left\{\sum_{\substack{m=0\\m\neq n}}^{\infty}\Omega_m(z)\cos(m-n)\theta\right\}. \tag{A.5}$$

Collecting up the terms from (A.2)–(A.5) and substituting into (A.1) gives

$$H(\theta,z) = \frac{\omega}{k}\sum_{n=0}^{\infty}\psi_n'(z)\cos n\theta - \frac{1}{4}\sum_{n=0}^{\infty}\left\{(\psi_n'(z))^2 + k^2n^2(\psi_n(z))^2 - 2\psi_n(z)\Omega_n(z)\right\}$$
$$-\frac{1}{4}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\psi_n'(z)\psi_m'(z) - nmk^2\psi_n(z)\psi_m(z) - 2\psi_n(z)\Omega_m(z)\right]\cos(n+m)\theta$$
$$-\frac{1}{4}\sum_{n=0}^{\infty}\sum_{\substack{m=0\\m\neq n}}^{\infty}\left[\psi_n'(z)\psi_m'(z) + nmk^2\psi_n(z)\psi_m(z) - 2\psi_n(z)\Omega_m(z)\right]\cos(m-n)\theta. \tag{A.6}$$

A.2. **The function** $F(\theta,z)$**.** From (4.7) and (4.8) the function $F(\theta,z)$ is given by

$$\begin{aligned}F(\theta,z) &= \Psi\nabla^2\Psi_\theta = \Psi\frac{\partial}{\partial\theta}\left[\nabla^2\Psi\right]\\ &= -\left\{\sum_{n=0}^{\infty}\psi_n(z)\cos n\theta\right\}\left\{\sum_{m=0}^{\infty}m\Omega_m(z)\sin m\theta\right\}\\ &= -\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}m\psi_n(z)\Omega_m(z)\cos n\theta\sin m\theta\\ &= -\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}m\psi_n(z)\Omega_m(z)\left[\sin(m+n)\theta + \sin(m-n)\theta\right]\\ &= -\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}m\psi_n(z)\Omega_m(z)\sin(m+n)\theta - \frac{1}{2}\sum_{n=0}^{\infty}\sum_{\substack{m=1\\m\neq n}}^{\infty}m\psi_n(z)\Omega_m(z)\sin(m-n)\theta.\end{aligned}$$

Accordingly,

$$\int^{\theta}F(\theta,z)d\theta = \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{m}{m+n}\psi_n(z)\Omega_m(z)\cos(m+n)\theta$$
$$+\frac{1}{2}\sum_{n=0}^{\infty}\sum_{\substack{m=1\\m\neq n}}^{\infty}\frac{m}{m-n}\psi_n(z)\Omega_m(z)\cos(m-n)\theta. \tag{A.7}$$

A.3. **The function** $G(\theta, z)$**.** From (4.7) and (4.8), $G(\theta, z)$ is composed of two components. The first is

$$\frac{\omega}{k}\nabla^2\Psi = \frac{\omega}{k}\Omega = \frac{\omega}{k}\sum_{n=0}^{\infty}\Omega_n(z)\cos n\theta,$$

and the second is

$$\begin{aligned}
\Psi\nabla^2\Psi_z &= \left\{\sum_{n=0}^{\infty}\psi_n(z)\cos n\theta\right\}\left\{\sum_{m=0}^{\infty}\Omega_m'(z)\cos m\theta\right\} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\psi_n(z)\Omega_m'(z)\cos n\theta\cos m\theta \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\psi_n(z)\Omega_m'(z)\left[\cos(m+n)\theta + \cos(m-n)\theta\right].
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\int^z G(\theta, z)dz &= \frac{\omega}{k}\sum_{n=0}^{\infty}\left\{\int^z \Omega_n(z)dz\right\}\cos n\theta \\
&\quad + \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left\{\int^z \psi_n(z)\Omega_m'(z)dz\right\}\left[\cos(m+n)\theta + \cos(m-n)\theta\right] \\
&= \frac{\omega}{k}\int^z \Omega_0(z)dz + \frac{\omega}{k}\sum_{n=1}^{\infty}\left\{\int^z \Omega_z(z)dz\right\}\cos n\theta \\
&\quad + \frac{1}{2}\left[\int^z \psi_0(z)\Omega_0'(z) + \sum_{n=1}^{\infty}\left\{\int^z \left[\psi_0(z)\Omega_n'(z) + \psi_n\Omega_0'(z)\right]dz\right\}\cos n\theta \right. \\
&\qquad\qquad \left. + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left\{\int^z \psi_n(z)\Omega_m'(z)dz\right\}\cos(m+n)\theta\right] \\
&\quad + \frac{1}{2}\left[\sum_{n=0}^{\infty}\left\{\int^z \psi_n(z)\Omega_n'(z)dz\right\} + \sum_{n=0}^{\infty}\sum_{\substack{m=0\\ m\neq n}}^{\infty}\left\{\int^z \psi_n(z)\Omega_m'(z)dz\right\}\cos(m-n)\theta\right].
\end{aligned} \tag{A.8}$$

Appendix B. Appendix 2

B.1. **Pressure version of the Rayleigh equation** (5.2)**.** There exists an alternative approach for computing the amplification factor $Q(z)$ to that described in Section 5. Rather than working with the Rayleigh equation expressed in terms of the streamfunction, (5.2), we may derive a version (cf. Peregrine [31]) in which the linear dynamic pressure is the unknown. Denoting $p_1(z) = P(z)$, we must solve

$$P''(z) + \left[\frac{2kU'}{\omega - kU}\right] P'(z) - k^2 P(z) = 0 \tag{B.1}$$

subject to the boundary conditions

$$\begin{aligned} P(z) &= P_b && \text{on } z = -h \\ P'(z) &= 0 && \text{on } z = -h \\ (\omega - kU)^2 &= g\frac{P'(z)}{P(z)} && \text{on } z = 0\,. \end{aligned} \tag{B.2}$$

The conditions (B.2) are analogues of (5.3), with the second and third conditions given in [31], whereas the first is not but is required here to derive the solution. With the first two conditions imposed, the surface condition in (B.2) is equivalent to the dispersion relation and is required to obtain the wavenumber k.

Peregrine notes that (B.1) can only be solved analytically for a few simple profiles. The two simplest for a non-zero current are the constant current $U(z) = U_c$ and the linear profile $U(z) = U_s + \Omega z$, though many profiles can be approximated by a number of linear components and with appropriate matching conditions applied at the interfaces. If $U(z)$ is arbitrary, a numerical solution to (B.1) is required; the imposed bottom boundary conditions in (B.2) ensure an initial-value problem that should not cause numerical difficulties. This is in contrast to the method employed by Thomas [35] when the solution for arbitrary $U(z)$ is driven by the given surface elevation amplitude a.

We note that the description of the water wave problem given in terms of the pressure, (B.1)–(B.2), is appropriate only for linear waves; it does not extend readily to higher order. Remarkably, however, the two-dimensional equations (B.1)–(B.2) do have a generalisation in three-dimensions, cf. [28]. In order to work with a higher-than-linear order wave, in two-dimensions the streamfunction approach outlined in (4) may be implemented, or more generally (and, in three-dimensions, necessarily) a numerical method may be used.

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