On the pressure transfer function for solitary water waves with vorticity

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Abstract

In this paper we analyse the role which the pressure function on the sea-bed plays in determining solitary waves with vorticity. We prove that the pressure function on the flat bed determines a unique, real analytic solitary wave solution to the governing equations, given a real analytic vorticity distribution. In particular, the pressure function on the flat bed prescribes a unique surface profile for the resulting solitary water wave.

Keywords: Solitary wave, pressure, vorticity.
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1 Introduction

In this paper we investigate the role which the pressure function plays in determining solitary waves with vorticity. This question, which is highly significant from a theoretical viewpoint, furthermore is of great importance in practical terms. The pressure function plays a key role, both in qualitative studies of travelling waves [6, 7, 8, 10, 11, 16, 17], and in quantitative studies. In field experiments, the free-surface profile of water waves is commonly calculated by way of the so-called pressure transfer function [3, 21, 27], which recovers the free-surface elevation using pressure measurements on the sea-bed. The key to this approach is in deriving a suitable candidate for the pressure transfer function, an issue which is the subject of a large body of research, and which to this point has focussed entirely on irrotational travelling water waves, primarily in the linear setting. Although the linear framework presents less complications when deriving the pressure transfer function [14], it results in significant errors for waves of large amplitude [4]. Accounting for nonlinear effects is vital in this regime, and recently there have been a number of advances in the nonlinear framework [5, 13, 24] in obtaining a pressure recovery function for irrotational solitary water waves. At this point, the work in [5] represents the most advanced theoretical progress, as it presents an explicit exact transfer formula for the surface profile in terms of the pressure on the flat bed, for fully nonlinear, large amplitude, irrotational solitary water waves.

There is a striking paucity of literature concerning the role the pressure function plays in flows with vorticity, chiefly due to the severe mathematical complications which are inherent in rotational flows. However, it is well known that flows with vorticity are highly physically relevant, being vital in the modelling of wave-current interactions, among other phenomena [25]. This is particularly pertinent when we take into account that the pressure sensors providing the data for the transfer function are located on the sea-bed— since the near-bed region is a location which commonly experiences currents (accounting for sediment transport, for instance). It is therefore highly desirous to extend the theoretical investigations of the pressure transfer function from irrotational flows to flows with vorticity. Although allowing for a general vorticity distribution destroys the possibility of obtaining an explicit recovery formula for the free-surface, this paper represents a first step in the theoretical analysis of the relationship between the pressure function on the flat bed, and the free-surface profile, for solitary water waves with vorticity. In this paper we restrict our attention to solitary waves with arbitrary vorticity distributions, the existence of which was rigorously proven by differing methods in [15, 18]— in [18], the existence of small-amplitude solitary waves was proven by way of a Nash-Moser type theorem, whereas [15] uses spatial dynamical techniques. Unlike for the irrotational case [2], the question of the existence of large amplitude solutions to the solitary wave problem with vorticity has not yet been solved.

The aim of this paper is to prove our main result, which may be stated as follows.

Theorem 1.1 *Let h be a solution of the system* (12)*, representing a steady solitary water wave over a flow with real analytic vorticity function γ, such that the wave speed exceeds the horizontal fluid velocity. Then the solution is real analytic, and it is uniquely determined by the pressure function P for the flow on the flat-bed. In particular, the wave surface profile of a solitary wave with vorticity is uniquely determined by the pressure function on the flat bed.*

2 Preliminaries

We consider two-dimensional steady travelling waves, propagating on the surface of an inviscid and incompressible fluid. The motion being steady implies a functional dependence on the independent variables of the form $(X - ct, Y)$, suggesting we transform to the reference frame moving with speed c via the change of variables

$$\begin{cases} x = X - ct, \\ y = Y. \end{cases}$$

The fluid domain $\mathcal{D}_\eta$ in the moving frame is bounded by the unknown free-surface $y = \eta(x)$ and the flat bed $y = -d$, and the motion throughout is governed by the mass conservation equation

$$u_x + v_y = 0 \qquad \text{in } \mathcal{D}_\eta, \tag{1a}$$

together with Euler's equation

$$(u - c)u_x + vu_y = -P_x, \tag{1b}$$
$$(u - c)v_x + vv_y = -P_y - g \qquad \text{in } \mathcal{D}_\eta. \tag{1c}$$

Here the velocity field is represented by $(u(x, y), v(x, y))$, $P(x, y)$ is the pressure function, and g the gravitational constant of acceleration. The kinematic and dynamic boundary conditions for the fluid are given by

$$v = (u - c)\eta_x \qquad \text{on } y = \eta(x), \tag{1d}$$
$$v = 0 \qquad \text{on } y = -d, \tag{1e}$$
$$P = P_{atm} \qquad \text{on } y = \eta(x), \tag{1f}$$

where P_{atm} is the constant atmospheric pressure. A solitary wave is, roughly-speaking, a localised lump of fluid which levels out in the far-field to a flat surface (the small-amplitude solitary waves derived in [15, 18] are waves of elevation which decay exponentially to a horizontal laminar flow at infinity). Hence, for solitary waves we additionally impose the far-field conditions

$$\begin{aligned} &\lim_{|x|\to\infty} \eta(x) \to 0, \\ &\lim_{|x|\to\infty} v(x, y) \to 0 \quad \text{uniformly in } y. \end{aligned} \tag{1g}$$

