# Statistical Properties of First-Order Bang-Bang PLL With Nonzero Loop Delay

Byungjin Chun, *Member, IEEE*, and Michael P. Kennedy, *Fellow, IEEE*

***Abstract*—A method to solve the stationary state probability is presented for the first-order bang-bang phase-locked loop (BBPLL) with nonzero loop delay. This is based on a delayed Markov chain model and a state flow diagram for tracing the state history due to the loop delay. As a result, an eigenequation is obtained, and its closed form solutions are derived for some cases. After obtaining the state probability, statistical characteristics such as mean gain of the binary phase detector and timing error variance are calculated and demonstrated.**

***Index Terms*—Bang-bang phase-locked loop (BBPLL), delayed Markov chain, nonzero loop delay.**

## I. Introduction

THE BANG-BANG phase-locked loop (BBPLL) is often used in communication systems, such as clock and data recovery (CDR) [1], mainly motivated by its high-speed operation capability. In general, the BBPLL does not allow for a linear system approach to characterize its performance due to its nonlinear element, the binary phase detector (BPD). Instead, the Markov chain theory [2] can be a tool to analyze the performance. In fact, [3] exploited it to derive the state probability of the first-order loop with *zero* loop delay and then solved the mean gain of the BPD. However, if components delay in the loop amounts to the order of clock cycles and/or some clock cycles of delay are intentionally introduced to the loop for pipelining, its effect on the system performance should be taken into account for fair characterization. Although [4] considered the delay, it was limited to the deterministic system setup.

The objective of this paper is to extend the work of [3] to the case of *nonzero* loop delay. However, the stability issue associated with loop delay is beyond in the scope of this paper.

For notational clearness, vectors and matrices are denoted as boldface lower case letters and boldface upper case letters, respectively. The $[\cdot]^T$ means the transpose operator. The $\mathcal{Z}$ means the set of integers. $P\{E\}$ and $P\{E_1|E_2\}$ mean the probability of event $E$ and the conditional probability of event $E_1$ given event $E_2$, respectively. The probabilities are assumed to be stationary without any comment.

In Sections II and III, the system model and the proposed method are described, respectively. Then, in Sections IV and V, simulation results and concluding remarks are given.

Manuscript received September 26, 2007; revised January 25, 2008. Current version published October 15, 2008. This work was supported by the Enterprise-Ireland. This paper was recommended by Associate Editor S. Callegari.

The authors are with the Department of Microelectronic Engineering, University College Cork, Cork, Ireland (e-mail: bjaychun@yahoo.co.kr).

Digital Object Identifier 10.1109/TCSII.2008.924371

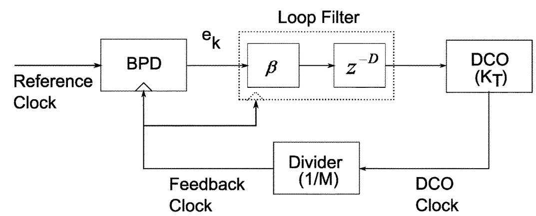


Fig. 1. Block diagram of the considered digital BBPLL with loop delay $D$.

## II. System Model

Consider a first-order BBPLL shown in Fig. 1. The BBPLL is composed of a BPD, a loop filter with gain $\beta$ and delay $D$, a digitally controlled oscillator (DCO) with gain $K_T$, and a $1/M$ frequency divider. The BPD and loop filter are clocked by the output of the frequency divider (feedback clock). This diagram is basically the same digital BBPLL discussed in [3] and [4]. However, it was modified by just ignoring the integral path in the loop filter but keeping nonzero delay (or latency) due to the pipelining.

Referring to [3][1] and Fig. 1, the dynamics of the BBPLL can be expressed by the following set of equations:

$$\Delta t^*_{k+1} = \Delta t^*_k - K e_{k-D} \tag{1}$$
$$\Delta t_k = \Delta t^*_k + \eta_k \tag{2}$$
$$e_k = \mathrm{sgn}(\Delta t_k). \tag{3}$$

Here, $k$ is the feedback clock time index, $\Delta t_k \equiv t_{r,k} - t_{f,k}$ is the difference between the rising edges of the jittered reference $(t_{r,k})$ and feedback clocks $(t_{f,k})$, $\Delta t^*_k$ is the value of $\Delta t_k$ in the case of unjittered reference, $\eta_k$ is the temporally uncorrelated reference clock jitter with the probability distribution function (pdf) $f_\eta(\eta)$, $e_k$ is the binary timing error signal, which is 1 if $\Delta t_k > 0$ and $-1$, otherwise, and $K \equiv \beta M K_T$. The $\Delta t^*_k$ can take only discrete values $Kn + t^*_0$ with $n \in \mathcal{Z}$, but $t^*_0$ is put to zero assuming the BBPLL is precisely centered when locked.

