ON THE EXISTENCE OF EQUATORIAL WIND WAVES

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Abstract. In this paper we apply local bifurcation theory to prove the existence of steady, periodic, two-dimensional surface water waves in the equatorial region which have a general underlying vorticity distribution. Furthermore, we derive explicit dispersion relations for the flow in the case where the vorticity is constant.

1. Introduction

In this paper we use local bifurcation theory to prove the existence of equatorial wind waves which have general underlying vorticity distributions without stagnation points. Due to the direction of prevailing winds, equatorial wind-generated surface waves propagate predominantly in a westerly direction [3, 15, 16, 33]. Accordingly, in this paper we prove the existence of steady periodic two-dimensional geophysical surface waves propagating westwards in the equatorial region.

Geophysical fluid dynamics is the study of large-scale physical phenomena where the effect of the Earth's rotation plays a significant role and therefore must be taken into account through the presence of Coriolis forces in the governing equations [15, 33]. Geophysical processes which occur in the equatorial region are of particular interest for a number of reasons. Physically, the equator has the remarkable property of acting as a natural wave guide, whereby equatorially trapped zonal waves decay exponentially away from the equator in the oceans [2–5, 16, 18]. Allied to this, large-scale currents and wave-current interactions play a major role in the geophysical dynamics of the equatorial region [34, 35]. For instance, the El Niño and La Niña phenomena have recently been ascribed to the interplay between equatorial currents in the ocean and atmosphere [27].

Among the most spectacular in scale is the Equatorial Undercurrent (EUC) which extends practically throughout the Pacific Ocean, 13000km, and which exists in a relatively shallow layer tens of metres beneath the surface. The EUC was fortuitously discovered in the 1950's when scientists on a research vessel noticed that, in the equatorial region, the prevalent westerly surface flow reverses direction at a depth of tens of metres, resulting in an easterly-flowing current. The EUC has since been the subject of much geophysical research [3, 15, 16, 23, 34, 35]. It is non-uniform with depth and is confined to a region less than 200m deep. Furthermore, the strength and form of the EUC is variable. Given the

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enormous zonal scale of the EUC, the width of the meridional region that it occupies is remarkably thin, about 300km in width, and the EUC is symmetric about the equator.

Mathematically, the study of equatorial waves is appealing due to a number of factors. Along the Equator, the Coriolis forces vanish, and in a region close to the equator we may approximate the governing equations by the so-called $f-$plane approximation [15, 33]. Furthermore, since the equator acts as a natural waveguide, equatorial waves are predominantly zonal, allowing us to approximate the flows as being two-dimensional. It has recently been shown that a number of qualitative analytical features, which apply to gravity waves with vorticity, similarly apply to equatorial waves [5–7, 9, 12, 26, 28, 32]. Furthermore, a number of explicit exact solutions of the $\beta-$plane geophysical governing equations have been derived for different physical settings [2, 4, 10, 18, 29–31]. The aim of this paper is to extend the recent rigorous mathematical work proving existence of gravity waves with general vorticity distributions, [11, 12, 19, 20, 24, 25], to the geophysical setting.

In order to model general non-uniform currents, wave-current interactions, or flows generated by wind-shears, we need to allow for a sufficiently general vorticity distribution [6, 36]. Recently, the EUC was modelled by ascribing a constant negative vorticity distribution to the flow, and the author then proved the existence of waves using bifurcation theory [3]. The main result of this paper may be stated as follows.

Theorem 1.1. *If a general vorticity distribution satisfies condition* (3.12)*, then there exist non-trivial steady periodic equatorial wind-waves, without stagnation points, propagating over an underlying flow which possesses that particular vorticity distribution.*

We will show that every negative vorticity distribution satisfies condition (3.12) *a priori*, thereby proving as a corollary to Theorem 1.1 that wind-generated waves exist whose underlying flow may prescribe any given negative vorticity distribution. The generality of this existence result allows us greater freedom in modelling the non-uniformity and variability of the EUC, and other equatorial currents, and their interactions with wind-generated waves. Geophysical flows are generally rotational, whereby vorticity plays a major role in the dynamics. However, we prove in this paper that in the equatorial region it is possible to have irrotational flows, which represent either uniform currents or an absence of underlying currents. In the final Section of this paper we derive the explicit dispersion relation for equatorial wind-waves when the underlying vorticity is constant.

In the complimentary paper [23] the authors have proven that these equatorial wind waves possess strong symmetry properties, along the lines of [6, 7, 9, 32].

2. The governing equations

We model the wind-generated equatorial waves using a two layer model, the top layer consisting of the fluid region to which the underlying current is confined, with its top boundary being the free-surface of the wave, and the bottom horizontal boundary having no vertical motion permeating it. We choose a reference frame with the origin located at a point on earth's surface and which is rotating with the earth, setting the $x-$axis to be the longitudinal variable (horizontal due east), the $y-$axis to be the latitudinal variable (horizontal due north) and the $z-$axis to be vertically upwards. We employ the $f-$plane approximation to the full geophysical governing equations, which is valid in the latitudinal region near the Equator, cf. [15]. We let $z=\eta(t,x,y)$ be the surface of the ocean, and set $z=0$ to be the mean surface level for the flow, with $z=-d$ denoting the lower boundary of the layer to which the equatorial undercurrent is confined. In the region $-d\le z\le \eta(t,x,y)$ the $f-$plane governing equations [15,33] comprise the Euler equations

$$\left\{\begin{array}{rcl} u_t+uu_x+vu_y+wu_z+2\omega w &=& -P_x/\rho,\\ v_t+uv_x+vv_y+wv_z &=& -P_y/\rho,\\ w_t+uw_x+vw_y+ww_z-2\omega u &=& -P_z/\rho-g, \end{array}\right. \tag{2.1a}$$

and, under the assumption of constant density, the equation of mass conservation takes the form of the continuity equation