In the current paper we admit flows which have a general vorticity distribution, and for two-dimensional flows the vorticity distribution is given by

$$\omega = u_y - v_x. \tag{1h}$$

We furthermore make the assumption that there are no stagnation points throughout the fluid by insisting that

$$u(x, y) < c \quad (x, y) \in \overline{\mathcal{D}_\eta}. \tag{2}$$

This hypothesis expresses the absence of stagnation points throughout the fluid, and is a physically reasonable assumption for water waves, without underlying currents containing strong non-uniformities, and which are not near breaking [6]. We use equation (1a) to define the stream function ψ up to a constant by

$$\psi_y = u - c, \quad \psi_x = -v, \tag{3}$$

and we fix the constant by setting $\psi = 0$ on $y = \eta(x)$. If we define the relative mass flux p_0 by

$$p_0 = \int_{-d}^{\eta(x)} (u(x, y) - c)dy < 0,$$

then it follows by direct calculation that p_0 is a constant of the flow, and furthermore $\psi = -p_0$ on $y = -d$. Hence

$$\psi(x, y) = -p_0 + \int_{-d}^{y} (u(x, s) - c)ds.$$

The level sets of $\psi(x, y)$ are the streamlines of the fluid motion. Furthermore, once we assume that (2) holds, it follows that the vorticity is a function of the streamline alone,

$$\omega = \gamma(\psi), \tag{4}$$

where γ is the vorticity function. We note that, owing to the boundary decay conditions (1g), the rotational solitary wave is a particularly illustrative archetype for describing the role which vorticity plays in wave-current interactions. Since in the far-field the flow is laminar, the velocity field there is a shear flow, or pure current:

$$\lim_{|x|\to\infty} (u(x, y) - c, v(x, y)) = (U(y) - c, 0)\,, \tag{5}$$

where U is an arbitrary (sufficiently regular) function, and $\omega = U'(y)$. Thus, in the far-field laminar region the vorticity is generated purely by the underlying current (5), and in the region of wave-current interaction where the solitary wave causes a disturbance the vorticity is prescribed by (4) (condition (2) implies that, for fixed x, the map $y \mapsto \psi(x, y)$ is an isomorphism). As a consequence of (1g) we get

$$p_0 = \int_{-d}^{0} (U(y) - c)dy, \tag{6}$$

and in particular for irrotational flows $p_0 = -cd$.

If we recast Euler's equation in terms of the stream function we derive Bernoulli's law, which states that the expression

$$E := |\nabla\psi|^2 + 2gy + 2P + 2\Gamma(-\psi) \tag{7}$$

is constant throughout the fluid, where $\Gamma(s) = \int_0^s \gamma(-s')ds'$ for $s \in [p_0, 0]$. We take advantage of the far-field conditions (1g) to evaluate this expression on the surface:

$$E = c_0^2 + 2P_{atm}, \quad \text{for } c_0 = c - U(0). \tag{8}$$

Using the stream function, we can reformulate the governing equations (1) in the moving frame as an elliptic equation with nonlinear boundary conditions, namely

$$\begin{aligned} \Delta\psi &= \gamma(\psi) && \text{in } \mathcal{D}_\eta, && (9a)\\ |\nabla\psi|^2 + 2g\eta &= c_0^2 && \text{on } y = \eta(x), && (9b)\\ \psi &= 0 && \text{on } y = \eta(x), && (9c)\\ \psi &= -p_0 && \text{on } y = -d, && (9d) \end{aligned}$$

where

$$\begin{array}{l} \lim_{|x|\to\infty} \eta(x) \to 0, \\ \lim_{|x|\to\infty} \nabla\psi \to (0, U(y) - c) \quad \text{for } -d \le y \le 0. \end{array} \tag{10}$$