$n$ is called the state number (shortly, *state*), and the event that the state at time $k$ is $n$ is denoted as $s_k = n$. The state probability $q_n \equiv P\{s_k = n\}$ plays a central role in determining various statistical properties of the BBPLL. $q_n$ can be found by modelling the BBPLL as a *delayed* Markov chain with the state transition probability $p_{m|n} \equiv P\{s_{k+1} = m | s_k = n\}$. The delayed Markov chain in this paper is defined as a state-space model whose state transition probability is determined not by the present state but by the state $D$ times before.[2]

[1]Equation (5) in this reference is misleading, although it does not affect on the overall context of the paper. It may suggest that the jitter accumulates in $\Delta t_k$ as the time elapses, which is not the case due to the binary decision of the BPD.

[2]$p_{m|n}$ is actually an expected value over all possible states $D$ times before conditioned on the current state $n$.

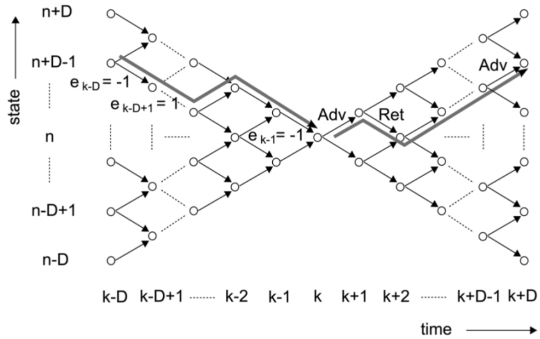


Fig. 2. State flow diagram around the current event $s_k = n$ (the center node) in the case of nonzero loop delay $D$. The two thick arrow lines represent examples of the past state flow (left) and the future state flow (right), respectively. In this example, the error signals $e_{k-D} = -1$, $e_{k-D+1} = 1, \ldots$, and $e_{k-1} = -1$ along the past state flow drive the future states to advance, retard, …, advance, in the order.

When $D = 0$, it is straightforward to calculate $p_{m|n}$ given $f_\eta(\eta)$, and the work in [3] derived $q_n$ based on the corresponding Markov chain model. For nonzero $D$, however, $p_{m|n}$ is not so obvious. Therefore, a new method, rather than directly relying on $p_{m|n}$, is needed to solve $q_n$, as will be discussed in Section III.

## III. Derivation of State Probability

### A. General Formulation

A *state flow diagram* around the current event $s_k = n$ is shown in Fig. 2. This shows how each state (marked as a node) flows with a state transition (marked as an arrow) as the time elapses. With *nonzero* loop delay $D$ ($D = 1, 2, 3, \ldots$), all possible *past state flows* from the time $k - D$ reaching the current event are limited to $2^D$ exclusive ones inside the fan-shaped area, and *vice versa* for all possible *future state flows* from the current event to the time $k + D$.

Consider a future state flow taking the path $(s_k = n) \rightarrow (s_{k+1} = n_1) \rightarrow \cdots \rightarrow (s_{k+D} = n_D)$. Due to the stationary condition, its probability can be expressed (defined) simply as $P\{n, n_1, \ldots, n_D\} \equiv r_{n,n_1,\ldots,n_D}$ with arguments arranged in order. Assume a past state flow taking the path $(s_{k-D} = m_D) \rightarrow \cdots \rightarrow (s_{k-1} = m_1) \rightarrow (s_k = n)$. In the same way as before, its probability can be expressed as $P\{m_D, \ldots, m_1, n\} = r_{m_D,\ldots,m_1,n}$.

Using the total probability theorem [2], the above two probabilities can be related through

$$r_{n,n_1,\ldots,n_D} = \sum_{(m_D,\ldots,m_1)\in S} r_{m_D,\ldots,m_1,n} \times P\{n, n_1, \ldots, n_D | m_D, \ldots, m_1, n\}. \quad (4)$$

Here, $P\{n, n_1, \ldots, n_D | m_D, \ldots, m_1, n\}$ means the conditional probability of the future state flow given the past state flow, and $S$ means the set of all the legitimate past state sequence $(m_D, \ldots, m_1)$.

Note that, according to (1), it is the past error signal sequence $(e_{k-D}, e_{k-D+1}, \ldots, e_{k-1})$ that drives the future state flow $n \rightarrow n_1 \rightarrow \cdots \rightarrow n_D$. Therefore, the conditional probability can be expressed as

$$\begin{aligned} &P\{n, n_1, \ldots, n_D | m_D, \ldots, m_1, n\} \\ &\quad = P\{e_{k-D} = h(n, n_1) | s_{k-D} = m_D\} \\ &\qquad \times P\{e_{k-D+1} = h(n_1, n_2) | s_{k-D+1} = m_{D-1}\} \\ &\qquad \times \cdots \times P\{e_{k-1} = h(n_{D-1}, n_D) | s_{k-1} = m_1\} \quad (5) \end{aligned}$$

where $h(n_i, n_{i+1})$ is defined as $-1$ if $n_i < n_{i+1}$ (i.e., state *advance*) and 1 if $n_i > n_{i+1}$ (i.e., state *retard*) with $i = 0, 1, \ldots, D-1$ and $n_0 \equiv n$.