$$u_x+v_y+w_z=0. \tag{2.1b}$$

Here t represents time, (u,v,w) is the fluid velocity, $\omega=73\cdot 10^{-6}rad/s$ is the (constant) rotational speed of the Earth[1] round the polar axis towards east, ρ is the (constant) density of the water, $g=9.8m/s^2$ is the (constant) gravitational acceleration at the Earth's surface, and P is the pressure. At the wave surface, the pressure of the fluid matches the atmospheric pressure P_{atm} :

$$P=P_{atm} \qquad \text{on} \quad z=\eta(t,x,y). \tag{2.1c}$$

Moreover, at each moment of time, the free surface of the wave consists of the same fluid particles, so that we obtain the kinematic boundary condition

$$w=\eta_t+u\eta_x+v\eta_y \qquad \text{on} \quad z=\eta(t,x,y). \tag{2.1d}$$

Since for finite depth waves the bottom of the ocean is assumed to be impermeable, we impose the no-flux condition

$$w=0 \qquad \text{on} \quad z=-d. \tag{2.1e}$$

In this paper we seek two-dimensional flows, moving in the zonal direction along the equator independent of the $y-$coordinate and therefore $v\equiv 0$ throughout the flow. Furthermore, we consider travelling waves, whereby the velocity field, the pressure, and the free surface exhibit an $(x,t)-$dependence of the form $(x-ct)$, where $c<0$ is the surface wave phase-speed in the westward direction. The vertical vorticity distribution is given by the equation

$$\gamma=u_z-w_x. \tag{2.1f}$$

We consider the fluid layer

$$\Omega_\eta=\{(x,y)\,:\,x\in\mathbb{R} \text{ and } -d<y<\eta(x)\}$$

which is bounded from above by the graph of the unknown free boundary η and from below by the line $z=-d$, where d is sufficiently deep that the equatorial undercurrent in

[1]Taken to be a perfect sphere of radius $6371km$.

contained in Ω_η. We consider solutions of (2.1) which are periodic in the $x-$variable, that is u, w, P, η are all $L-$ periodic in x. Furthermore, we assume the absence of stagnation points— the latter property is satisfied if

$$c < u \qquad \text{in } \Omega_\eta. \tag{2.2}$$

The periodicity assumption, coupled with physical considerations, means that at any fixed time t_0 we have

$$\int_0^L \eta(x, t_0)dx = 0. \tag{2.3}$$

The governing equations for equatorial water waves which are propagating in the westerly direction are given by the nonlinear free-boundary problem (2.1) with additional constraints (2.2) and (2.3). Accordingly, we require a priori that the solutions have the following Hölder regularity

$$\eta \in C^{2,\alpha}(\mathbb{R}) \text{ and } (u, w, P) \in \left(C^{1,\alpha}(\overline{\Omega}_\eta)\right)^3. \tag{2.4}$$

2.1. Equivalent formulations of the problem (2.6). We may define the stream function ψ up to a constant by

$$\psi_z = u - c, \quad \psi_x = -w, \tag{2.5}$$

and we fix the constant by setting $\psi = 0$ on $z = \eta(x)$. Relations (2.1d) and (2.1e) imply that ψ is constant on both boundaries of Ω_η, and so it follows from integrating (2.5) and using (2.2) that $\psi = m$ on $z = -d$, where

$$m = \int_{-d}^{\eta(x)} (c - u(x, z))dz < 0. \tag{2.6}$$

The above expression is usually referred to as the relative mass flux. Since

$$\psi(x, z) = m + \int_{-d}^{z} (u(x, s) - c)ds,$$

we deduce that ψ is a periodic function, with period L, and the level sets of the stream function $\psi(x, z)$ describe the streamlines of the flow. Condition (2.2) enables us also to introduce new variables by means of the Dubreil-Jacotin hodograph transformation $\mathcal{H} : \Omega_\eta \to \Omega$, defined by

$$\mathcal{H}(x, z) := (q, p)(x, z) := (x, \psi(x, z)) \qquad (x, z) \in \overline{\Omega}_\eta.$$

Here

$$\Omega := \{(q, p) : m < p < 0, q \in \mathbb{R}\}.$$

The mapping $\mathcal{H}$ is a diffeomorphism and as in [6] we have $\partial_q(\gamma \circ \mathcal{H}^{-1}) = 0$ in Ω, which means that the vorticity $\gamma = \gamma(p)$ is prescribed as a function of the streamlines: $\gamma(x, z) = \gamma(\psi(x, z))$ for all $(x, z) \in \Omega_\eta$. Finally, if we define the hydraulic head by the expression

$$E := \frac{(u - c)^2 + w^2}{2} + (g - 2\omega c)z + \frac{P}{\rho} - 2\omega\psi + \int_0^\psi \gamma(s)\, ds \qquad \text{in } \Omega_\eta,$$

we can show that E is constant in Ω_η by taking the curl of equations (2.1a). This is the $f-$plane version of Bernoulli's law. Therefore, we can reformulate (2.1) in terms of (η, ψ) for the following problem,

$$\Delta\psi = \gamma(\psi) \qquad \text{in} \quad -d < z < \eta(x), \tag{2.7a}$$

$$|\nabla\psi|^2 + 2(g - 2\omega c)z = Q \qquad \text{on } z = \eta(x), \tag{2.7b}$$

$$\psi = 0 \qquad \text{on } z = \eta(x), \tag{2.7c}$$

$$\psi = m \qquad \text{on } z = -d, \tag{2.7d}$$

where γ is the function which describes the vorticity distribution.

