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TECHNICAL APPENDIX

Inference in the spatial autoregressive efficiency model with an application to Dutch dairy farms

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1 Spatial autoregressive efficiency model

1.1 Setup

Consider the model:

$$\mathbf{y}_t = \mathbf{a} + \mathbf{X}_t\boldsymbol{\beta} + \mathbf{v}_t + \log(\mathbf{TE}_t) \tag{1}$$

$$\mathbf{s}_t = \rho\mathbf{W}\mathbf{s}_{t-1} + \mathbf{Z}\boldsymbol{\delta} + \boldsymbol{\xi}_t \tag{2}$$

$$\mathbf{s}_0 = (\mathbf{I}-\rho\mathbf{W})^{-1}\mathbf{Z}\boldsymbol{\delta} + (\mathbf{I}-\rho\mathbf{W})^{-1}\boldsymbol{\xi}_0 \tag{3}$$

where $\mathbf{s}_t = \log\left(\frac{\mathbf{TE}_t}{1-\mathbf{TE}_t}\right)$. This relationship implies that $\mathbf{TE}_t = \frac{e^{\mathbf{s}_t}}{1+e^{\mathbf{s}_t}}$ and $\log(\mathbf{TE}_t) = \mathbf{s}_t - \log(1+e^{\mathbf{s}_t})$. The following assumptions are made for the model's error components:

- $\mathbf{v}_t \sim \mathcal{N}(0, \tau\mathbf{I})$
- $\mathbf{a} \sim \mathcal{N}(0, \omega\mathbf{I})$
- $\boldsymbol{\xi}_t \sim \mathcal{N}(0, \phi\mathbf{I})$
- $\boldsymbol{\xi}_0 \sim \mathcal{N}(0, \phi(\mathbf{I}-\rho\mathbf{W})^2)$

where τ, ω and ϕ are precision parameters, while $\mathbf{I}$ is a $N \times N$ identity matrix[1].

Let $\boldsymbol{\theta} = [\omega, \boldsymbol{\beta}, \tau, \rho, \boldsymbol{\delta}, \phi]'$. Based on this setup, the complete-data likelihood is written as:

$$\begin{aligned}
p(\mathbf{y}_t, \{\mathbf{a}\}, \{\mathbf{s}_t\}|\boldsymbol{\theta}, \mathbf{X}_t, \mathbf{Z}) &= p(\mathbf{y}_t|\{\mathbf{a}\}, \{\mathbf{s}_t\}, \boldsymbol{\beta}, \tau, \mathbf{X}_t) \times p(\{\mathbf{s}_t\}|\rho, \boldsymbol{\delta}, \phi, \mathbf{Z}) \times p(\{\mathbf{a}\}|\omega) \\
&= \frac{\tau^{NT/2}}{(2\pi)^{NT/2}} \exp\left\{-\frac{\tau}{2}\sum_{t=0}^{T-1}[\mathbf{y}_t - \mathbf{a} - \mathbf{X}_t\boldsymbol{\beta} - \log(\mathbf{TE}_t)]^2\right\} \\
&\times \frac{\boldsymbol{\phi}_0^{N/2}}{(2\pi)^{N/2}} \exp\left\{-\frac{\boldsymbol{\phi}_0}{2}[\mathbf{s}_0 - (\mathbf{I}-\rho\mathbf{W})^{-1}\mathbf{Z}\boldsymbol{\delta}]^2\right\} \\
&\times \frac{\phi^{N(T-1)/2}}{(2\pi)^{N(T-1)/2}} \exp\left\{-\frac{\phi}{2}\sum_{t=1}^{T-1}(\mathbf{s}_t - \rho\mathbf{W}\mathbf{s}_{t-1} - \mathbf{Z}\boldsymbol{\delta})^2\right\} \\
&\times \frac{\omega^{N/2}}{(2\pi)^{\frac{N}{2}}} \exp\left\{-\frac{\omega}{2}\mathbf{a}^2\right\}
\end{aligned} \tag{4}$$

where $\boldsymbol{\phi}_0 = \phi(\mathbf{I}-\rho\mathbf{W})^2$.