Meanwhile, the conditional pdf of $\Delta t_k = \tau$ given $s_k = n$ can be expressed as $f_\eta(\tau - Kn)$ from (2). Therefore, denoting $A_n \equiv P\{e_k = -1 | s_k = n\}$ (i.e., probability of state *advance* from $n$) and $R_n \equiv P\{e_k = 1 | s_k = n\}$ (i.e., probability of state *retard* from $n$), and using (3), they can be written as

$$\begin{cases} A_n = \int_{-\infty}^{0} f_\eta(\tau - Kn) d\tau \\ R_n = \int_{0}^{\infty} f_\eta(\tau - Kn) d\tau. \end{cases} \quad (6)$$

Using (6), (5) can be calculated as

$$H_{m_D} \times H_{m_{D-1}} \times \cdots \times H_{m_1} \; (\equiv \mathcal{H}_{m_D,\ldots,m_1}) \quad (7)$$

with

$$H_{m_i} \equiv \begin{cases} A_{m_i}, & \text{if } n_{D-i} < n_{D-i+1}, \; (i = D, \ldots, 1), \\ R_{m_i}, & \text{if } n_{D-i} > n_{D-i+1}. \end{cases}$$

Consequently, from (4)–(7), we have

$$r_{n,n_1,\ldots,n_D} = \sum_{(m_D,\ldots,m_1)\in S} \mathcal{H}_{m_D,\ldots,m_1} r_{m_D,\ldots,m_1,n}. \quad (8)$$

Assuming the number of states is $N$ ($N$ may be infinite), the above equation forms a system of $2^D N$ equations with respect to $2^D N$ unknowns $\{r_{n,n_1,\ldots,n_D}\}$[3] as $n$ takes $N$ different values.

The system of equations can be put in a matrix equation

$$\mathbf{r} = \mathbf{H}\mathbf{r} \quad (9)$$

where $\mathbf{r}$ is a $2^D N$-by-1 vector composed of $\{r_{n,n_1,\ldots,n_D}\}$, and $\mathbf{H}$ is a $2^D N$-by-$2^D N$ matrix relating elements of $\mathbf{r}$ according to (8). Recognizing that each element of $\mathbf{H}$ is nonnegative (therefore, $\mathbf{H}$ is a nonnegative matrix), the eigenequation (9) has a nonnegative eigenvector (i.e., the desired solution) associated with the largest eigenvalue (here, 1) by Perron–Frobenius theorem [5].

Once $\mathbf{r}$ is obtained, the state probability is calculated as

$$q_n = \sum_{(n_D,\ldots,n_1)\in S} r_{n,n_1,\ldots,n_D}. \quad (10)$$

In general, the eigenvector solution of (9) may require a numerical method (e.g., Matlab) due to its high complexity.[4] However, we can find a closed-form solution when $D$ is small, as will be described in Section III-B.

[3]These unknowns coincide with $\{r_{m_D,\ldots,m_1,n}\}$ as $n$ varies.

[4]The complexity may be reduced through an approximation of $A_n$ and $R_n$ under small jitter variance conditions as described in Section III-D.

### B. Case of $D = 1$

With $D = 1$, $r_{n,n_1,\ldots,n_D}$ $(= r_{n,n_1})$ in (8) can take only two variables $r_{n,n\pm1}$, and $\mathcal{H}_{m_D,\ldots,m_1}$ $(= H_{m_1})$ takes only four values $A_{n\pm1}$ and $R_{n\pm1}$ according to $m_1$ $(= n \pm 1)$ and its relation to $n$ [see (7)]. Defining

$$\begin{cases} r_n^+ \equiv r_{n,n+1} = q_n p_{n+1|n} \\ r_n^- \equiv r_{n,n-1} = q_n p_{n-1|n} \end{cases} \tag{11}$$

for notational convenience, (8) can be expressed as

$$\begin{cases} r_n^+ = A_{n-1} r_{n-1}^+ + A_{n+1} r_{n+1}^- \\ r_n^- = R_{n-1} r_{n-1}^+ + R_{n+1} r_{n+1}^-. \end{cases} \tag{12}$$

Also, from (10) and (11), we obtain

$$q_n = r_n^+ + r_n^-. \tag{13}$$

To calculate $r_n^+$ and $r_n^-$ recursively, we need to change (12) to a causal form with respect to $n$. Noting that the variables in (12) can be partitioned to two groups $\{r_{n-1}^+, r_n^-\}$ and $\{r_n^+, r_{n+1}^-\}$, and the latter is the one state-advanced version of the former, we can set up a recursive equation in a causal form as follows:

$$\mathbf{x}_{n+1} = \mathbf{S}_n \mathbf{x}_n \tag{14}$$

with

$$\mathbf{S}_n \equiv (1/R_{n+1}) \begin{bmatrix} 1 - R_{n-1} - A_{n+1} & A_{n+1} \\ -R_{n-1} & 1 \end{bmatrix}$$
$$\mathbf{x}_n \equiv [\, r_{n-1}^+ \quad r_n^- \,]^T.$$

Here, we used $A_n + R_n = 1$ from (6).