2.1.1. *The height function formulation.* To obtain a second equivalent formulation of the original problem, we introduce the height function $h : \Omega \to \mathbb{R}$ by the relation

$$h(q, p) = z + d \qquad \text{for } (q, p) \in \Omega.$$

We remark, from the definition of $\mathcal{H}$, that

$$h_q = -\frac{\psi_x}{\psi_z} = \frac{w}{u - c}, \qquad h_p = \frac{1}{\psi_z} = \frac{1}{u - c}, \tag{2.8}$$

and assumption (2.2) implies then that $h_p > 0$. It follows readily from the definition of h and of the coordinate transformation $\mathcal{H}$ that h solves the following equations:

$$\begin{cases} (1 + h_q^2)h_{pp} - 2h_p h_q h_{pq} + h_p^2 h_{qq} + \gamma(p)h_p^3 = 0 & \text{in } \Omega, \\ 1 + h_q^2 + [2(g - 2\omega c)(h - d) - Q]h_p^2 = 0 & \text{on } p = 0, \\ h = 0 & \text{on } p = m. \end{cases} \tag{2.9}$$

Note that h is even and has period L in the $q-$variable. The problem (2.9) is an elliptic problem, since $h_p > 0$, with nonlinear boundary conditions. We look for solutions $h \in X$ where

$$X = \{h \in C^{2,\alpha}_{per}(\overline{\Omega}),\ h = 0 \text{ on } p = m\}. \tag{2.10}$$

Here the subscript *per* denotes that h is periodic in the $q-$variable. From now on we assume without loss of generality that the period $L = 2\pi$, since performing the following scaling of variables

$$(x, z, t, g, \gamma, \omega, \eta, u, w, P, c) \mapsto (\kappa x, \kappa z, \kappa t, \kappa^{-1} g, \kappa^{-1}\gamma, \kappa^{-1}\omega, \kappa\eta, u, w, P, c) \tag{2.11}$$

where $\kappa = \frac{2\pi}{L}$ is the wavenumber, we end up with a $2\pi-$periodic system in the new variables identical to (2.1b)–(2.1e) except g, ω, γ are replaced by $\kappa^{-1}g, \kappa^{-1}\omega, \kappa^{-1}\gamma$.

3. The Bifurcation Setting

In this paper we use the Crandall-Rabinowitz [14] local bifurcation theorem to prove, for general vorticity functions γ which satisfy (3.12), the existence of non-trivial solutions to (2.9), as stated in Theorem 1.1. We now state the Crandall-Rabinowitz theorem in it simplest form— we refer to [1, 6] for a detailed discussion of local bifurcation theory, including a proof of Theorem 3.1.

Theorem 3.1 (Crandall-Rabinowitz). *Let X, Y be Banach spaces and let $\mathcal{F} \in C^k(X\times\mathbb{R}, Y)$ with $k \geq 2$ satisfy:*

(i) $\mathcal{F}(0,\lambda) = 0$ *for all* $\lambda \in \mathbb{R}$;

(ii) The Fréchet derivative $\mathcal{F}_x(0,\lambda^*)$ *is a Fredholm operator of index zero with a one-dimensional kernel:*

$$\ker(\mathcal{F}_x(0,\lambda^*)) = \{sx_0 : s \in \mathbb{R}, 0 \neq x_0 \in X\};$$

(iii) The tranversality condition holds:

$$\mathcal{F}_{\lambda x}(0,\lambda^*)[(x_0,1)] \notin range(\mathcal{F}_x(0,\lambda^*)).$$

Then λ^* *is a bifurcation point in the sense that there exists* $\epsilon_0 > 0$ *and a branch of solutions*

$$\{(x,\lambda) = \{(s\chi(s), \Lambda(s)) : s \in \mathbb{R}, |s| < \epsilon_0\} \subset X \times \mathbb{R}, \},$$

with $\mathcal{F}(x,\lambda) = 0$, $\Lambda(0) = 0, \chi(0) = x_0$, *and the maps*

$$s \mapsto \Lambda(s) \in \mathbb{R}, \qquad s \mapsto s\chi(s) \in X,$$

are of class C^{k-1} *on* $(-\epsilon_0, \epsilon_0)$. *Furthermore there exists an open set* $U_0 \in X \times \mathbb{R}$ *with* $(0,\lambda_0) \in U_0$ *and*

$$\{(x,\lambda) \in U_0 : \mathcal{F}(x,\lambda) = 0, x \neq 0\} = \{(s\chi(s), \Lambda(s)) : 0 < |s| < \epsilon_0\}.$$

3.1. Laminar flows, bifurcation parameter, and operator reformulation. In the process of finding the laminar flow solutions of (2.9) we are lead naturally to a bifurcation parameter λ and a reformulation of (2.9) in terms of a Banach space operator which is suitable for applying Theorem 3.1. Laminar flows are solutions $H(p)$ of (2.9) that have no $q-$dependence, and so the streamlines of the resulting physical flow are purely horizontal. Accordingly, $H(p)$ solves

$$\begin{aligned} \frac{H_{pp}}{H_p^3} &= -\gamma(p) && \text{in} \quad m < p < 0, \\ H_p^2(0) &= [Q - 2(g-2\omega c)(H(0)-d)]^{-1} && \text{on} \quad p = 0, \\ H &= 0, && \text{on} \quad p = m. \end{aligned} \tag{3.1}$$

We solve to get

$$H(p;\lambda) = \int_m^p \frac{ds}{\sqrt{\lambda + 2\Gamma(s)}} \tag{3.2}$$

where the parameter $\lambda > 0$ is defined by

$$\sqrt{\lambda} = u(0;\lambda) - c,$$

and

$$\Gamma(p) = \int_0^p \gamma(s)ds. \tag{3.3}$$

Here $u(0;\lambda)$ denotes the physical speed of the particular laminar fluid flow on the free surface. For $\lambda > -2\Gamma_{min}$ we have the relation

$$\lambda = Q - 2(g - 2\omega c)[H(0;\lambda) - d].$$

If $H(p,\lambda)$ are the laminar flows, then for

$$h(q,p) = H(p;\lambda) + f(q,p) \quad \text{ with } f \in X,$$

and for $\lambda > -2\Gamma_{min}$, the system (2.9) can be expressed in operator form

$$\mathcal{F}(f,\lambda) = 0 \quad \text{ with } f \in X.$$

Here $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) : X \times (-2\Gamma_{min}, \infty) \to Y$, for $Y := Y_1 \times Y_2 = C^{1,\alpha}_{per}(\overline{\Omega}) \times C^{2,\alpha}_{per}(\mathbb{S})$, is given by