[1] Note that in contrast to the manuscript where the second moment of the Normal distribution is parameterized in terms of variance (as this is more familiar to the audience), the current technical appendix uses the precision parameterization because it is customary in Bayesian analysis to work with precisions instead of variances. In practical terms, the precision is simply the inverse of the variance. Specifically, $\tau = \frac{1}{\sigma_v^2}$, $\omega = \frac{1}{\sigma_\alpha^2}$ and $\phi = \frac{1}{\sigma_\xi^2}$.

1.2 Priors

The following priors are imposed on the model's parameters:

- $p(\boldsymbol{\beta}) = \frac{|\mathbf{P}|^{1/2}}{(2\pi)^{K/2}} \exp\Big\{ -\frac{1}{2}(\boldsymbol{\beta}-\mathbf{m})'\mathbf{P}(\boldsymbol{\beta}-\mathbf{m})\Big\}$
- $p(\boldsymbol{\delta}) = \frac{|\mathbf{R}|^{1/2}}{(2\pi)^{L/2}} \exp\Big\{ -\frac{1}{2}(\boldsymbol{\delta}-\mathbf{q})'\mathbf{R}(\boldsymbol{\delta}-\mathbf{q})\Big\}$
- $p(\tau) = \frac{b_\tau^{a_\tau}}{\Gamma(a_\tau)}\tau^{a_\tau-1}e^{-b_\tau \tau}$
- $p(\omega) = \frac{b_\omega^{a_\omega}}{\Gamma(a_\omega)}\omega^{a_\omega-1}e^{-b_\omega \omega}$
- $p(\phi) = \frac{b_\phi^{a_\phi}}{\Gamma(a_\phi)}\phi^{a_\phi-1}e^{-b_\phi \phi}$
- $p(\rho) = \frac{\rho^{a-1}(1-\rho)^{b-1}}{B(a,b)}$

1.3 Posterior

The posterior density of the model's parameters and the latent data is:

$$\pi(\boldsymbol{\theta}, \{\mathbf{a}\}, \{\mathbf{s}_t\}|\mathbf{y}_t, \mathbf{X}_t, \mathbf{Z}) \propto p(\mathbf{y}_t, \{\mathbf{a}\}, \{\mathbf{s}_t\}|\boldsymbol{\theta}, \mathbf{X}_t, \mathbf{Z}) \times p(\boldsymbol{\theta}) \tag{5}$$

where $p(\mathbf{y}_t, \{\mathbf{a}\}, \{\mathbf{s}_t\}|\boldsymbol{\theta}, \mathbf{X}_t, \mathbf{Z})$ is the complete-data likelihood defined in subsection 1.1 and $p(\boldsymbol{\theta})$ is the product of all prior densities defined in subsection 1.2.

1.4 Full conditionals

For convenience, the following are defined: $\mathbf{y}_t^* = \mathbf{y}_t - \mathbf{a} - \log(\mathbf{TE}_t)$, $\mathbf{s}_t^* = \mathbf{s}_t - \rho\mathbf{W}\mathbf{s}_{t-1}$, $\mathbf{d}_t^* = \mathbf{s}_t - \mathbf{Z}\boldsymbol{\delta}$, $\mathbf{g}_t^* = \mathbf{y}_t - \mathbf{a} - \mathbf{X}_t\boldsymbol{\beta}$.