If $f_\eta(\eta)$ is symmetrical around 0, it is obvious that $A_{-n} = 1 - A_n$, $R_{-n} = 1 - R_n$, $r_{-n}^+ = r_n^-$, and $q_{-n} = q_n$ for any $n$. Then, by putting $n = 0$ into (12) and (13), and using the above relations, we can show that $r_0^+ = r_0^- = r_{-1}^+ = r_1^- = q_0/2$. As a result, by putting $\mathbf{x}_0 = [\, r_{-1}^+ \quad r_0^- \,]^T = (q_0/2)\,[1 \quad 1]^T$ and calculating (14) recursively from $n = 0$ [then (13)], we obtain

$$r_{n-1}^+ = r_n^- = \prod_{i=0}^{n-1} \left( \frac{1 - R_{i-1}}{R_{i+1}} \right) \frac{q_0}{2} \tag{15}$$

$$q_n = q_{-n} = \left( \frac{1 - R_{n-1} + R_{n+1}}{R_{n+1}} \right) \prod_{i=0}^{n-1} \left( \frac{1 - R_{i-1}}{R_{i+1}} \right) \frac{q_0}{2} \tag{16}$$

for $n \geq 1$. Here, $q_0$ can be determined as

$$\left[ 1 + \sum_{n=1}^{\infty} \left( \frac{1 - R_{n-1} + R_{n+1}}{R_{n+1}} \right) \prod_{i=0}^{n-1} \left( \frac{1 - R_{i-1}}{R_{i+1}} \right) \right]^{-1} \tag{17}$$

from the normalization condition

$$\sum_{n=-\infty}^{\infty} q_n = q_0 + 2 \sum_{n=1}^{\infty} q_n = 1. \tag{18}$$

Futhermore, from (11), (15), and (16), $p_{n\pm1|n}$ are calculated as

$$\begin{cases} p_{n+1|n} = \frac{r_n^+}{q_n} = \frac{1 - R_{n-1}}{1 - R_{n-1} + R_{n+1}} \\ p_{n-1|n} = \frac{r_n^-}{q_n} = \frac{R_{n+1}}{1 - R_{n-1} + R_{n+1}}. \end{cases} \tag{19}$$

### C. Case of $D = 2$

With $D = 2$, $r_{n,n_1,\ldots,n_D}$ $(= r_{n,n_1,n_2})$ in (8) can take four variables $r_{n,n+1,n+2}$, $r_{n,n+1,n}$, $r_{n,n-1,n}$, and $r_{n,n-1,n-2}$. Following the same way as in $D = 1$ and defining

$$\begin{cases} r_n^{++} \equiv r_{n,n+1,n+2} = q_n p_{n+1|n} p_{n+2|n+1|n} \\ r_n^{+-} \equiv r_{n,n+1,n} = q_n p_{n+1|n} p_{n|n+1|n} \\ r_n^{-+} \equiv r_{n,n-1,n} = q_n p_{n-1|n} p_{n|n-1|n} \\ r_n^{--} \equiv r_{n,n-1,n-2} = q_n p_{n-1|n} p_{n-2|n-1|n} \end{cases} \tag{20}$$

(8) can be written as

$$\begin{cases} r_n^{++} = A_{n-2}A_{n-1} \cdot r_{n-2}^{++} + A_n A_{n+1} \cdot r_n^{+-} \\ \qquad + A_n A_{n-1} \cdot r_n^{-+} + A_{n+2} A_{n+1} \cdot r_{n+2}^{--} \\ r_n^{+-} = A_{n-2} R_{n-1} \cdot r_{n-2}^{++} + A_n R_{n+1} \cdot r_n^{+-} \\ \qquad + A_n R_{n-1} \cdot r_n^{-+} + A_{n+2} R_{n+1} \cdot r_{n+2}^{--} \\ r_n^{-+} = R_{n-2} A_{n-1} \cdot r_{n-2}^{++} + R_n A_{n+1} \cdot r_n^{+-} \\ \qquad + R_n A_{n-1} \cdot r_n^{-+} + R_{n+2} A_{n+1} \cdot r_{n+2}^{--} \\ r_n^{--} = R_{n-2} R_{n-1} \cdot r_{n-2}^{++} + R_n R_{n+1} \cdot r_n^{+-} \\ \qquad + R_n R_{n-1} \cdot r_n^{-+} + R_{n+2} R_{n+1} \cdot r_{n+2}^{--}. \end{cases} \tag{21}$$