$$\begin{aligned}
\mathcal{F}_1(f,\lambda) = {} & \left(1 + f_q^2\right)(H_{pp} + f_{pp}) - 2f_q f_{qp}(H_p + f_p) \\
& + f_{qq}(H_p + f_p)^2 + \gamma(p)(H_p + f_p)^3, \\
\mathcal{F}_2(f,\lambda) = {} & \left(1 + f_q^2 + [2(g - 2\omega c)(H + f - d) - Q](H_p + f_p)^2\right)_T,
\end{aligned}$$

where the subscript T denotes the trace operator. It follows that $\mathcal{F}(0,\lambda) = 0$ when $\lambda > -2\Gamma_{min}$ since H satisfies (3.1), and so $\mathcal{F}$ is in the appropriate form for applying the Crandall-Rabinowitz theorem with laminar flow solutions assuming the role of the trivial solution in part (i) of Theorem 3.1. In order to prove existence of non-trivial solutions for the full water wave problem (2.9), we must establish that there exists a critical value λ^* of the bifurcation parameter for which the conditions $(ii) - (iii)$ in Theorem 3.1 hold at the point $(H(p), \lambda^*)$. The local bifurcation curve at this point will then consist of non-trivial (non-laminar) water waves. Whether such a critical value λ^* exists depends on the distribution of the vorticity function: in (3.12) we present a criteria for the vorticity distribution which is sufficient for local bifurcation to occur.

3.2. The Fréchet derivative $\mathcal{F}_x(h,\lambda)$. To apply the Crandall-Rabinowitz theorem, we must perform a careful analysis of the relevant Fréchet derivatives of $\mathcal{F}$. We begin by examining the linearisation of the problem, that is, we look for solutions of (2.9) which are of the form $h(q,p) = H(p;\lambda) + \varepsilon m(q,p)$, where $m \in C^{2,\alpha}_{per}(\overline{\Omega})$ is even in q, and study the equations where the parameter ϵ is at the first order. If we set $a(p;\lambda) = H_p^{-1} = \sqrt{\lambda + 2\Gamma(p)}$, for $\lambda > -2\Gamma_{min}$, then $a_p = \gamma(p)a^{-1}$. The resulting equations look like

$$(a^3 f_p)_p + a f_{qq} = 0 \quad \text{in } \Omega \tag{3.4a}$$

$$a^3 f_p = (g - 2\omega c) f \quad \text{on } p = 0, \tag{3.4b}$$

$$f = 0 \quad \text{on } p = m. \tag{3.4c}$$

Since the even function f has a Fourier series representation

$$f(q,p) = \sum_{k=0}^{\infty} f_k(p)\cos(kq) \in C^2_{per}(\overline{\Omega}), \tag{3.5}$$

it follows that f solves (3.4) if and only if for each value k the function $f_k(p)$ solves the following Sturm-Liouville problem

$$(a^3 f_p)_p = k^2 a f \qquad \text{in} \quad m < p < 0, \tag{3.6a}$$
$$a^3 f_p = (g - 2\omega c) f \qquad \text{on} \quad p = 0, \tag{3.6b}$$
$$f = 0 \qquad \text{on} \quad p = m. \tag{3.6c}$$

We associate to (3.6) a minimisation problem concerning its Rayleigh quotient

$$\mu(\lambda) = \inf_{\phi \in H^1(m,0),\ \phi(m)=0,\ \phi \not\equiv 0} \mathbb{F}(\phi, \lambda), \tag{3.7}$$
$$\text{with} \quad \mathbb{F}(\phi,\lambda) = \frac{-(g-2\omega c)\phi^2(0) + \int_m^0 a^3 \phi_p^2 dp}{\int_m^0 a\phi^2 dp}.$$

Here $H^1(m,0)$ is the Sobolev space of square summable functions on $[m,0]$. It follows from standard variational considerations (cf. [6,11,12,19]) that the variational problem (3.7) is well-posed, that is, for fixed λ we have $\mu(\lambda) > -\infty$. Additionally, it can be shown [6] that the limit $\mu(\lambda)$ is attained by a function $\phi = M \in C^{2,\alpha}[m,0]$ which is a classical solution of the weighted Sturm-Liouville problem

$$(a^3 M_p)_p = -\mu(\lambda) a M \qquad \text{in} \quad m < p < 0 \tag{3.8a}$$
$$a^3 M_p = (g - 2\omega c) M \qquad \text{on} \quad p = 0 \tag{3.8b}$$
$$M = 0 \qquad \text{on} \quad p = m. \tag{3.8c}$$

Since we seek periodic solutions $f(q,p)$ of (3.4) which have period 2π, we restrict ourselves to the case where $k = 1$ in (3.6). This function f represents a $2\pi-$periodic solution of the linearisation of the water wave problem (2.9), that is, $\mathcal{F}_x(H(p),\lambda)f = 0$. For a given vorticity distribution γ, the existence of a solution of (3.4) is ensured if we can show that $\mu(\lambda^*) = -1$ in (3.7) for some critical value λ^*.