The full conditionals of the structural parameters and the latent states are:

- The full conditional for $\boldsymbol{\beta}$ is a multivariate Normal:

  $$\pi(\boldsymbol{\beta}|\cdot) = \frac{|\tilde{\mathbf{P}}|^{1/2}}{(2\pi)^{K/2}} \exp\Big\{ -\frac{1}{2}(\boldsymbol{\beta}-\tilde{\mathbf{m}})'\tilde{\mathbf{P}}(\boldsymbol{\beta}-\tilde{\mathbf{m}})\Big\}$$

  where:

  - $\tilde{\mathbf{m}} = (\tau\mathbf{X}_t'\mathbf{X}_t + \mathbf{P})^{-1}(\tau\mathbf{X}_t'\mathbf{y}_t^* + \mathbf{P}\mathbf{m})$
  - $\tilde{\mathbf{P}} = \tau\mathbf{X}_t'\mathbf{X}_t + \mathbf{P}$

- The full conditional for $\boldsymbol{\delta}$ is a multivariate Normal:

  $$\pi(\boldsymbol{\delta}|\cdot) = \frac{|\tilde{\mathbf{R}}|^{1/2}}{(2\pi)^{L/2}} \exp\Big\{ -\frac{1}{2}(\boldsymbol{\delta}-\tilde{\mathbf{q}})'\tilde{\mathbf{R}}(\boldsymbol{\delta}-\tilde{\mathbf{q}})\Big\}$$

  where:

  - $\tilde{\mathbf{q}} = (\phi\mathbf{Z}'\mathbf{Z} + \mathbf{R})^{-1}(\phi\mathbf{Z}'\mathbf{s}_t^* + \mathbf{R}\mathbf{q})$
  - $\tilde{\mathbf{R}} = \phi\mathbf{Z}'\mathbf{Z} + \mathbf{R}$

- The full conditional for τ is a Gamma:

$$\pi(\tau|\cdot) = \frac{\tilde{b}_\tau^{\tilde{a}_\tau}}{\Gamma(\tilde{a}_\tau)} \tau^{\tilde{a}_\tau - 1} e^{-\tilde{b}_\tau \tau}$$

  where:

  - $\tilde{a}_\tau = \frac{NT}{2} + a_\tau$
  - $\tilde{b}_\tau = \frac{1}{2}(\mathbf{y}_t^* - \mathbf{X}_t\boldsymbol{\beta})'(\mathbf{y}_t^* - \mathbf{X}_t\boldsymbol{\beta}) + b_\tau$

- The full conditional for ω is a Gamma:

$$\pi(\omega|\cdot) = \frac{\tilde{b}_\omega^{\tilde{a}_\omega}}{\Gamma(\tilde{a}_\omega)} \omega^{\tilde{a}_\omega - 1} e^{-\tilde{b}_\omega \omega}$$

  where:

  - $\tilde{a}_\omega = \frac{N}{2} + a_\omega$
  - $\tilde{b}_\omega = \frac{1}{2}\mathbf{a}^2 + b_\omega$

- The full conditional for ϕ is a Gamma:

$$\pi(\phi|\cdot) = \frac{\tilde{b}_\phi^{\tilde{a}_\phi}}{\Gamma(\tilde{a}_\phi)} \phi^{\tilde{a}_\phi - 1} e^{-\tilde{b}_\phi \phi}$$

  where:

  - $\tilde{a}_\phi = \frac{NT}{2} + a_\phi$
  - $\tilde{b}_\phi = \frac{(\mathbf{I}-\rho\mathbf{W})^2}{2}[\mathbf{s}_0 - (\mathbf{I} - \rho\mathbf{W})^{-1}\mathbf{Z}\boldsymbol{\delta}]^2 + \frac{1}{2}(\mathbf{s}_t - \rho\mathbf{W}\mathbf{s}_{t-1} - \mathbf{Z}\boldsymbol{\delta})^2 + b_\phi$

- The full conditional for ρ does not belong to any known family:

$$\begin{aligned}\pi(\rho|\cdot) = \exp&\left\{ -\frac{\boldsymbol{\phi}_0}{2}\Big[[(\mathbf{I} - \rho\mathbf{W})^{-1}\mathbf{Z}\boldsymbol{\delta}]^2 - 2\mathbf{s}_0(\mathbf{I} - \rho\mathbf{W})^{-1}\mathbf{Z}\boldsymbol{\delta}\Big]\right\} \\ \times \exp&\left\{ -\frac{\phi}{2}\sum_{t=1}^{T-1}\Big[(\rho\mathbf{W}\mathbf{s}_{t-1})^2 - 2\mathbf{d}_t^*\rho\mathbf{W}\mathbf{s}_{t-1}\Big]\right\} \\ \times\, &\rho^{a-1}(1-\rho)^{b-1}\end{aligned}$$