Here, $p_{a|b|c}$ means $p_{a|b}$ conditioned again on the previous state $c$. Also, from (10) and (20), we obtain

$$q_n = r_n^{++} + r_n^{+-} + r_n^{-+} + r_n^{--}. \tag{22}$$

Equation (21) can be put as follows:

$$\mathbf{u}_n = \mathbf{U}_n \mathbf{w}_n + \mathbf{V}_n \mathbf{v}_n \tag{23}$$
$$\mathbf{v}_n = \mathbf{W}_n \mathbf{w}_n + \mathbf{X}_n \mathbf{v}_n \tag{24}$$

where

$$\mathbf{u}_n \equiv [\, r_n^{++} \quad r_n^{--} \,]^T, \quad \mathbf{v}_n \equiv [\, r_n^{+-} \quad r_n^{-+} \,]^T$$
$$\mathbf{w}_n \equiv [\, r_{n-2}^{++} \quad r_{n+2}^{--} \,]^T$$
$$\mathbf{U}_n \equiv \begin{bmatrix} A_{n-2}A_{n-1} & A_{n+2}A_{n+1} \\ R_{n-2}R_{n-1} & R_{n+2}R_{n+1} \end{bmatrix}$$
$$\mathbf{V}_n \equiv \begin{bmatrix} A_n A_{n+1} & A_n A_{n-1} \\ R_n R_{n+1} & R_n R_{n-1} \end{bmatrix}$$
$$\mathbf{W}_n \equiv \begin{bmatrix} A_{n-2}R_{n-1} & A_{n+2}R_{n+1} \\ R_{n-2}A_{n-1} & R_{n+2}A_{n+1} \end{bmatrix}$$
$$\mathbf{X}_n \equiv \begin{bmatrix} A_n R_{n+1} & A_n R_{n-1} \\ R_n A_{n+1} & R_n A_{n-1} \end{bmatrix}.$$

From (24), we have

$$\mathbf{v}_n = \mathbf{Y}_n \mathbf{w}_n \tag{25}$$

where $\mathbf{Y}_n \equiv (\mathbf{I}_{(2\times2)} - \mathbf{X}_n)^{-1} \mathbf{W}_n$ and $\mathbf{I}_{(2\times2)}$ are the $2\times2$ identity matrix, and, by putting (25) into (23), we obtain

$$\mathbf{u}_n = \mathbf{Z}_n \mathbf{w}_n \tag{26}$$

with $\mathbf{Z}_n \equiv \begin{bmatrix} z_{11,n} & z_{12,n} \\ z_{21,n} & z_{22,n} \end{bmatrix} \equiv \mathbf{U}_n + \mathbf{V}_n \mathbf{Y}_n$. In the similar way that (12) was changed to (14), we can convert (26) to

$$\mathbf{x}_{n+2} = \mathbf{S}_n \mathbf{x}_n \tag{27}$$

with $\mathbf{S}_n \equiv (1/z_{22,n}) \begin{bmatrix} z_{11,n}z_{22,n} - z_{12,n}z_{21,n} & z_{12,n} \\ -z_{21,n} & 1 \end{bmatrix}$ and $\mathbf{x}_n \equiv [\, r_{n-2}^{++} \quad r_n^{--} \,]^T$. Note that we need to iterate (27) from

TABLE I
APPROXIMATE SOLUTION OF $q_n$ FOR $D = 2$

| $n$ | $\mathbf{S}_n$ | $\mathbf{x}_n = \begin{bmatrix} r_{n-2}^{++} \\ r_n^{--} \end{bmatrix}$ | $\mathbf{u}_n = \begin{bmatrix} r_n^{++} \\ r_n^{--} \end{bmatrix}$ | $\mathbf{v}_n = \begin{bmatrix} r_n^{+-} \\ r_n^{-+} \end{bmatrix}$ | $q_n$ |
|---|---|---|---|---|---|
| 0 | $\begin{bmatrix} \frac{1}{A_1+1} & \frac{A_1}{A_1+1} \\ \frac{-A_1}{A_1+1} & \frac{2A_1+1}{A_1+1} \end{bmatrix}$ | $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ | $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ | $c_1 \begin{bmatrix} 2A_1 \\ 2A_1 \end{bmatrix}$ | $c_1(4A_1+2)$ |
| 1 | $\begin{bmatrix} \frac{1}{2(A_1+1)} & 0 \\ \frac{-2A_1-1}{2(A_1+1)} & 1 \end{bmatrix}$ | $c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ | $c_2 \begin{bmatrix} \frac{1}{2(A_1+1)} \\ 1 \end{bmatrix}$ | $c_2 \begin{bmatrix} \frac{1}{2R_1(A_1+1)} \\ \frac{A_1}{A_1+1} \end{bmatrix}$ | $c_2 \frac{4R_1A_1+3R_1+1}{2R_1(A_1+1)}$ |
| 2 | $\begin{bmatrix} \frac{A_1}{2} & 0 \\ \frac{A_1-2}{2} & 1 \end{bmatrix}$ | $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ | $c_1 \begin{bmatrix} \frac{A_1}{2} \\ 1 \end{bmatrix}$ | $c_1 \begin{bmatrix} \frac{R_1}{2} \\ \frac{A_1}{2R_1} \end{bmatrix}$ | $c_1 \frac{2R_1+1}{2R_1}$ |
| 3 | $\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$ | $c_2 \begin{bmatrix} \frac{1}{2(A_1+1)} \\ \frac{1}{2(A_1+1)} \end{bmatrix}$ | $c_2 \begin{bmatrix} 0 \\ \frac{1}{2(A_1+1)} \end{bmatrix}$ | $c_2 \begin{bmatrix} \frac{A_1}{2(A_1+1)} \\ 0 \end{bmatrix}$ | $c_2 \frac{1}{2}$ |
| 4 | $\begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$ | $c_1 \begin{bmatrix} \frac{A_1}{2} \\ \frac{A_1}{2} \end{bmatrix}$ | $c_1 \begin{bmatrix} 0 \\ \frac{A_1}{2} \end{bmatrix}$ | $c_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ | $c_1 \frac{A_1}{2}$ |
| $\geq 5$ | . | $c_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ | $c_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ | $c_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ | 0 |