3.3. The groundstate μ dependence on λ. In this Section we prove that the groundstate $\mu(\lambda)$ has a real-analytic dependence on the parameter λ, and we show that $\mu(\lambda)$ is monotonically increasing when $\mu(\lambda) < 0$. Therefore any value λ for which $\mu(\lambda) = -1$ must be unique. We may normalise the functions ϕ in the variational problem (3.7) so that $\phi(0) = 1$, and we deduce from the formulation (3.7) of $\mu(\lambda)$ that the functional relation

$$G(\mu,\lambda) = \mu(\lambda)\int_m^0 aM^2 dp + (g - 2\omega c) - \int_m^0 a^3 M_p^2 dp = 0$$

holds, with

$$G_\mu(\mu,\lambda) = \int_m^0 aM^2 dp + 2\mu \int_m^0 aMM_\mu dp - 2\int_m^0 a^3 M_p M_{p\mu} dp.$$

Differentiating (3.8a) with respect to μ, multiplying the resulting equation by M and integrating gives us

$$\mu \int_m^0 aMM_\mu dp - \int_m^0 a^3 M_p M_{\mu p} dp = - \int_m^0 aM^2 dp \tag{3.9}$$

hence

$$G_\mu(\mu, \lambda) = - \int_m^0 aM^2 dp < 0$$

and the implicit function theorem for real analytic functions implies that $\lambda \mapsto \mu(\lambda)$ is real analytic. As this mapping is real analytic, it follows from the smooth dependence of solutions on parameters that the mapping $\lambda \mapsto M(\cdot, \lambda)$ is smooth, since $M(p, \lambda) = \phi(p, \lambda, \mu(\lambda))$ is the unique solution of the linear differential equation

$$(a^3 \phi_p)_p = -\mu a \phi \qquad \text{in } (m, 0),$$

with initial data $\phi(0) = 1, \phi'(0) = \frac{g}{a^3(0)}$. If $\dot{a}$ is the derivative of a with respect to λ then we have the relations

$$\dot{a} = \frac{\partial a}{\partial \lambda} = \frac{1}{2a}, \quad \dot{a}_p = -\frac{a_p}{2a^2} = -\frac{\gamma(p)}{2a^3}.$$

Differentiating equations (3.8) we get

$$(a^3 \dot{M}_p)_p + \frac{3}{2}(aM_p)_p = -\dot{\mu} aM - \mu \frac{M}{2a} - \mu a \dot{M}, \qquad p \in (m, 0), \tag{3.10a}$$

$$\frac{3}{2} aM_p + a^3 \dot{M}_p = (g - 2\omega c)\dot{M}, \qquad p = 0, \tag{3.10b}$$

$$\dot{M} = 0, \qquad p = m. \tag{3.10c}$$

Multiplying the above equation by M and (3.8a)–(3.8b) by $\dot{M}$, integrating both equations on $(m, 0)$ and subtracting the outcomes we obtain

$$\dot{\mu} \int_m^0 aM^2 dp = -\mu \int_m^0 \frac{M^2}{2a} dp + \frac{3}{2} \int_m^0 aM_p^2 dp.$$

Proposition 1. *The map $\lambda \mapsto \mu(\lambda)$ is increasing on any interval where it is negative and therefore the solution λ^* to $\mu(\lambda) = -1$, if it exists, is unique.*

Remark 1. For any given vorticity function γ there exists some value $\lambda > -2\Gamma_{min}$ such that $\mu(\lambda) > -1$.

Proof. For $\lambda > (g - 2\omega c) - 2\Gamma_{min}$, we have $a > \sqrt{g - 2\omega c}$, and also

$$\begin{aligned} M^2(0)(g - 2\omega c) = 2(g - 2\omega c) \int_m^0 MM_p dp \leq \sqrt{g - 2\omega c} \int_m^0 (g - 2\omega c) M_p^2(s) ds \\ + \sqrt{g - 2\omega c} \int_m^0 M^2(s) ds \leq \int_m^0 a^3 M_p^2(s) ds + \int_m^0 aM^2(s) ds, \end{aligned}$$

and so, for λ sufficiently large, we have

$$\mathbb{F}(M,\lambda) = \mu(\lambda) > -1.$$

$\square$

Corollary 1. *A solution λ^* to $\mu(\lambda) = -1$ exists if and only if*

$$\lim_{\lambda \downarrow -2\Gamma_{min}} \mu(\lambda) < -1.$$

Remark 2. If the vorticity $\gamma > 0$ is a positive constant, and sufficiently large such that

$$\gamma > \frac{1}{2} + \frac{(g-2\omega c)^2}{2|m|^3}\left(\frac{3}{2} - m\right)^2, \tag{3.11}$$

where m is the mass-flux of the resulting flow. Then $\mu(\lambda) > -1$ for all $\lambda > -2\Gamma_{min}$, hence local bifurcation cannot occur.

Proof. In this setting, $a > \sqrt{2\gamma(p-m)}$, and pursuing estimates along the lines of Remark 1, we find that $\mu(\lambda) > -1$ for all λ. $\square$

The following main result gives a sufficiency condition for the vorticity function γ to ensure that $\mu(\lambda) < -1$ for some λ, thereby ensuring that the linearised system (3.6) has a solution. The proof relies on estimates which follow directly along the lines of water waves without Coriolis effects, and we refer the reader to [6].

Proposition 2. *Suppose that*

$$g > 2\omega c + \frac{\sqrt{2}}{3}\gamma_\infty^{\frac{3}{2}}|p_1|^{\frac{1}{2}} + \frac{2\sqrt{2}}{5}\gamma_\infty^{\frac{1}{2}}|p_1|^{\frac{3}{2}}, \tag{3.12}$$

with $\gamma_\infty = \|\gamma\|_{C([m,0])}$ and $p_1 = \min\{p \in [m,0] : \Gamma(p) = \Gamma_{min}\}$. Then there exist non-trivial solutions to the linearised problem (3.8).

Remark 3. An immediate consequence of this result is that local bifurcation always occurs when the vorticity $\gamma \leq 0$, for in this case $p_1 = 0$. Furthermore, since $c < 0$ we see that condition (3.12) is slightly less-restrictive than the analogous condition which holds for water waves without Coriolis effects [6].

4. Local bifurcation

The linearised operator $\mathcal{F}_\phi = (\mathcal{F}_{1,\phi}, \mathcal{F}_{2,\phi})$, formed by taking the Fréchet derivative of $\mathcal{F}$ with respect to ϕ, is given at $\phi = 0$ by

$$\mathcal{F}_{1,\phi}(0,\lambda) = \partial_{pp} + H_p^2\partial_q^2 + 3\gamma(p)H_p^2\partial_p \quad \text{in } \Omega, \tag{4.1}$$

$$\mathcal{F}_{2,\phi}(0,\lambda) = 2\left((g-2\Omega c)\lambda^{-1} - \lambda^{\frac{1}{2}}\partial_p\right)\Big|_T. \tag{4.2}$$

It follows immediately that a solution f to the linear eigenvalue problem (3.8) belongs to the nullspace of $\mathcal{F}_\phi(0,\lambda)$.