- The full conditional for $\mathbf{a}$ is Normal:

$$\pi(\mathbf{a}|\cdot) = \frac{\tilde{t}^{N/2}}{(2\pi)^{N/2}} \exp\left\{ -\frac{\tilde{t}}{2}(\mathbf{a} - \tilde{\mathbf{c}})\right\}$$

  where:

  - $\tilde{\mathbf{c}} = \frac{\tau}{\tilde{\mathbf{t}}}\sum_{t=1}^{T}[\mathbf{y}_t - \mathbf{X}_t\boldsymbol{\beta} - \log(\mathbf{TE}_t)]$
  - $\tilde{t} = \tau T + \omega$

- The full conditional for $\mathbf{s}_0$ does not belong to any known family:

$$\begin{aligned}\pi(\mathbf{s}_0|\cdot) = \exp&\left\{ -\frac{\tau}{2}\Big[[\mathbf{s}_0 - \log(1 + e^{s_0})]^2 - 2\mathbf{g}_0^*[\mathbf{s}_0 - \log(1 + e^{s_0})]\Big]\right\} \\ \times \exp&\left\{ -\frac{1}{2}\Big[\mathbf{s}_0^2(\boldsymbol{\phi}_0 + \phi\rho^2\mathbf{W}^2) - 2\mathbf{s}_0\Big[(\mathbf{I} - \rho\mathbf{W})^{-1}\mathbf{Z}\boldsymbol{\delta}\boldsymbol{\phi}_0 + \phi\rho\mathbf{W}(\mathbf{s}_1 - \mathbf{Z}\boldsymbol{\delta})\Big]\Big]\right\}\end{aligned}$$

- The full conditional for $\mathbf{s}_t$ does not belong to any known family:

$$\pi(\mathbf{s}_t|\cdot) = \exp\Bigg\{ -\frac{\tau}{2}\sum_{t=0}^{T-1}\Big[[\mathbf{s}_t - \log(1+e^{s_t})]^2 - 2\mathbf{g}_t^*[\mathbf{s}_t - \log(1+e^{s_t})]\Big]\Bigg\}$$
$$\times \exp\Bigg\{ -\frac{\phi}{2}\sum_{t=1}^{T-1}\Bigg[\mathbf{s}_t^2(\mathbf{I}+\rho^2\mathbf{W}^2) - 2\mathbf{s}_t\Big[(\mathbf{I}-\rho\mathbf{W})\mathbf{Z}\boldsymbol{\delta} + \rho\mathbf{W}(\mathbf{s}_{t+1}+\mathbf{s}_{t-1})\Big]\Bigg]\Bigg\}$$

- The full conditional for $\mathbf{s}_{t-1}$ does not belong to any known family:

$$\pi(\mathbf{s}_{t-1}|\cdot) = \exp\Bigg\{ -\frac{\tau}{2}\sum_{t=1}^{T-1}\Big[[\mathbf{s}_{t-1} - \log(1+e^{s_{t-1}})]^2 - 2\mathbf{g}_t^*[\mathbf{s}_{t-1} - \log(1+e^{s_{t-1}})]\Big]\Bigg\}$$
$$\times \exp\Bigg\{ -\frac{\phi}{2}\sum_{t=2}^{T-1}\Bigg[\mathbf{s}_{t-1}^2 - 2\mathbf{s}_{t-1}[\mathbf{Z}\boldsymbol{\delta} + (\mathbf{I}-\rho\mathbf{W})\mathbf{s}_{t-2}]\Bigg]\Bigg\}$$