$n = 0$ and 1 independently to obtain $\mathbf{x}_n$ for even $n$ and odd $n$, respectively. This happens because $D$ is an even number.

If $f_\eta(\eta)$ is symmetrical around 0, it is obvious $r_{-n}^{++} = r_n^{--}$, $r_{-n}^{+-} = r_n^{-+}$ and $q_{-n} = q_n$ for any $n$. Therefore, the initial conditions for even $n$ and odd $n$ can be set as $\mathbf{x}_0 = [r_{-2}^{++} \quad r_0^{--}]^T = c_1 [1 \quad 1]^T$ (put $n = 0$ in (21) to check), and $\mathbf{x}_1 = [r_{-1}^{++} \quad r_1^{--}]^T = c_2 [1 \quad 1]^T$ for some constants $c_1$ and $c_2$, respectively. From $\mathbf{x}_0$ and $\mathbf{x}_1$, $\mathbf{x}_n$ (therefore, $\mathbf{w}_n$, $\mathbf{u}_n$) and $\mathbf{v}_n$ can be calculated recursively using (27) and (25), respectively. Then, $q_n$ is obtained using (22).

Finally, we need to fix the constants $c_1$ and $c_2$ using some conditions to get the complete solution. One constant can be eliminated using a relationship from the total probability theorem [2]

$$q_n = \sum_m p_{n|m} q_m = p_{n|n+1} q_{n+1} + p_{n|n-1} q_{n-1} \tag{28}$$

for any $n$. Here, the state transition probabilities are obtained by $p_{n+1|n} = (r_n^{++} + r_n^{+-})/q_n$ and $p_{n-1|n} = (r_n^{--} + r_n^{-+})/q_n$ from (20). Another constant can be fixed by applying (18).

Due to the complexity in expressing each element of $\mathbf{S}_n$ in (27), simple closed-form expression for $q_n$ seems to be hard to find. Instead, an approximate closed-form solution is possible, as will be explained next.

### *D. Approximate Solution Example* ($D = 2$)

If $f_\eta(\eta)$ is symmetrical around 0 and has a small variance compared with $K^2$, $A_n$ and $R_n$ in (6) can be approximated as $A_0 = R_0 = 1/2$, $A_n = 0$, and $R_n = 1$ for $n \geq 2$, and $A_n = 1$ and $R_n = 0$ for $n \leq -2$. Then, it is enough to evaluate $\mathbf{U}_n$, $\mathbf{V}_n$, $\mathbf{W}_n$, $\mathbf{X}_n$, $\mathbf{Y}_n$, $\mathbf{Z}_n$, and $\mathbf{S}_n$ in (23)–(27) for $0 \leq n \leq 4$ only. The final $\mathbf{S}_n$ is listed in the second column of Table I. The remaining steps are similar to those explained in the previous subsection. At first, $\mathbf{x}_n$ is calculated recursively from $\mathbf{x}_0$ and $\mathbf{x}_1$ using (27). Then, the elements in $\mathbf{x}_n$ are rearranged to form $\mathbf{w}_n$ and $\mathbf{u}_n$, and $\mathbf{v}_n$ is calculated using (25).

As a result, an approximate solution of $q_n$ can be obtained by summing all of the elements in $\mathbf{u}_n$ and $\mathbf{v}_n$. The result of the above calculations is summarized in Table I. Here, the remaining constants can be fixed as $c_2 = c_1(A_1 + 1)$ and $c_1 = R_1/(10R_1A_1 + 8R_1 + 2)$ using (28) and (18), respectively.

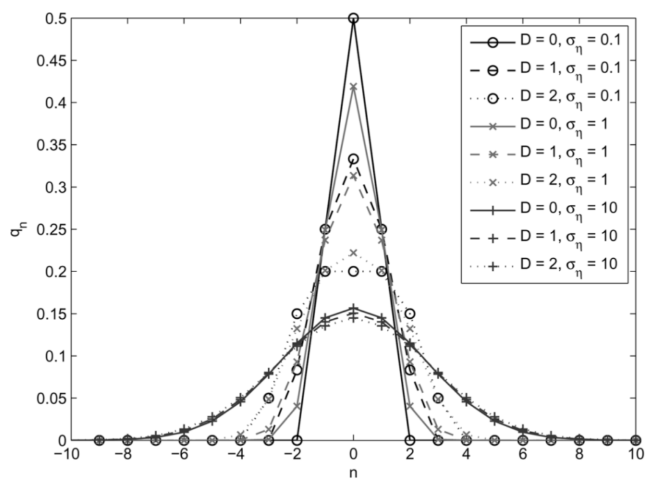


Fig. 3. Theoretically calculated stationary state probability ($q_n$).