4.1. The null space of $\mathcal{F}_w$. We showed in Section 3.2 that when (3.12) holds there exists a unique value $\lambda^* > -2\Gamma_{min}$ with $\mu(\lambda^*) = -1$. It follows that the null space of $\mathcal{F}_w(0, \lambda^*)$ contains at least one element $m^*(q,p) = M(p)\cos(q)$, where $M \in C^{2,\alpha}([m, 0])$ is the unique eigenfunction of (3.8) corresponding to the eigenvalue $\mu(\lambda^*) = -1$. In fact the null space is generated by this element, and so is one-dimensional: this follows from uniqueness properties of the minimiser of the Rayleigh quotient (3.7), cf. [6].

4.2. The range. First, we show that the derivative $\mathcal{F}_\phi : X \to Y$ is a Fredholm operator of index zero for all $\lambda > -2\Gamma_{min}$. This property will enable us to identify when $\lambda = \lambda^*$ the image of $\mathcal{F}_\phi$ and the complement of it.

Lemma 4.1. Given $\lambda > -2\Gamma_{min}$, the operator $\mathcal{F}_\phi(0, \lambda) : X \to Y$ is a Fredholm operator of index zero.

Proof. Let thus $\lambda > -2\Gamma_{min}$ be given. Then we can write

$$\mathcal{F}_\phi(0, \lambda) = -2\lambda^{-1/2}(L_0, \operatorname{tr}\partial_p) + 2(g - 2\omega c)\lambda^{-1}(0, \operatorname{tr}),$$

whereby $L_0 := -2^{-1}\lambda^{1/2}\big(\partial_{pp} + H_p^2\partial_q^2 + 3\gamma(p)H_p^2\partial_p\big) \in \mathcal{L}(X, Y_1)$. By standard elliptic theory [17], it follows that $(L_0, \operatorname{tr}\partial_p) : X \to Y$ is an isomorphism. Moreover, because the embedding $C^{1+\alpha}(\mathbb{S}) \hookrightarrow C^\alpha(\mathbb{S})$ is compact, the operator $(0, \operatorname{tr}) : X \to Y$ is itself compact. Consequently, $\mathcal{F}_\phi(0, \lambda)$ is a Fredholm operator with the same index as $(L_0, \operatorname{tr}\partial_p)$. $\square$

We are now in the position of identify the range of $\mathcal{F}_\phi$ at $\lambda = \lambda^*$.

Lemma 4.2. The vector $(f, \varphi) \in Y$ belongs to $\operatorname{Im}\mathcal{F}_\phi(0, \lambda^*)$ if and only if we have

$$\int_\Omega a^3 m^* f \, d(q,p) + \int_{\mathbb{S}} \frac{a^2 m^* \varphi}{2} \, dq = 0, \tag{4.3}$$

whereby m^* is a vector which spans the kernel of $\mathcal{F}_\phi(0, \lambda^*)$ and $a = a(\lambda^*)$.

Proof. Assume first that $(f, \varphi) \in \operatorname{Im}\mathcal{F}_\phi(0, \lambda^*)$. Then, there exists a vector $h \in X$ such that $\mathcal{F}_\phi(0, \lambda^*)[h] = (f, \varphi)$. Using integration by parts, we find by using the formulation (3.4), that

$$\begin{aligned}
&\int_\Omega a^3 m^* f \, d(q,p) + \int_{\mathbb{S}} \frac{a^2 m^* \varphi}{2} \, dq \\
&= \int_\Omega (a^3 h_p)_p m^* + a h_{qq} m^* \, d(q,p) + \int_{\mathbb{S}} \frac{a^2 m^* \varphi}{2} \, dq \\
&= -\int_\Omega a^3 h_p m_p^* + a h_q m_q^* \, d(q,p) + (g - 2\omega c)\int_{\mathbb{S}} h m^* \, dq.
\end{aligned} \tag{4.4}$$

Additionally, we know that $\mathcal{F}_\phi(0, \lambda^*)[m^*] = 0$, and therefore

$$\begin{aligned}
0 &= \int_\Omega (a^3 m_p^*)_p h + a m_{qq}^* h \, d(q,p) + (g - 2\omega c)\int_{\mathbb{S}} m^* h \, dq - \int_{\mathbb{S}} a^3 m_p^* h \, dq \\
&= -\int_\Omega a^3 h_p m_p^* + a h_q m_q^* \, d(q,p) + (g - 2\omega c)\int_{\mathbb{S}} h m^* \, dq.
\end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5) we obtain that (f,φ) satisfy (4.3). Observing that the relation (4.3) defines a closed subspace of Y which has codimension one and which contains $\operatorname{Im}\mathcal{F}_\phi(0,\lambda^*)$, we obtain in view of $\lambda=\lambda^*$ and of Lemma 4.1, that $\operatorname{Im}\mathcal{F}_\phi(0,\lambda^*)$ is characterized by the property (4.3). This completes our argument. □