3 Semi-Lagrangian hodograph transformation

Since a major part of the difficulty of the water wave problem (9) is connected to the boundary at the free-surface being unknown, we can bypass this to some extent by applying the following semi-Lagrangian hodograph transformation, which transforms from the fluid domain $\mathcal{D}_\eta$ to the fixed semi-infinite rectangular domain $R = \mathbb{R} \times [p_0, 0]$:

$$\begin{cases} q = x, \\ p = -\psi(x, y). \end{cases} \tag{11}$$

For the height function in the new (q, p)-variables,

$$h(q, p) = y + d,$$

the change of variables (11) transforms the system of equations (9a)-(9d) on an unknown domain into the following system for the function $h(q, p)$ in the fixed rectangular domain R:

$$(1+h_q^2)h_{pp} - 2h_qh_ph_{pq} + h_p^2h_{qq} = \gamma(-p)h_p^3 \quad \text{in } R, \tag{12a}$$

$$1 + h_q^2 + \left[2g(h-d) - \frac{1}{c_0^2}\right] h_p^2 = 0, \quad p = 0, \tag{12b}$$

$$h = 0, \quad p = p_0, \tag{12c}$$

with the boundary conditions

$$\lim_{|q|\to\infty} h(q,0) = d, \ \lim_{|q|\to\infty} h_q(q,p) = 0. \tag{13}$$

If $h \in C^{3,\alpha}(\overline{R})$, then the equivalence of the systems (1), (9) and (12) follows as for periodic waves [12, 18]. We note that the non-stagnation condition (2) implies that

$$h_p(q,p) = \frac{1}{c-u} > 0, \tag{14}$$

and this function has a limiting value at infinity which is given by (10). Now, equation (12a) is uniformly elliptic (due to (14)), and the boundary conditions satisfy the complementing condition (cf. [6] for discussions of the complementing condition). Hence it is in a suitable form to derive the following regularity result, which states that if the vorticity function γ is real analytic in p, then the corresponding solution h of the water wave problem (12) is *a priori* real analytic in the $(q,p)-$variables throughout $\overline{R}$.

Lemma 3.1 *Let $\gamma \in C^\omega([p_0,0])$ and consider the corresponding solution $h \in C^{3,\alpha}(\overline{R})$ of the governing equations* (12)*, representing a travelling solitary water wave with vorticity such that the wave speed exceeds the horizontal fluid velocity throughout the flow. Then $h \in C^\omega(\overline{R})$.*

Proof As the quasilinear system (12) is uniformly elliptic, and the boundary conditions satisfy the complementing condition (in the sense of [1]), if $\gamma \in C^\omega([p_0,0])$ we may apply the elliptic regularity results of [23] to infer that $h \in C^\omega(\overline{R})$.

The first analytic regularity result for irrotational travelling waves was provided in the paper [22]; see the discussion in the survey paper [26]. For recently obtained results regarding the analyticity of periodic travelling waves with vorticity cf. [9]. For the remainder of this article we will assume that γ is real analytic in p.

4 Proof of Theorem 1.1

We express Bernoulli's law (7) in terms of the height function,

$$\frac{1+h_q^2}{h_p^2} + 2g(h-d) + 2P + 2\Gamma(p) = E, \tag{15}$$

where E is the constant given by (8) above. On the flat bed (where $h = 0$) this expression becomes

$$\frac{1}{h_p^2(q,p_0)} - 2gd + 2P(q,p_0) + 2\Gamma(p_0) = E, \tag{16}$$

where p_0 is given by (6), and so it follows that the pressure function P on the flat bed determines h_p on the flat-bed. Let us now define a further transformation in terms of the independent variables h_q, h_p as follows:

$$F = \frac{h_q}{h_p}, \quad G = \frac{1}{h_p}. \tag{17}$$

This transformation (which is feasible due to (14)) reformulates the second order, uniformly elliptic equation (12a), as the following nonlinear first order system:

$$F_p = \frac{F}{G}\left(\frac{GF_q + FG_q - G\gamma(-p)}{F^2+G^2}\right) - \frac{G_q}{G}, \tag{18a}$$

$$G_p = \frac{GF_q + FG_q - G\gamma(-p)}{F^2+G^2} \qquad \text{in} \quad R. \tag{18b}$$

Although a consequence of this transformation is a loss of ellipticity in the system, we can now work as follows. Let h be a solution of (12) satisfying the boundary conditions (13). If follows, from (12c) and (17), that $F = 0$ on the flat-bed $p = p_0$. Furthermore, it is a consequence of (16) that the pressure function P determines G on the flat-bed (at $p = p_0$). Since we assume that γ is real analytic in p, the above system satisfies the conditions of the Cauchy-Kowalevski theorem [19]. This assures us of the existence of solutions F, G of the system (18), which are real analytic in the (q,p)-variables, in an open neighbourhood $\mathcal{N}$ of the flat bed. We deduce, from standard results for real anaytic functions (cf. [20]), that the corresponding functions h_p, h_q, and in turn h, given by (17), are real analytic functions in the $(q,p)-$variables, in the same neighbourhood $\mathcal{N}$. Furthermore, the Cauchy-Kowalevski theorem ensures that the solutions F, G of (18) are unique in the class of real analytic functions on $\mathcal{N}$. Since, from Lemma 3.1, all solutions h of (12) are real analytic throughout $\overline{R}$, by applying unique continuation results [20] for functions which are real analytic throughout R, and which match on $\mathcal{N}$, the Cauchy-Kowalevski theorem implies the uniqueness of solutions h to (12), which satisfy (16), once the pressure function is prescribed on the flat bed. Our result follows.

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