## IV. SIMULATION RESULTS

The proposed method to solve $q_n$ is demonstrated by simulation for various loop delay and reference clock jitter conditions. The $f_\eta(\eta)$ was assumed to be Gaussian with zero mean and variance $\sigma_\eta^2$ (i.e., $f_\eta(\eta) = 1/(\sqrt{2\pi}\sigma_\eta)\exp\left(-\eta^2/\left(2\sigma_\eta^2\right)\right), -\infty < \eta < \infty$). Throughout the simulations, $K = 1$ was assumed and the Markov chain was limited to 21 states.

In Figs. 3 and 4, the theoretical and estimated $q_n$ are plotted for given conditions, respectively. To obtain the estimates, Monte Carlo (MC) method was applied with $10^5$ iterations per each condition. The two results show a good agreement, verifying the validity of the proposed method. The distributions tend to spread as $D$ and $\sigma_\eta$ increase. Also, the distributions become nearly insensitive to $D$ for larger $\sigma_\eta$.

In Fig. 5(a)–(c), the theoretical $q_n$ for $D = 2$ according to the precise solution (Section III-C) and the approximate solution (Section III-D) are compared with each other. They match almost perfectly for $\sigma_\eta = 0.1$ and 1, as shown in Fig. 5(a) and (b), respectively. However, their mismatch increases considerably as $\sigma_\eta$ reaches 10 [Fig. 5(c)].

Once $q_n$ is obtained, characteristics of the BBPLL such as the mean gain of the BPD ($K_{\mathrm{bpd}}$) and the stationary timing error variance $\left(\sigma_{\Delta t}^2\right)$ can be evaluated through $K_{\mathrm{bpd}} = 2f_{\Delta t}(0) = 2\sum_{n=-\infty}^{\infty} q_n f_\eta(-Kn)$ according to (6) in [3], and $\sigma_{\Delta t}^2 = K^2\sigma_q^2 + \sigma_\eta^2$ from (2), respectively.

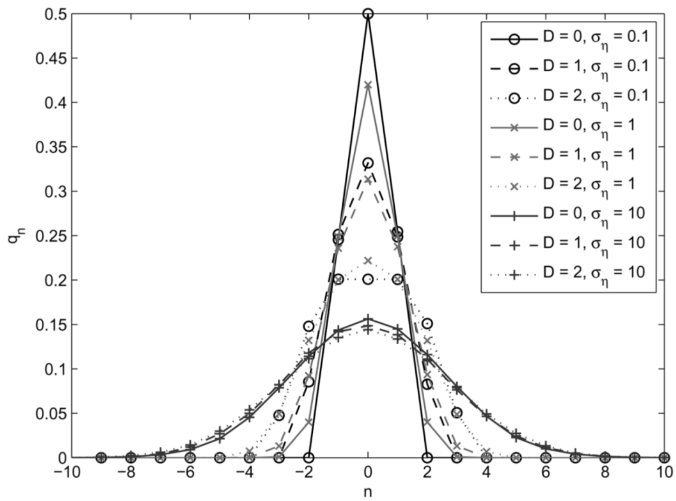


Fig. 4. Estimated stationary state probability $(q_n)$ by MC method.

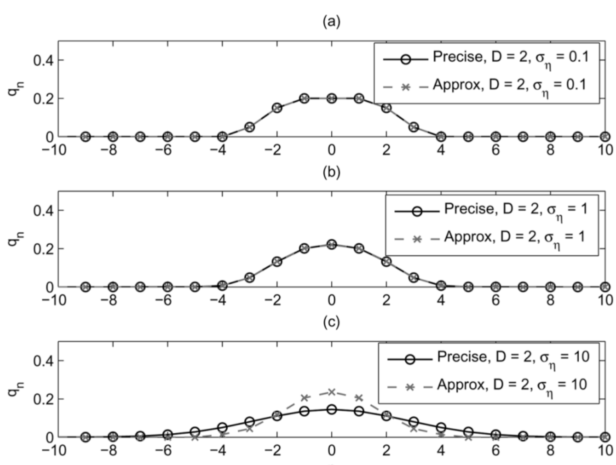


Fig. 5. Precise (solid line) and approximate (dashed line) $q_n$ for $D = 2$.