4.3. The transversality condition. The last remaining step before we may apply Theorem 3.1 is to check that the transversality condition (iii) is satisfied. We infer from (4.1) and (4.2) that

$$\mathcal{F}_{1,\phi\lambda}(0,\lambda^*)=-\frac{1}{a^4}\partial_q^2-\frac{3\gamma}{a^4}\partial_p \quad \text{in } \Omega, \tag{4.6}$$

$$\mathcal{F}_{2,\phi\lambda}(0,\lambda^*)=-\frac{2(g-2\omega c)}{(\lambda^*)^2}\,\mathrm{tr}-\frac{1}{(\lambda^*)^{1/2}}\,\mathrm{tr}\,\partial_p, \tag{4.7}$$

with a being evaluated at λ^* in (4.6). Therefore, we have that

$$\begin{aligned}
&\int_\Omega a^3m^*\mathcal{F}_{1,\phi\lambda}(0,\lambda^*)\,d(q,p)+\int_{\mathbb{S}}\frac{a^2m^*\mathcal{F}_{2,\phi\lambda}(0,\lambda^*)}{2}\,dq\\
&=\int_\Omega-\frac{m^*m^*_{qq}}{a}-\frac{3\gamma m^*m^*_p}{a}\,d(q,p)+\int_{\mathbb{S}}-\frac{(g-2\Omega c)(m^*)^2}{\lambda^*}-\frac{(\lambda^*)^{1/2}m^*m^*_p}{2}\,dq\\
&=\int_\Omega\frac{|m^*_q|^2}{a}-3a_pm^*m^*_p\,d(q,p)-\int_{\mathbb{S}}\frac{3(g-2\Omega c)(m^*)^2}{2\lambda^*}\,dq,
\end{aligned}\tag{4.8}$$

where we used the fact that $\mathcal{F}_{\phi\lambda}(0,\lambda^*)[m^*]=0$ for the last equality. Similarly, we find that

$$\begin{aligned}
&\int_\Omega a_pm^*m^*_p\,d(q,p)\\
&=-\int_\Omega a|m^*_p|^2d(q,p)-\int_\Omega am^*m^*_{pp}d(q,p)+\int_{\mathbb{S}}am^*m^*_p\,dq\\
&=\frac{(g-2\Omega c)}{\Lambda^*}\int_{\mathbb{S}}|m^*|^2\,dq-\int_\Omega a|m^*_p|^2d(q,p)-\int_\Omega m^*\Big(-3a_pm^*_p-\frac{m^*_{qq}}{a}\Big)d(q,p)
\end{aligned}$$

and so

$$\begin{aligned}
&\int_\Omega a_pm^*m^*_p\,d(q,p)\\
&=-\frac{(g-2\Omega c)}{2\lambda^*}\int_{\mathbb{S}}|m^*|^2\,dq+\int_\Omega\frac{a|m^*_p|^2}{2}d(q,p)+\int_\Omega\frac{|m^*_q|^2}{2a}d(q,p).
\end{aligned}\tag{4.9}$$

Combining (4.8) and (4.9), we end up with

$$\begin{aligned}
&\int_\Omega a^3m^*\mathcal{F}_{1,\phi\lambda}(0,\lambda^*)\,d(q,p)+\int_{\mathbb{S}}\frac{a^2m^*\mathcal{F}_{2,\phi\lambda}(0,\lambda^*)}{2}\,dq\\
&=-\int_\Omega\frac{|m^*_q|^2}{2a}+\frac{3a|m^*_p|^2}{2}\,d(q,p)<0.
\end{aligned}\tag{4.10}$$

We have proved the following lemma.

Lemma 4.3 (The transversality condition). For each $0 \neq m^* \in \operatorname{Ker} \mathcal{F}_\phi(0, \lambda^*)$, we have

$$\mathcal{F}_{\phi\lambda}(0, \lambda^*)[m^*] \notin \operatorname{Im} \mathcal{F}_\phi(0, \lambda^*). \tag{4.11}$$

Compiling the preceding results, we deduce that if the vorticity distribution is such that (3.12) holds, then the requisite criteria of the Crandall-Rabinowitz local bifurcation theorem are fulfilled. Accordingly we have proven Theorem 1.1, ensuring the existence of non-trivial small amplitude equatorial water waves which possess the given vorticity distribution.

5. Dispersion relations

In this Section we derive the dispersion relations for small-amplitude equatorial wind waves in the special cases of irrotational waves, and waves which have constant non-zero vorticity. The dispersion relation provides a formula describing how the phase-speed c of the wave varies with respect to certain physical parameters, such as the fixed mean-depth d of the flow, the wavelength L, and the constant vorticity distribution γ [6, 12, 13, 19]. Furthermore, the dispersion relation will include the horizontal speed of the flow at the bottom of the fluid layer, which we label u_d. In the particular settings of irrotational and constant vorticity flows we can explicitly compute the critical value $\sqrt{\lambda^*}$ at which bifurcation occurs, and in the process we elicit the dispersion relation using the identity $\sqrt{\lambda^*} = u_0 - c$, where u_0 is the horizontal speed of the laminar flow at the surface at the point of bifurcation.

5.1. Irrotational flow.

For irrotational flow we have $a = \sqrt{\lambda}$, and equations (3.8) become

$$M_{pp} = \lambda^{-1} M \qquad \text{in} \quad m < p < 0 \tag{5.1a}$$

$$\lambda^{3/2} M_p = (g - 2\omega c) M \qquad \text{on} \quad p = 0 \tag{5.1b}$$

$$M = 0 \qquad \text{on} \quad p = m. \tag{5.1c}$$

The general solution of (5.1a) which satisfies (5.1c) is

$$M(p) = A \sinh\left(\frac{p - m}{\sqrt{\lambda}}\right) \qquad m < p < 0.$$

To satisfy (5.1b) we must have

$$\tanh\left(\frac{-m}{\sqrt{\lambda^*}}\right) = \frac{\lambda^*}{g - 2\omega c}.$$

Now, from (2.6), we compute

$$m = -d\sqrt{\lambda^*},$$

and since the laminar flow is irrotational, $u_z = 0$, we must have $u_0 = u_d$. Therefore

$$\sqrt{\lambda^*} = u_d - c = \sqrt{(g - 2\omega c)\tanh d}.$$

and

$$c^2 - 2(u_d - \omega \tanh d)c + u_d^2 - g \tanh d = 0.$$

We find that the unique negative root of this quadratic is given by

$$c = u_d - \omega \tanh d - \sqrt{(g - 2u_d\omega)\tanh d + \omega^2 \tanh^2 d}. \tag{5.2}$$

Taking into account the scaling (2.11) we see that, for waves with period L, the dispersion relation is given by

$$c = u_d - \frac{\omega}{\kappa}\tanh(\kappa d) - \sqrt{\frac{g - 2u_d\omega}{\kappa}\tanh(\kappa d) + \left(\frac{\omega}{\kappa}\right)^2 \tanh^2(\kappa d)}. \tag{5.3}$$

From (5.3) we see that waves of different lengths travel at different speeds, this is the dispersive effect.

5.2. Constant vorticity. For constant vorticity $\gamma \neq 0$ we have $a = \sqrt{\lambda + 2\gamma p}$, and making the substitution

$$M(p) = \frac{1}{a} M_0\left(\frac{a}{\gamma}\right)$$

transforms (3.8) into $M_0'' = M_0,\ m \leq p \leq 0$, with $M_0(m) = 0$. The solution $M(p)$ of (3.8) is therefore given by

$$M(p) = \frac{1}{\sqrt{\lambda^* + 2\gamma p}} \sinh\left(\frac{\sqrt{\lambda^* + 2\gamma p} - \sqrt{\lambda^* + 2\gamma m}}{\gamma}\right)$$

where the boundary condition (3.8b) is satisfied if

$$\frac{\lambda^*}{\gamma\sqrt{\lambda^*} + g - 2\omega c} = \tanh\left(\frac{\sqrt{\lambda^*} - \sqrt{\lambda^* + 2\gamma m}}{\gamma}\right). \tag{5.4}$$

By (3.2) the laminar flow is given by

$$H(p;\lambda) = \frac{\sqrt{\lambda + 2\gamma p} - \sqrt{\lambda + 2\gamma m}}{\gamma}.$$

In the physical variables, the laminar flow with vorticity γ has $v \equiv 0$ and also

$$(u(z) - c) - (u - c)|_{flat\ surface} = \int_0^z u_z dz = \gamma z,$$

therefore

$$(u(z) - c, v(z)) = (\sqrt{\lambda^*} + \gamma z, 0).$$

In particular, if $u_d = u(-d)$ is the constant horizontal speed at the bottom of the fluid layer, we have

$$\sqrt{\lambda^*} = u_d - c + \gamma d. \tag{5.5}$$

By (2.6) we have

$$m = \int_{-d}^{0} (c - u(z))dz = -\sqrt{\lambda^*}z - \frac{\gamma}{2}z^2\Big|_{-d}^{0} = -\sqrt{\lambda^*}d + \frac{\gamma}{2}d^2,$$

which we solve to get

$$d = \frac{\sqrt{\lambda^*} - \sqrt{\lambda^* + 2\gamma m}}{\gamma} > 0.$$

Hence relation (5.4) may be expressed as

$$\lambda^* - \gamma \tanh d \sqrt{\lambda^*} - g \tanh d + 2\omega c \tanh d = 0,$$

and applying (5.5) we get

$$c^2 + (\gamma \tanh d + 2\omega \tanh d - 2(u_d + \gamma d))c + (u_d + \gamma d)^2 - (\gamma(u_d + \gamma d) + g) \tanh d = 0.$$

The unique negative root $c < 0$ of the above quadratic equation take the form

$$c = u_d + \gamma d - \frac{(\gamma + 2\omega) \tanh d}{2} - \sqrt{\left(\frac{(\gamma + 2\omega) \tanh d}{2}\right)^2 + (g - 2\omega(u_d + \gamma d)) \tanh d}, \tag{5.6}$$

where the choice of signs is dictated by (2.2). We note that (5.6) is the correct form of the dispersion relation, and it differs slightly from the relation provided in [3] due to a minor error in [3]. We see that upon letting $\gamma \to 0$ that we recover the dispersion relation (5.2) for irrotational flow. Once more applying the scaling (2.11) the dispersion relation (5.6) becomes

$$c = u_d + \gamma d - \frac{(\gamma + 2\omega) \tanh(\kappa d)}{2\kappa} - \sqrt{\left(\frac{(\gamma + 2\omega) \tanh(\kappa d)}{2\kappa}\right)^2 + (g - 2\omega(u_d + \gamma d)) \frac{\tanh(\kappa d)}{\kappa}}.$$

We note that (5.4) provides us with a way of inferring a necessary and sufficient condition for local bifurcation to occur when the vorticity is positive and constant— when the vorticity is negative, (3.12) implies that local bifurcation always occurs. Furthermore, (3.11) gives us a condition on $\gamma > 0$ where bifurcation cannot occur. If local bifurcation occurs, then

$$f(\lambda) = \frac{\lambda}{\gamma\sqrt{\lambda} + g - 2\omega c} - \tanh\left(\frac{\sqrt{\lambda} - \sqrt{\lambda + 2\gamma m}}{\gamma}\right)$$

has a unique root. As $\gamma > 0$, we have $\lim_{\lambda \to \infty} f(\lambda) = \infty$. Furthermore, $f : (-2\gamma m, \infty) \to \mathbb{R}$ is a strictly increasing function. Therefore, if $\lim_{\lambda \to -2\gamma m} f(\lambda) < 0$ then bifurcation occurs, otherwise it cannot. So a necessary and sufficient condition for the existence of equatorial wind waves with constant positive vorticity is that

$$\frac{2\gamma |m|}{\gamma\sqrt{2\gamma |m|} + g - 2\omega c} < \tanh\left(\sqrt{\frac{2|m|}{\gamma}}\right). \tag{5.7}$$

For waves of general wavelength L, the condition above under the scaling (2.11) becomes

$$\frac{2\gamma |m| \kappa}{\gamma\sqrt{2\gamma |m|} + g - 2\omega c} < \tanh\left(\kappa\sqrt{\frac{2|m|}{\gamma}}\right),$$

and we see that for waves of very short wavelength, with $\kappa \gg 1$, bifurcation cannot occur.

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