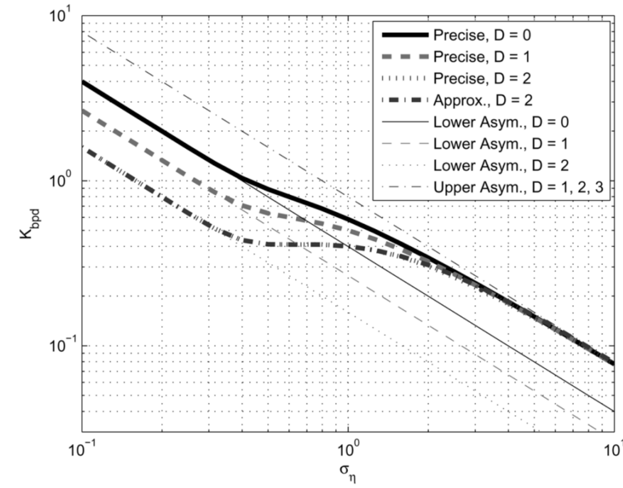


Fig. 6. Mean gain of the binary phase detector $(K_{\mathrm{bpd}})$.

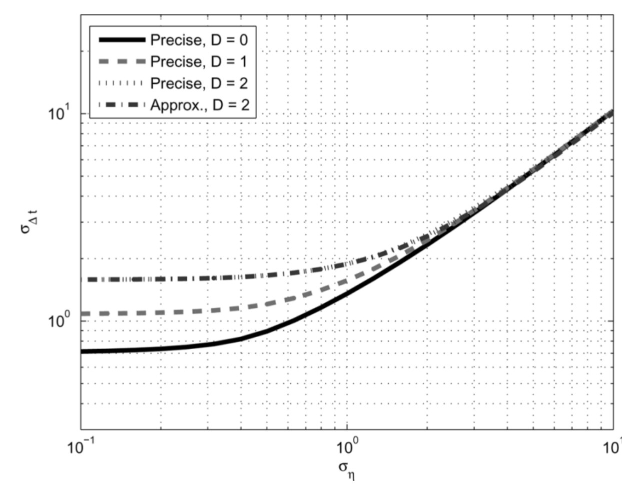


Fig. 7. Standard deviation of the timing error $(\sigma_{\Delta t})$.

Here, $f_{\Delta t}(\tau)$ is the pdf of $\Delta t$, and $\sigma_q^2$ is the variance of $q_n$. According to [3], the upper asymptotic $K_{\mathrm{bpd}}$ for $\sigma_\eta \gg K$ can be expressed as $K_{\mathrm{bpd}}^{(\mathrm{upper})} = 2/(\sqrt{2\pi}\sigma_\eta)$. Also, the lower asymptotic $K_{\mathrm{bpd}}$ for $\sigma_\eta \ll K$ can be written as $K_{\mathrm{bpd}}^{(\mathrm{lower})} = 2q_0 f_\eta(0) = 2q_0/(\sqrt{2\pi}\sigma_\eta)$ using the above expression of $K_{bpd}$. Here, $q_0$ for very small $\sigma_\eta$ can be well approximated as 1/2, 1/3, 1/5 for $D = 0$, 1, 2, respectively (see [4, Table I]).

In Figs. 6 and 7, the theoretical $K_{\mathrm{bpd}}$ and $\sigma_{\Delta t}$ are plotted as a function of $\sigma_\eta$, respectively. In general, $K_{\mathrm{bpd}}$ gets smaller, and $\sigma_{\Delta t}$ gets larger as $D$ increases. As mentioned before, however, they get insensitive to $D$ for larger $\sigma_\eta$. The difference between precise and approximate solutions for $D = 2$ are negligibly small throughout the whole $\sigma_\eta$ range. The $K_{\mathrm{bpd}}$ follows $K_{\mathrm{bpd}}^{(\mathrm{lower})}$ and $K_{\mathrm{bpd}}^{(\mathrm{upper})}$ (thin lines) asymptotically for lower and upper $\sigma_\eta$ values. It is interesting to observe that $K_{\mathrm{bpd}}$ shows a flat curve in the middle for a wider range of $\sigma_\eta$ as $D$ increases. As a result, the loop bandwidth of the BBPLL [1] may be stabilized against the wider range of $\sigma_\eta$ as $D$ increases.

## V. Conclusion

A method to calculate the stationary state probability of the first-order BBPLL with nonzero loop delay was presented. Various statistical properties were evaluated using the state probability, and the effect of the delay on the properties was investigated through simulations.

## References

[1] Y. Choi, D.-K. Jeong, and W. Kim, "Jitter transfer analysis of tracked oversampling techniques for multigigabit clock and data recovery," *IEEE Trans. Circuits Syst. II, Analog Digit. Signal Process.*, vol. 50, no. 11, pp. 775–783, Nov. 2003.

[2] A. Papoulis, *Probability, Random Variables, and Stochastic Processes*, 3rd ed. New York: McGraw-Hill, 1991.

[3] N. D. Dalt, "Markov chains-based derivation of the phase detector gain in bang-bang PLLs," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 53, no. 11, pp. 1–5, Nov. 2006.

[4] N. D. Dalt, "A design-oriented study of the nonlinear dynamics of digital bang-bang PLLs," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 52, no. 1, pp. 21–31, Jan. 2005.

[5] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 1985.